A note on Hölder regularity of weak solutions to linear elliptic equations

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Abstract

In this paper, we show that weak solutions of

$$-\operatorname{div} \mathbb{A}(x)\nabla u = 0 \qquad \text{where} \qquad A(x) = A(x)^T \text{ and } \lambda|\zeta|^2 \leq \langle A(x)\zeta,\zeta\rangle \leq \Lambda|\zeta|^2,$$

and $\mathbb{A}(x) \equiv \mathbb{A}$ is a constant matrix are Hölder continuous $u \in C^{\alpha}_{\text{loc}}$ with $\alpha \geq \frac{1}{2}\left(-(n-2) + \sqrt{(n-2)^2 + \frac{4(n-1)\lambda}{\Lambda}}\right)$. This implies that the example constructed by Piccinini - Spagnolo is sharp in the class of constant matrices $\mathbb{A}(x) \equiv \mathbb{A}$. The proof of Hölder regularity does not go through a reduction of oscillation type argument and instead is achieved through a monotonicity formula.

In the case of general matrices $\mathbb{A}(x)$, we obtain the same regularity under some additional hypothesis.

Keywords: Hölder regularity, Pohožaev identity, De Giorgi - Nash - Moser theory
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Contents

1. Introduction

In this paper, we shall consider weak solutions $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ solving

$$-\operatorname{div} \mathbb{A}(x)\nabla u = 0 \qquad \text{with} \qquad \lambda|\xi|^2 \leq \langle \mathbb{A}(x)\xi,\xi\rangle \leq \Lambda|\xi|^2, \tag{1.1}$$

where $\mathbb{A}(x)$ is assumed to be symmetric and $\lambda, \Lambda \in (0,\infty)$ are any two fixed positive constants.

The Hölder regularity results were first proved in [3, 8] and subsequently, a different proof was given in [5, 6, 7]. All three approaches provided the tools to prove a much stronger Harnack inequality and Hölder regularity was deduced from this. In [1], a sharp form of Harnack inequality was proved which implied that the Hölder exponent had exponential dependence on $\frac{\lambda}{\Lambda}$. Since Harnack inequality obtained in [1] was sharp, this implied that the approaches developed in [3, 8, 5] could at best give the Hölder exponent that had exponential dependence on $\frac{\lambda}{\Lambda}$.

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In $\mathbb{R}^2$, a new proof using a monotonicity formula which is an optimized version of the 'hole filling' technique was given in [9] and they obtained the sharp Hölder exponent to be $\sqrt{\frac{\lambda}{\Lambda}}$ and conjectured that in higher dimensions, the exponent should have the form

$$\alpha \geq \frac{1}{2}\left(-(n-2)+\sqrt{(n-2)^2+\frac{4(n-1)\lambda}{\Lambda}}\right).$$

In this regard, we also mention [2, 12] and references therein where the 'hole filling' technique was extended to higher dimensional linear elliptic and parabolic equations using Green's function estimates.

In this paper, we prove the following theorem:

Theorem 1.1. *Any local weak solutions of* (1.1) *with* $\mathbb{A}(x) \equiv \mathbb{A}$ *is Hölder continuous* $u \in C^{\alpha}_{\text{loc}}(\mathbb{R}^n)$ *with*

$$\alpha \geq \frac{1}{2}\left(-(n-2)+\sqrt{(n-2)^2+\frac{4(n-1)\lambda}{\Lambda}}\right).$$

Corollary 1.2. *Any local weak solutions of* (1.1) *with a general* $\mathbb{A}(x)$ *and* $\mathbb{E}_{\mathrm{r}} \geq 0$ *is Hölder continuous* $u \in C^{\alpha}_{\text{loc}}(\mathbb{R}^n)$ *with*

$$\alpha \geq \frac{1}{2}\left(-(n-2)+\sqrt{(n-2)^2+\frac{4(n-1)\lambda}{\Lambda}}\right).$$

The quantity $\mathbb{E}_{\mathrm{r}}$ *is as obtained in Lemma 4.1 and the regularity is obtained in Remark 4.5.*

It suffices to prove the following theorem instead:

Theorem 1.3. *Let* S_1 *be the sphere of unit radius in* $\mathbb{R}^n$ *and* B_1 *be the unit ball in* $\mathbb{R}^n$*. Then, any weak solution of* (1.1) *satisfies*

$$\int_{S_1} \langle \mathbb{A}\nabla u, \nabla u\rangle \, d\sigma \geq \sqrt{(n-2)^2+\frac{4(n-1)\lambda}{\Lambda}} \int_{B_1} \langle \mathbb{A}\nabla u, \nabla u\rangle \, dx.$$

