Identifying the topological order of quantized half-filled Landau levels through their daughter states

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(Dated: May 8, 2024)

Fractional quantum Hall states at a half-filled Landau level are believed to carry an integer number $\mathcal{C}$ of chiral Majorana edge modes, reflected in their thermal Hall conductivity. We show that this number determines the primary series of Abelian fractional quantum Hall states that emerge above and below the half-filling point. On a particular side of half-filling, each series may originate from two consecutive values of $\mathcal{C}$, but the combination of the series above and below half-filling uniquely identifies $\mathcal{C}$. We analyze these states both by a hierarchy approach and by a composite fermion approach. In the latter, we map electrons near a half-filled Landau level to composite fermions at a weak magnetic field and show that a bosonic integer quantum Hall state is formed by pairs of composite fermions and plays a crucial role in the state's Hall conductivity.

The nature of the ground state of quantum Hall states at half-integer filling factors, such as $\nu = 5/2$ [1], remains an open question of great interest due to the potential for hosting non-Abelian excitations useful for topological quantum computation. Many candidate states [2–4] were proposed over time, including Moore-Reed Pfaffian [5], anti-Pfaffian [6], and particle-hole-Pfaffian (PH-Pfaffian) [7]. It was later realized [8] that these states belong to an infinite series of states, all sharing an electrical Hall conductance of $\sigma_{xy} = \frac{e^2}{2h}$ but differenting by thermal Hall conductances $\kappa_{xy} = \frac{\pi^2 T}{6h}(2 + \mathcal{C})$. The topological index $\mathcal{C}$ is an integer; the state is Abelian if $\mathcal{C}$ is even and carries non-Abelian Ising anyons [9] if $\mathcal{C}$ is odd.

While numerical works seem to favor the Pfaffian ($\mathcal{C} = 1$) and anti-Pfaffian ($\mathcal{C} = -3$) states [10–12] and early quasiparticle tunneling experiments [13] indicated an anti-Pfaffian order, there is now accumulating experimental evidence in favor of the PH-Pfaffian ($\mathcal{C} = -1$) in narrow-well GaAs samples [14–18], possibly stabilized by disorder [19–22]. Despite numerous proposed and performed experiments [4, 23–36], the value of $\mathcal{C}$ and the precise identification of the half-filled states in different systems are still under intensive study.

A promising direction has been opened up by recent experiments on high-mobility GaAs and graphene samples, attempting to identify the state at a half-filled Landau level (the "parent state") through the quantum Hall states occurring in filling fractions close to it (the "daughter states") [37–39]. Levin–Halperin hierarchy [38] of states spanning from $\nu = 5/2$ Pffafian (and, by particle-hole conjugation, anti-Pfaffian) state is particularly interesting, with recent experiments in wide GaAs wells [40, 41] and bilayer graphene [42–45] observing fractions corresponding to the daughter states of Pfaffian and anti-Pfaffian states.

In this work, we calculate the series of daughter states that emerge from a parent state with an arbitrary $\mathcal{C}$. While we start the calculation using the hierarchy approach employed in earlier works, we then show how the

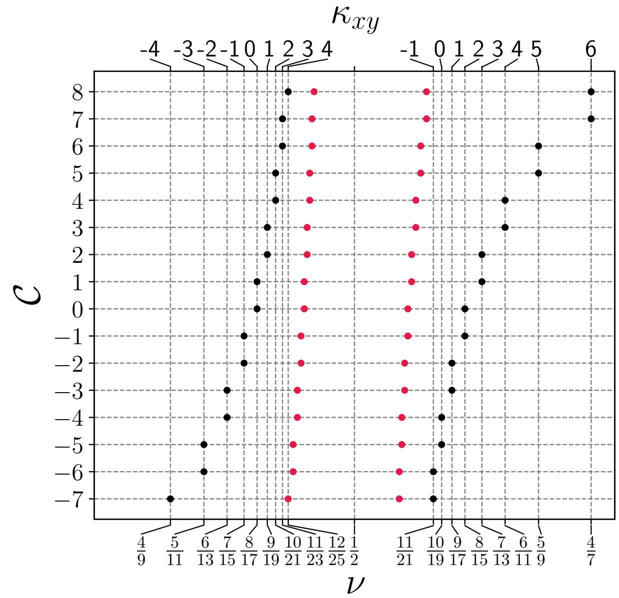

FIG. 1: The daughter states on the first two levels of hierarchy. The thermal conductance refers to the state at the first level of the hierarchy (black dots).

daughter states can be understood in terms of flux attachment [46] that maps electrons at a half-filled Landau level to composite fermions at zero magnetic field. This approach, in which the index $\mathcal{C}$ corresponds to the Chern number of the composite fermion superconductor, allows us to elucidate the relation between the infinite number of possible values of $\mathcal{C}$ and the so-called "sixteen-fold way", which introduces a 16-fold periodicity in $\mathcal{C}$ [9, 47].

Specifically, we find that for *Abelian* parent states (even $\mathcal{C}$), the series of daughter states is parametrized by two integers, $\mathcal{C}/2$ and $m > 0$. Denoting $\mathcal{C}' = \mathcal{C} + 16k$ for some integer k such that $-7 \leqslant \mathcal{C}' \leqslant 8$, the filling factors satisfy

$$\nu^{-1} = 2 \pm \left(8m \pm \mathcal{C}'/2\right)^{-1}, \tag{1}$$

while the thermal Hall conductance is

$$\kappa_{xy} = \left(\frac{\mathcal{C}}{2} + 1 \pm 1\right)\kappa_0, \tag{2}$$

where $\pm$ corresponds to holes and particles and $\kappa_0 = \frac{\pi^2 T}{3h}$.

For non-Abelian parent states (odd $\mathcal{C}$), we find that each daughter state has the same filling fraction and anyonic content as a daughter state of an Abelian parent state. In particular, for $\nu > 1/2$, Eqs. (1) and (2) hold for daughter states of the non-Abelian parent states with $\mathcal{C}+1$ instead of $\mathcal{C}$ and for $\nu < 1/2$ with $\mathcal{C}-1$ instead of $\mathcal{C}$. The filling fractions of all the daughter states are visualized in Fig. 1.

Notably, these results imply that identifying two series of daughter states—above and below the half-filled Landau level—is sufficient to identify the parent state from which they emerge. However, an identification of one series is consistent with two values of $\mathcal{C}$, separated by one, corresponding to one Abelian and one non-Abelian state.

Eq. (1) is written in a suggestive way, drawing an analogy to the composite fermion theory of the Jain series [46]. In that theory, $\nu^{-1} = 2 + \nu_{\mathrm{CF}}^{-1}$, where ν_{CF} is the composite fermions filling factor, and the Jain series corresponds to integer ν_{CF}. Indeed, all fractions included in (1) are Jain fractions. However, the thermal Hall conductance of a Jain state $\nu = (2 + \frac{1}{p})^{-1}$ is $\kappa_{xy} = p$. Remarkably, in the series we consider here, the variation of the density or magnetic field changes m but keeps $\mathcal{C}$ constant. Thus, it changes the electric Hall conductance while keeping the thermal Hall conductance fixed. As we show below, the m-dependence of Eq. (1) can be understood as originating from an integer quantum Hall state of bosons comprised of pairs of composite fermions.

The sixteen-fold way is apparent in Eq. (1), in which shifting $\mathcal{C}$ by 16 yields the same filling fractions. The two daughter states have a thermal Hall conductance that differs by $8\kappa_0$. Since their anyon content is identical, the difference in thermal conductance can be described as originating from the attachment of decoupled layers of the E_8 state, a bosonic Abelian state with no anyons [48, 49].

Hierarchy construction—The daughter states of the Abelian parents can be obtained from the parent states using the Haldane–Halperin hierarchical construction [50, 51].