Remark 1.4. *In the rest of the paper, we will assume* $\mathbb{A}(x)$ *and* u *are smooth and obtain a priori estimates. Such an assumption can be made to hold by a standard approximation procedure, see [4, Section 4] for the details. The proof also shows that for general matrices* $\mathbb{A}(x)$*, we can obtain the same regularity as Theorem 1.1 provided* $\mathbb{E}_{\mathrm{r}} \geq 0$*, where* $\mathbb{E}_{\mathrm{r}}$ *is as obtained in Lemma 4.1.*

2. Ideas of Piccinini-Spagnolo revisited

In [9], the authors proved the following result:

Theorem 2.1. *When* $n = 2$*, weak solutions of* (1.1) *are* C^{α}_{loc} *regular with* $\alpha = \sqrt{\frac{\lambda}{\Lambda}}$*. In particular, the following estimate holds for any* $r > 0$*:*

$$\int_{B_r} \langle \mathbb{A}(x)\nabla u, \nabla u\rangle \, dx \leq \frac{r\sqrt{\frac{\Lambda}{\lambda}}}{2} \int_{S_r} \langle \mathbb{A}(x)\nabla u, \nabla u\rangle \, d\sigma, \tag{2.1}$$

where B_r *is the ball of raidus* r *and* $S_r = \partial B_r$ *is the boundary of* B_r*.*

Proof. We shall change to polar co-ordinates and use the following notation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}, \quad \mathbb{J}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad \mathbb{P}(x) = \mathbb{J}(\theta)^{\top}\mathbb{A}(x)\mathbb{J}(\theta), \quad \underbrace{\begin{bmatrix} u_x \\ u_y \end{bmatrix}}_{:=\nabla u} = \mathbb{J}(\theta) \underbrace{\begin{bmatrix} u_r \\ \frac{1}{r}u_\theta \end{bmatrix}}_{:=\bar{\nabla} u} := \begin{bmatrix} u_N \\ u_T \end{bmatrix}.$$

Following [9], we have

$$\begin{aligned} g(r) &:= \int_{B_r} \langle \mathbb{A}(x)\nabla u, \nabla u\rangle \, dx = \int_{S_r} (u-k)\left\langle \mathbb{P}(x)\bar{\nabla} u, \vec{e_1}\right\rangle d\sigma = \int_{S_r} (u-k)\left(p_{11}u_\rho + p_{12}\tfrac{1}{r}u_\theta\right) d\sigma, \\ g'(r) &:= \int_{S_r} \left\langle \mathbb{P}(x)\bar{\nabla} u, e_1\right\rangle d\sigma, \end{aligned} \tag{2.2}$$

where to obtain the first equality in the definition of $g(r)$, we made use of the fact that u solves (1.1).

With the choice $k = \fint_{S_r} u\, d\sigma$ and $L := \frac{\Lambda}{\lambda}$, we shall estimate $g(r)$ as follows:

$$
\begin{aligned}
g(r) &\overset{(a)}{\leq} \left(\int_{S_r} p_{11}(u-k)^2\, d\sigma\right)^{\frac{1}{2}} \left(\int_{S_r} \left(\sqrt{p_{11}}u_r + \frac{p_{12}}{\sqrt{p_{11}}}\frac{1}{r}u_\theta\right)^2 d\sigma\right)^{\frac{1}{2}} \\
&\overset{(b)}{\leq} r\sqrt{L}\left(\lambda \int_{S_r} u_T^2\, d\sigma\right)^{\frac{1}{2}} \left(\int_{S_r} \left(\sqrt{p_{11}}u_N + \frac{p_{12}}{\sqrt{p_{11}}}u_T\right)^2 d\sigma\right)^{\frac{1}{2}} \\
&\overset{(c)}{\leq} \frac{r\sqrt{L}}{2}\left[\int_{S_r}\left(p_{22} - \frac{p_{12}^2}{p_{11}}\right)u_T^2\, d\sigma + \int_{S_r}\left(p_{11}u_N^2 + \frac{p_{12}^2}{p_{11}}u_T^2 + 2p_{12}u_N u_T\right) d\sigma\right] \\
&= \frac{r\sqrt{L}}{2}\int_{S_r} \left\langle P\bar{\nabla}u, \bar{\nabla}u\right\rangle\, d\sigma = \frac{r\sqrt{L}}{2} g'(r),
\end{aligned}
\tag{2.3}
$$

where to obtain (a), we applied Hölder's inequality, to obtain (b), we applied Proposition 3.1 along with the bound $\lambda \leq p_{11}(x) \leq \Lambda$ and finally to obtain (c), we made use of the matrix inequality $\lambda \leq p_{22}(x) - \dfrac{p_{12}^2(x)}{p_{11}(x)} \leq \Lambda$ along with Young's inequality. Recalling (2.2), the estimate in (2.3) is exactly the claim in (2.1) and this completes the proof of the theorem.