We first briefly review the hierarchical construction. The wavefunction of the daughter state with electrons at positions $\{\mathbf{r}_k\}$ can be written as [3]

$$\Psi_{\mathrm{d}}(\{\mathbf{r}_k\}) = \int \mathrm{d}\boldsymbol{\eta}\, \Phi^*(\{\boldsymbol{\eta}_j\})\Psi_{\mathrm{p}}(\{\boldsymbol{\eta}_j\},\{\mathbf{r}_k\}) \tag{3}$$

where Ψ_{p} is the parent state at filling $1/2$ with N quasiparticles at positions $\{\boldsymbol{\eta}_j\}$ and $\mathrm{d}\boldsymbol{\eta} = \prod_j \mathrm{d}\boldsymbol{\eta}_j$. At a low density of quasiparticles, the coordinates $\{\boldsymbol{\eta}_j\}$ are fixed in a Wigner crystal structure. When the density is sufficiently high, the quasiparticles condense to form the next level of the FQH hierarchy. Then Φ, which is called a pseudo wavefunction (since it is not single-valued), can be written as

$$\Phi^*(\{\boldsymbol{\eta}_j\}) = P(\{w_k\})Q(\{w_k\})e^{-\sum_k \frac{|q||w_k|^2}{4\ell_0^2}}. \tag{4}$$

Here, $w = \eta_x \mp i\eta_y$ is a complex coordinate, with the sign depending on the sign of the quasiparticle charge $q = 1/4$. The term Q is

$$Q(\{w\}) = \prod_{j<k}(w_k - w_j)^{\mp 1/\lambda}. \tag{5}$$

As a result, when two quasiparticles are exchanged, the wavefunction will change by a phase factor $(-1)^{\pm 1/\lambda}$, as expected from the particles with fractional statistics. P is a symmetric polynomial, which, in the spirit of the Laughlin argument [52], is chosen to be

$$P(\{w\}) = \prod_{j<k}(w_k - w_j)^{2m} \tag{6}$$

to ensure high-degree zeros when two quasiparticles are brought close together. $|\Psi_{\mathrm{d}}(\{\mathbf{r}\})|^2$ is then describing a two-dimensional plasma at inverse temperature $\beta = \lambda_{\mathrm{d}}$. The value of λ_{d} is given by

$$\lambda_{\mathrm{d}} = 2m \pm \lambda^{-1}, \tag{7}$$

and from the angular momentum of Φ, $L_{\max} \sim \lambda_{\mathrm{d}}^2 N$, we conclude that the the filling of the quasiparticles is $\nu_{\mathrm{anyon}} = 1/\lambda_{\mathrm{d}}^2$. The filling fraction of the state that is formed is then

$$\nu = \frac{1}{2} \pm \frac{1}{16\lambda_{\mathrm{d}}}. \tag{8}$$

We first apply this procedure to Abelian $\nu = 1/2$ parent states. Substituting $q = 1/4$, and using the exchange phase of quasiparticles in the parent state $\lambda^{-1} = \frac{\mathcal{C}+1}{8}$ [9], we get

$$\lambda_{\mathrm{d}} = 2m \mp \frac{\mathcal{C}+1}{8} = \frac{16m \mp (\mathcal{C}+1)}{8} \tag{9}$$

$$\nu = \frac{1}{2} \pm \frac{1}{2}\cdot\frac{1}{16m \mp (\mathcal{C}+1)} = \frac{8m \mp \mathcal{C}/2}{16m \mp (\mathcal{C}+1)}, \tag{10}$$

where $\mp$ again corresponds to particles and holes.

In the non-Abelian case, the pseudo-wavefunction Φ depends on the conformal blocks α, which is defined by the pairwise fusion channels:

$$\Psi_{\mathrm{d}}(\{\mathbf{r}\}) = \int \mathrm{d}\boldsymbol{\eta} \sum_{\alpha} \Phi_\alpha^*(\{\boldsymbol{\eta}\})\Psi_{\mathrm{p},\alpha}(\{\boldsymbol{\eta}\},\{\mathbf{r}\}), \tag{11}$$

The pseudo wavefunction can be split into a similar product as in Eq. (4) with an additional term, Y_α that captures the dependence on α:

$$\Phi^*_\alpha(\{\boldsymbol{\eta}_j\}) = Y_\alpha(\{w_j\})P(\{w_j\})Q(\{w_j\})e^{-\sum_j \frac{|q||w_j|^2}{4\ell_0^2}}. \tag{12}$$

Similarly to the Abelian case, we want Φ^*_α to transform in opposite manner to the Ψ_α under braiding: if $\Psi_\alpha \mapsto U_{\alpha\beta}\Psi_\beta$, then $\Phi^*_\alpha \mapsto \Phi^*_\alpha U^*_{\beta\alpha}$. The factor $Y_{n,\alpha}$ can be expressed as a conformal field theory (CFT) correlator [53, 54].