□

3. Proof of Theorem 1.3 when $\mathbb{A}(x) = \mathbb{I}$ is identity matrix - case of harmonic functions

Let us first recall the Poincaré inequality on spheres (see [11, Chapter IV, Section 2] for the details):

Proposition 3.1. *Let $u \in W^{1,2}_{\mathrm{loc}}(S_r)$ where S_r is the unit sphere in $\mathbb{R}^n$ with $n \geq 2$ and $\bar{\nabla}u = (u_N, u_T)$ be the tangential and normal derivative of u. Then the following sharp form of Poincaré's inequality holds:*

$$\int_{S_r} (u-k)^2\, d\sigma \leq \left(\frac{r^2}{n-1}\right)\int_{S_r} |u_T|^2\, d\sigma,$$

where $k = \fint_{S_r} u\, d\sigma$.

We shall also recall the well known Pohožaev identity for harmonic functions obtained in [10].

Proposition 3.2. *Let u to be a smooth, weak solution of $-\Delta u = 0$ (i.e., u is locally harmonic), then the following identity holds:*

$$\int_{S_1} |u_T|^2\, d\sigma = \int_{S_1} |u_N|^2\, d\sigma + (n-2)\int_{B_1} |\nabla u|^2\, dx.$$

Proof. Simple calculations implies

$$\Delta u \left\langle \nabla u, x\right\rangle = \operatorname{div}(\nabla u \left\langle \nabla u, x\right\rangle) - \frac{1}{2}\operatorname{div}(|\nabla u|^2 x) + \frac{n-2}{2}|\nabla u|^2. \tag{3.1}$$

Using $-\Delta u = 0$ along with integrating (3.1) over B_1, we have the following identity

$$
\begin{aligned}
0 = \int_{B_1} \Delta u \left\langle \nabla u, x\right\rangle\, dx &= \int_{S_1} \left\langle \nabla u, x\right\rangle^2\, d\sigma - \frac{1}{2}\int_{S_1} |\nabla u|^2 \left\langle x, x\right\rangle\, d\sigma + \frac{n-2}{2}\int_{B_1} |\nabla u|^2\, dx \\
&= \int_{S_1} |u_N|^2\, d\sigma - \frac{1}{2}\int_{S_1} (|u_T|^2 + |u_N|^2)\, d\sigma + \frac{n-2}{2}\int_{B_1} |\nabla u|^2\, dx,
\end{aligned}
$$

which completes the proof. □

3.1. Some discussion

If we consider solutions of $-\Delta u = 0$ in $\mathbb{R}^n$ (i.e., $\mathbb{A}(x) = I$) for $n \geq 3$ and try to follow the calculations of Theorem 2.1, then we get

$$\int_{B_r} \left\langle \nabla u, \nabla u\right\rangle\, dx \leq \frac{r}{2\sqrt{n-1}}\int_{S_r} \left\langle \nabla u, \nabla u\right\rangle\, d\sigma, \tag{3.2}$$

which fails to prove Lipschitz regularity (i.e., the constant in (3.2) should be $\frac{r}{n}$) when $n = 3$ and for $n \geq 7$, we have $2\sqrt{n-1} < n-2$ and thus no regularity follows.

On the other hand, if we consider the simple example of harmonic functions which satisfy $|\nabla u| = C$ for some constant C, then we have

$$\frac{\int_{B_r} \langle \nabla u, \nabla u \rangle \, dx}{\int_{S_r} \langle \nabla u, \nabla u \rangle \, d\sigma} = \frac{\omega_n r^n}{n\omega_n r^{n-1}} = \frac{r}{n},$$

which implies Lipschitz regularity. But the above calculations shows that the factor $\frac{1}{n}$ is coming from the ratio of $|B_r|$ and $|S_r|$ which is a property of the fact that we are in $\mathbb{R}^n$ and not necessarily from the property of the equation and solution.

This suggest that some additional cancellations should hold that can further improve (3.2) and these cancellations should be obtained from the fact that we are in $\mathbb{R}^n$. We shall formalize this observation for harmonic functions in the following theorem:

Theorem 3.3. *Let u be a local weak solution of $-\Delta u = 0$, then we have*

$$\int_{B_r} \langle \nabla u, \nabla u \rangle \, dx \leq \frac{r}{n} \int_{S_r} \langle \nabla u, \nabla u \rangle \, d\sigma,$$

holds for all $r > 0$ and in particular $u \in C^{0,1}_{\text{loc}}$.