The filling fraction of the daughter state is determined by the maximal angular momentum of Φ_α, i.e., by its scaling as $w_i \to \infty$. The scaling of Y_α depends on the relative sign of $(\nu - 1/2)$ and $\mathcal{C}$. If the relative sign is negative (i.e., quasiholes for $\mathcal{C} > 0$ and quasiparticles for $\mathcal{C} < 0$), then Y_α is just a correlator of the Ising theory of the opposite chirality to the one appearing in $\Psi_{\mathrm{p},\alpha}$. For example, for $\mathcal{C} > 0$ the α-depending part of $\Psi_{\mathrm{p},\alpha}$ is of the form $\left\langle \prod_j \sigma(w_j)\right\rangle_\alpha$. For quasiholes, Φ_α needs to be anti-holomorphic, thus we can choose $Y_\alpha = \left\langle \prod_j \sigma'(\bar{w}_j)\right\rangle_\alpha$, which will leave Eq. (11) invariant under braiding. Thus, for the negative relative sign between $(\nu - 1/2)$ and $\mathcal{C}$, the contribution of the Y_α part to the angular momentum does not scale with the system size and hence does not affect the filling fraction. The scaling of Φ_α as $w_i \to \infty$ is given by $\lambda_\mathrm{d}^2 N$ (10) with

$$\lambda = \frac{\mathcal{C} + 1 - \mathrm{sgn}(\mathcal{C}')}{8}, \tag{13}$$

which is identical to the case of the Abelian parent with Chern number $\mathcal{C} - \mathrm{sgn}(\mathcal{C}')$. When $(\nu - 1/2)$ and $\mathcal{C}$ have the same sign, we show below that Y_α contributes additional $\mathrm{sgn}(\mathcal{C}')/4$ to λ, giving a total of

$$\lambda = \frac{\mathcal{C} + 1 + \mathrm{sgn}(\mathcal{C}')}{8}, \tag{14}$$

corresponding to the case of the Abelian parent with Chern number $\mathcal{C} + \mathrm{sgn}(\mathcal{C}')$, from which we can get the filling factor using Eq. (10). In the rest of the paper, we limit the values of $\mathcal{C}$ to be between -7 and 8. The discussion can be easily generalized to other values of $\mathcal{C}$.

Daughter states of Abelian parent states—The topological properties of the Abelian daughter states are most concisely described using the combination of K-matrices and charge vector t [3, 55–57]. The K-matrices are symmetric and integer-valued, and their determinant counts the topologically distinct quasiparticles. The Hall conductivity is $t^\top K^{-1} t$. Quasiparticles are described by integer-valued vectors ℓ; the quasiparticle charge is $t^\top K^{-1}\ell$, and the mutual fractional statistics of two quasiparticles ℓ_1, ℓ_2 is $\ell_1^\top K^{-1}\ell_2$. Finally, the same state may be described by infinitely many pairs of K, t, related to one another by an SL($\mathbb{Z}$) transformation W, such that $K' = W^\top K W$, and $t' = W^\top t$.

We use two choices for K and t to describe each state. First, we use a symmetric charge vector $t^\top = \begin{pmatrix} 1 & 1 & \dots & 1\end{pmatrix}$ to construct the K-matrices of all Abelian parent states. We start from 113 state with K-matrix

$$K = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \tag{15}$$

This state has two counterpropagating modes and vanishing thermal Hall conductance, which corresponds, in our notation, to $\mathcal{C} = -2$.

To construct other Chern number states, we perform two operations alternately [47]: particle-hole conjugation $K \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -K \end{pmatrix}$ that maps $\mathcal{C} \mapsto -2 - \mathcal{C}$ and flipping the direction of the neutral modes by changing the direction of two fluxes ($K \mapsto \Sigma - K$, where $\Sigma_{ij} = 4$) that maps $\mathcal{C} \mapsto -\mathcal{C}$. These operations generate all even-$\mathcal{C}$ states.

Denoting by $K_\mathcal{C}$ the K-matrix of the Abelian parent states with $\mathcal{C}$, we now follow the prescription by Wen [57] to construct K, t for the daughter states. We write the K-matrix of the daughter state as

$$\begin{pmatrix} K_\mathcal{C} & \ell \\ \ell^\top & 2m \end{pmatrix}, \tag{16}$$

where ℓ is the integer-valued vector that generates a quarter-charge quasiparticle. Note that any choice of ℓ with the same charge and statistical phase would give the same state.

The filling fraction of the state, which is also its Hall conductivity, is given by Eq. (10). The elementary charges are $1/(16m \mp (\mathcal{C}+1))$; the statistical phase is $\pi \pm \frac{2\pi}{16m \mp (\mathcal{C}+1)}$ with $\mp$ and $\pm$ corresponding quasiparticles and quasiholes. Thermal Hall conductance is given by the thermal Hall conductance of the original state $\pm\kappa_0$, i.e., $\kappa_{xy} = (1 + \mathcal{C}/2 \pm 1)\kappa_0$.

The second choice we use for K and t allows us to describe the daughter states in terms of flux attachment. In this description, the attachment of two flux quanta to each electron maps a daughter state filling fraction to an integer filling fraction, and this integer is composed of a $\mathcal{C}/2$ integer quantum Hall state of electrons in parallel to a $2m$ integer quantum Hall state of charge-two bosons [49, 58, 59]. To show that, we use the SL transformations given in Supplementary material [60] to change the charge vector to be composed of $|\mathcal{C}|/2$ (2 for $K = 8$) entries of 1 (corresponding to single electrons), and two entries of 2 (corresponding to bosonic pairs of electrons), i.e., $t^\top = \begin{pmatrix} 1 & \dots & 1 & 2 & 2\end{pmatrix}$. The K-matrix then becomes $K = K_0 + \Phi$, where K_0 is made of a diagonal fermionic $\frac{|\mathcal{C}|}{2} \times \frac{|\mathcal{C}|}{2}$ block (2×2 for $K = 8$), describing an integer quantum Hall state of $\nu_f = \mathcal{C}/2$, and a bosonic 2×2 block

$$K_0 = \begin{pmatrix} K_f & 0 \\ 0 & K_b \end{pmatrix}, \tag{17}$$

with

$$K_b = \begin{pmatrix} 0 & 1 \\ 1 & 2(1-m) \end{pmatrix} \tag{18}$$

The matrix K_b, together with the corresponding elements of the charge vector, describe a bosonic integer quantum Hall state of $\nu_b = 2m$, whose contribution to the Hall conductivity is $8m$. The flux attachment part of the K-matrix is $\Phi = 2tt^\top$, such that $\Phi_{ij} = 2t_i t_j$. Two flux quanta are attached to each electron, such that a boson, which is a pair of electrons, carries four flux quanta.

In the study of the Jain series, flux attachment in the form of composite fermion theory has been successful in identifying an emergent length scale, the composite fermion cyclotron radius $R_c^* \propto \nu_{\mathrm{CF}}/k_F$, with k_F being the Fermi wave-vector [61]. This scale may also be written as $\hbar k_F/e^*B$, where e^* is the quasiparticle charge. Indeed, it was experimentally observed [62]. The mapping we have here suggests the existence of two length scales. The first is a scale that is inversely proportional to e^*, and hence proportional to $16m \mp (\mathcal{C}+1)$. This scale changes as m is varied by changing the distance of the electronic filling fraction from 1/2. The second scale that emerges from the fermionic $\nu_f = \mathcal{C}/2$ part, is $\propto \mathcal{C}/k_F$ and thus is independent of m. Interestingly, the corresponding momentum scale, $\hbar k_F/\mathcal{C}$ has a role in the parent state of the half-filled Landau level. It is the momentum scale over which the super-conducting order parameter winds for the pairing of angular momentum $\mathcal{C}$. A microscopic model is needed, however, to investigate the roles of these two scales, and we leave such an investigation to future work.