3.2. Discussion of the two proofs

We shall give two proofs of Theorem 3.3, both very similar, but have some crucial differences which are detailed in the following observations:

(i) In the first proof, we need to make use of (3.6) to obtain (3.7). This does not require equality to hold in (3.6), but it suffices if the following inequality holds:

$$\int_{S_1} |u_T|^2 \, d\sigma \leq \int_{S_1} |u_N|^2 \, d\sigma + (n-2) \int_{B_1} |\nabla u|^2 \, dx. \tag{3.3}$$

(ii) In the first proof, we need to make use of (3.6) a second time in (3.9). This again does not require equality to hold in (3.6), but it suffices if the following inequality holds:

$$\int_{S_1} |u_T|^2 \, d\sigma \geq \int_{S_1} |u_N|^2 \, d\sigma + (n-2) \int_{B_1} |\nabla u|^2 \, dx. \tag{3.4}$$

(iii) In particular, the first proof requires (3.4) and (3.3) to both hold, which is the equality form of the identity obtained in Proposition 3.2.

(iv) In the second proof, we only need to use the Pohožaev identity (3.11) once to obtain (3.12). In particular, the second proof works if we only had the following one sided inequality:

$$\int_{S_1} |\nabla u|^2 \, d\sigma \geq 2 \int_{S_1} |u_N|^2 \, d\sigma + (n-2) \int_{B_1} |\nabla u|^2 \, dx.$$

3.3. First proof

We shall use the notation $\bar{\nabla} u = (u_N, u_T)$ where $u_N = \langle \bar{\nabla} u, \vec{e_1} \rangle$ and $u_T = (\langle \bar{\nabla} u, \vec{e_2} \rangle, \ldots, \langle \bar{\nabla} u, \vec{e_n} \rangle)$ are the normal and tangential derivatives on the sphere respectively in polar coordinates. We can also take $r = 1$ since the required estimate can be obtained by scaling.

Remark 3.4. *We will be deliberately careless in switching between ∇u and $\bar{\nabla} u$, i.e., cartesian and polar coordinates in the proof, so as to better present the ideas of the proof, noting that this switch is just a transformation by orthogonal matrices and hence does not affect any of the calculations.*

The proof will now proceed in several steps:

Step 1: Since u is harmonic, we see that

$$\begin{aligned} \int_{B_1} \langle \nabla u, \nabla u \rangle \, dx &= \int_{S_1} (u - k) \langle \nabla u, \vec{n} \rangle \, d\sigma = \int_{S_1} (u-k) \langle \bar{\nabla} u, \vec{e_1} \rangle \, d\sigma \\ &\overset{(a)}{\leq} \left(\frac{1}{2(n-1)} \int_{S_1} |u_T|^2 \, d\sigma + \frac{1}{2} \int_{S_1} |u_N|^2 \, d\sigma \right), \end{aligned} \tag{3.5}$$

where to obtain (a), we applied Young's inequality followed by Proposition 3.1 with $k = \fint_{S_1} u \, d\sigma$.

Step 2: We recall Pohožaev identity from Proposition 3.2:

$$\int_{S_1} |u_T|^2 \, d\sigma = \int_{S_1} |u_N|^2 \, d\sigma + (n-2) \int_{B_1} |\nabla u|^2 \, dx. \tag{3.6}$$

Step 3: We shall substitute (3.6) into (3.5) and obtain

$$\int_{B_1} |\nabla u|^2 \, dx \leq \frac{1}{2(n-1)} \left(\int_{S_1} |u_N|^2 \, d\sigma + (n-2) \int_{B_1} |\nabla u|^2 \, dx \right) + \frac{1}{2} \int_{S_1} |u_N|^2 \, d\sigma, \tag{3.7}$$

which after simplification, becomes

$$\int_{B_1} |\nabla u|^2 \, dx \leq \int_{S_1} |u_N|^2 \, d\sigma. \tag{3.8}$$

Step 4: Let us add $\int_{S_1} |u_N|^2 \, d\sigma$ to both sides of (3.6) and then make use of (3.8) to get

$$\begin{aligned} \int_{S_1} |\nabla u|^2 \, d\sigma = \int_{S_1} (|u_T|^2 + |u_N|^2) \, d\sigma &\overset{(3.6)}{=} 2 \int_{S_1} |u_N|^2 \, d\sigma + (n-2) \int_{B_1} |\nabla u|^2 \, dx \\ &\overset{(3.8)}{\geq} 2 \int_{B_1} |\nabla u|^2 \, dx + (n-2) \int_{B_1} |\nabla u|^2 \, dx \\ &= n \int_{B_1} |\nabla u|^2 \, dx. \end{aligned} \tag{3.9}$$

This completes the proof of the theorem.