Daughters of non-Abelian parent states—The construction of the daughter states for a parent state with an odd $\mathcal{C}$ starts from the construction of the parent state wavefunction. This can be done using the CFT of the $\mathcal{C} = (2D+1)\mathrm{sgn}(\mathcal{C})$ Ising state [47], which includes D neutral bosons ϕ_i, a single downstream charge mode ϕ_ρ and an Ising CFT with Majorana mode ψ and spin field σ. The vertex operators $V_\beta(w) = e^{i\beta\phi(w)}$ satisfy

$$\left\langle \prod_j V_{\beta_j}(w_j) \right\rangle = \prod_{i<j} (w_i - w_j)^{\beta_i\beta_j/k}, \tag{19}$$

and have scaling dimension $\Delta_\beta = \beta^2/(2k)$. Here, ϕ_ρ has $k=2$, and all other bosonic modes have $k=1$.

There are $2D+1$ electronic operators

$$\psi_e = \psi e^{2i\phi_\rho} \quad \psi_e = e^{\pm i\phi_i} e^{2i\phi_\rho} \tag{20}$$

and 2^D quasihole operators

$$\psi_{\mathrm{qh}} = \sigma e^{i\phi_\rho/2} \prod_{i=1}^{D} e^{\pm i\phi_i/2}. \tag{21}$$

where σ is the Ising spin field. The field ϕ_ρ is always holomorphic (downstream), while the direction of σ, ψ, and ϕ_i depend on the $\mathrm{sgn}(\mathcal{C})$.

The parent wavefunction in Eq. (11) with electrons at positions z_k and $2N$ excitations at fixed positions w_j fusing to α is then

$$\Psi_{\mathrm{p},\alpha}(\{w_j\},\{z_j\}) = \tilde{\Psi}_{\mathrm{p},\alpha}(w_j, z_k) e^{-\frac{1}{4\ell^2}\sum_k |z_k|^2} \tag{22}$$

$$\tilde{\Psi}_{\mathrm{p},\alpha}(w_j, z_k) = \left\langle \prod_j \psi_{\mathrm{qh}}(w_j) \prod_k \psi_{\mathrm{e}}(z_k) \right\rangle_\alpha . \tag{23}$$

As mentioned earlier, Φ_α (12) can be expressed as a correlator in a CFT, specifically the product of a chiral Ising model σ' and a chiral boson ϕ', in conformal block α. For quasiparticle condensate, the fields σ' and ϕ' are holomorphic and $\Psi_\alpha(\{w_j\})$ is given by

$$\Phi^*_\alpha(\{w_j\}) = e^{-\frac{1}{16\ell^2}\sum_j |w_j|^2} \left\langle \prod_j \sigma'(w_j) e^{i\lambda\phi'(w_j)} \right\rangle_\beta R_{\beta\alpha}, \tag{24}$$

with the value of λ from Eq. (14) and $R_{\beta\alpha}$ is defined below. For the quasihole condensate, the fields are anti-holomorphic, and $\Psi_\alpha(\{\bar{w}_j\})$ is obtained from in Eq. (24) by replacing w_j by $\bar{w}_j$ and using λ from Eq. (13). As a result, σ appearing in Eq. (23) and σ' appearing in Y_α are copropagating if $(\nu - 1/2)$ and $\mathcal{C}$ have the same sign, and counterpropagating otherwise.

In the counterpropagating case, $R_{\alpha\beta} = \delta_{\alpha\beta}$; since σ and σ' have opposite chirality, they also have opposite braiding phases, which sum up to zero. In the copropagating case, σ and σ' have the same chirality, and thus, the phases do not sum up to zero (correlator involving σ' transforms with U rather than $U^\dagger$). To fix that, we apply $R_{\alpha\beta}$ given by the unique unitary matrix such that $R^{-1}UR = U^\dagger$. In the conformal block α, the σ particles are grouped into pairs, and every pair fuses to 1 or ψ. The matrix R swaps between these two fusion channels ($1 \leftrightarrow \psi$). The total phase is then always $-\frac{\pi}{8} + \frac{3\pi}{8} = \frac{\pi}{4}$. This is the source of the $\mathrm{sgn}(\mathcal{C}')/4$ factor in Eq. (14). The filling fractions are then obtained using Eqs. (7) and (8).

The full expression for the daughter state wavefunction is obtained by substituting Eq. (22) and Eq. (24) into Eq. (11). The sum over α results in the full correlator without branch cuts:

$$\Psi_{\mathrm{d}}(\{z_k\}) = \int \mathrm{d}w \, \tilde{\Psi}_{\mathrm{d}} \cdot e^{-\frac{1}{4\ell^2}\sum_i |z_k|^2 - \frac{1}{16\ell^2}\sum_i |w_j|^2} \tag{25}$$

$$\tilde{\Psi}_{\mathrm{d}}(\{z_k\}) = \left\langle \prod_j \tilde{\psi}_{\mathrm{qh}}(w_j) \prod_k \tilde{\psi}_{\mathrm{e}}(z_k) \right\rangle, \tag{26}$$

where $\tilde{\psi}_{\mathrm{qh}}$ and $\tilde{\psi}_{\mathrm{e}}$ are the modified operators acquired by gathering all terms depending on quasihole and electron coordinates, correspondingly, and $\mathrm{d}w = \prod_j \mathrm{d}w_j$.

The thermal Hall conductance is given by the central charge of the total CFT in Eq. (26). For the parent state, $\kappa_{xy} = (1+\mathcal{C}/2)\kappa_0$. The thermal Hall conductance of the added CFT is $\pm 3\kappa_0/2$ (it is a product of chiral boson ϕ' and Ising CFT σ'); in total, we get $\kappa_{xy} = (5+\mathcal{C})\kappa_0/2$ for quasiparticles and $\kappa_{xy} = (\mathcal{C}-1)\kappa_0/2$ for quasiholes.

Excitations of the daughter state correspond to the insertion of an additional operator ψ_{ex} into Eq. (26). For the correlator to be single-valued and non-singular, ψ_{ex} should be local relative to the modified quasihole $\tilde{\psi}_{\text{qh}}$ and electron $\tilde{\psi}_e$ operators (given explicitly below). By writing the operator-product expansion $\psi_{\text{ex}}(v)\tilde{\psi}_{\text{qh}}(w) \sim (v-w)^{\Delta_{\text{fused}}-\Delta_{\text{ex}}-\Delta_{\text{qh}}}\psi_{\text{fused}}$, we conclude that the scaling dimension difference $\Delta_{\text{fused}} - \Delta_{\text{ex}} - \Delta_{\text{qh}}$ should be non-negative integer for the wavefunction to be single-valued and non-singular. A similar procedure should be applied to $\psi_{\text{ex}}(v)\tilde{\psi}_{\text{e}}(w)$.

In the counter-propagating case, $\tilde{\psi}_{\text{qh}}$ is given by

$$\tilde{\psi}_{\text{qh}}(w_j,\bar{w}_j) = \sigma(w_j,\bar{w}_j)e^{i\phi_\rho/2+i\lambda\phi'}\prod_i e^{\pm i\phi_i/2} \tag{27}$$

where $\sigma(w_j,\bar{w}_j)$ is the spin field in the nonchiral Ising model, and $\tilde{\psi}_{\text{e}} = \psi_{\text{e}}$.

We write a general operator in the combined CFT, $\psi_{\text{ex}} = \chi e^{i(a\phi_\rho+\sum_j a_j\phi_j+b\phi')}$ for $\chi \in \{1,\psi,\sigma,\mu\}$, where μ is the disorder operator of Ising CFT. Denoting $b = (b'-a/4)/\lambda$, the excitations local relative to electrons and quasiholes have the following constraints: a and a_j are integer if $\chi \in \{1,\psi\}$ and half-integer otherwise, b' is integer if $\chi \in \{1,\sigma\}$ and half-integer otherwise.

When satisfying the above constraints, the excitation ψ_{ex} can be written as a product of $\mathcal{O} = \psi e^{i\phi'/2\lambda}$ and additional $\tilde{\psi}_{\text{qh}}$ and $\tilde{\psi}_e$. The elementary charge can be determined [38] by inserting $\mathcal{O}^2$ into Eq. (25); using Eq. (19) and the fact that $\psi^2 = 1$, this result in factor $\prod_j(w_0 - w_j)$. That means that $\mathcal{O}^2$ creates a Laughlin quasihole in the condensate. Since the charge of the condensed anyon is $e/4$, $\mathcal{O}^2$ creates the charge of $\nu_{\text{anyon}}e/4$, and since the anyon filling is $\nu_{\text{anyon}} = \frac{1}{2m+(2D\pm1)/8}$, the elementary charge is $\frac{e}{16m+2D\pm1}$. Calculating $\langle\mathcal{O}(w_1)\mathcal{O}(w_2)\rangle$ using Eq. (19) and $\langle\psi(w_1)\psi(w_2)\rangle = (w_1-w_2)^{-1}$ we find that topological phase is $\pi \pm \frac{2\pi}{16m+2D\pm1}$.

In the copropagating case, we make use of an identity relating correlator of Ising CFT and correlator of a chiral bosonic field ϕ'':

$$\left\langle \prod_j \sigma'(w_j) \right\rangle_\alpha \left\langle \prod_j \sigma(w_j) \prod_k \psi(z_k) \right\rangle_\beta R_{\alpha\beta} = \\ = \left\langle \prod_j \exp(i\phi''(w_j)/2) \prod_k \cos(\phi''(z_k)) \right\rangle. \tag{28}$$

This identity can be understood in two steps. First, the bosonization of the Dirac fermion formed by the two Majorana fermions ψ, ψ' [63] gives $\psi \sim \cos(\phi'')$. Second, the fusion channel of the two particles $(\sigma(w_1),\sigma'(w_1))$ and $(\sigma(w_2),\sigma'(w_2))$ is either $(\psi,1)$ or $(1,\psi')$, giving after summation the full Dirac fermion $\exp(i\phi'')$, which can be written as $\exp(i\phi''(w_1)/2)\exp(i\phi''(w_2)/2)$. In total, we get

$$\tilde{\psi}_{\text{qh}}(w_j,\bar{w}_j) = e^{\frac{i}{2}\phi_\rho+i\lambda\phi'+i\phi''/2}\prod_i e^{\pm i\phi_i/2} \tag{29}$$

$$\tilde{\psi}_{\text{e}}(z_k) = e^{i2\phi_\rho(z_k)}\cos(\phi''(z_k)). \tag{30}$$

In Eq. (29), ϕ_ρ is holomorphic, and ϕ', ϕ'', ϕ_i are holomorphic for $\mathcal{C} > 0$ and anti-holomorphic otherwise.

Repeating the procedure outlined above, we get the elementary excitation of the form $\mathcal{O} = \psi e^{i(\phi'/2\lambda-\phi'')}$, with the elementary charge of $\frac{e}{16m-2D-2\pm1}$ and topological phase $\pi \pm \frac{2\pi}{16m-2D-2\pm1}$.

Thus, since all the topological properties of the daughters of the non-Abelian states are identical to those of the daughters of the corresponding Abelian states ($\mathcal{C}+1$ for $\nu > 1/2$ and $\mathcal{C}-1$ for $\nu < 1/2$), we conclude that these are indeed the same states.

To summarize, we constructed the daughter states of quantized paired states of half-filled Landau levels. We showed that the daughter states formed around half-filling reflect the Chern number of the neutral modes of the half-filled state from which they emerge. Provided that no unexpected phase transition occurs as the filling is varied away from the half-filled level, the daughter states can be used to identify the topological order of the half-filled state.

Acknowledgments—We thank David Mross, Misha Yutushui, and Yuval Ronen for instructive discussions. EZ is supported by the Adams Fellowships Program of the Israel Academy of Sciences. NL is grateful for funding from the ISF Quantum Science and Technology program (2074/19). AS was supported by grants from the ERC under the European Union's Horizon 2020 research and innovation programme (Grant Agreements LEGOTOP No. 788715), the DFG (CRC/Transregio 183, EI 519/71), and by the ISF Quantum Science ands Technology (2074/19). and Humanities.

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W matrices between hierarchical and flux basis

To transform the K matrix of the Abelian state of the daughter state to the flux basis, we first write them in hierarchical basis, by transforming $K_{\mathcal{C}}$ with $W = \delta_{i,j} - \delta_{i+1,j}$ and applying procedure Eq. (16). Now we give here explicitly the transition matrices W for every dimension of the K matrix that map the hierarchical basis to the flux-attached basis, i.e., to K_0 defined in Eq. (17).

$$W_3 = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.1}$$

$$W_4 = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.2}$$

$$W_5 = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 \\ 0 & -1 & -1 & -2 & -2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{A.3}$$

$$W_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & -1 & -1 & 1 & -2 & -2 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{A.4}$$