3.4. Second proof

Using the same notation as the first proof, we will give a slightly modified proof of Theorem 3.3 which will proceed in several steps:

Step 1: Since u is harmonic, we see that

$$\begin{aligned} \int_{B_1} \langle \nabla u, \nabla u \rangle \, dx &= \int_{S_1} (u-k) \langle \nabla u, \vec{n} \rangle \, d\sigma = \int_{S_1} (u-k) \langle \bar{\nabla} u, \vec{e_1} \rangle \, d\sigma \\ &\overset{(a)}{\leq} \left(\frac{1}{2(n-1)} \int_{S_1} |u_T|^2 \, d\sigma + \frac{1}{2} \int_{S_1} |u_N|^2 \, d\sigma \right) \\ &\overset{(b)}{=} \left(\frac{1}{2(n-1)} \int_{S_1} |\nabla u|^2 \, d\sigma + \left(\frac{1}{2} - \frac{1}{2(n-1)} \right) \int_{S_1} |u_N|^2 \, d\sigma \right), \end{aligned} \tag{3.10}$$

where to obtain (a), we applied Young's inequality followed by Proposition 3.1 with $k = \fint_{S_1} u \, d\sigma$ and to obtain (b), we add and subtract $\frac{1}{2(n-1)} \int_{S_1} |u_N|^2 \, d\sigma$.

Step 2: We rewrite the Pohožaev identity from Proposition 3.2 as:

$$\int_{S_1} |\nabla u|^2 \, d\sigma = 2 \int_{S_1} |u_N|^2 \, d\sigma + (n-2) \int_{B_1} |\nabla u|^2 \, dx. \tag{3.11}$$

Step 3: We shall substitute (3.11) into (3.10) to get

$$\int_{B_1} |\nabla u|^2 \, dx \leq \frac{1}{2(n-1)} \int_{S_1} |\nabla u|^2 \, d\sigma + \left(\frac{1}{2} - \frac{1}{2(n-1)} \right) \frac{1}{2} \left(\int_{S_1} |\nabla u|^2 \, d\sigma - (n-2) \int_{B_1} |\nabla u|^2 \, dx \right), \tag{3.12}$$

which after simplification, becomes

$$\frac{n^2}{4(n-1)} \int_{B_1} |\nabla u|^2 \, dx \leq \frac{n}{4(n-1)} \int_{S_1} |\nabla u|^2 \, d\sigma.$$

This completes the proof of the lemma.

4. Proof of Theorem 1.3 when $\mathbb{A}(x) \equiv \mathbb{A}$ has constant entries

The first lemma we generalize is the Pohožaev identity from Proposition 3.2 to variable coefficient operators of the form considered in (1.1). This is well known in literature and we present all the details for sake of completeness.

Lemma 4.1. *Let u be a weak solution to (1.1) and assume both u and $\mathbb{A}(x)$ are sufficiently smooth. Then the following Pohožaev type identity holds:*

$$\begin{aligned}
\int_{S_1} \langle \mathbb{A}(x)\nabla u, \nabla u\rangle \langle \mathbb{A}(x)x, x\rangle \, d\sigma &= 2\int_{S_1} \langle \mathbb{A}(x)\nabla u, x\rangle^2 \, d\sigma + \int_{B_1} \left(\sum_i^n a_{ii}(x)\right) \langle \mathbb{A}(x)\nabla u, \nabla u\rangle \, dx \\
&\quad + \sum_{i,j,k,l} \int_{B_1} x_j(\partial_{x_i} a_{ij}) a_{kl} u_{x_k} u_{x_l} \, dx - 2\int_{B_1} \langle \mathbb{A}(x)\nabla u, \mathbb{A}(x)\nabla u\rangle \, dx \\
&\quad -2 \sum_{i,j,k,l} \int_{B_1} u_{x_i} a_{ij} u_{x_k} x_l (\partial_{x_j} a_{kl}) \, dx \\
&\quad + \sum_{i,j,m,n} \int_{B_1} a_{ij} x_j u_{x_m} u_{x_n} (\partial_{x_i} a_{mn}) \, dx \\
&= 2\int_{S_1} \langle \mathbb{A}(x)\nabla u, x\rangle^2 \, d\sigma + \int_{B_1} \left(\sum_i^n a_{ii}(x)\right) \langle \mathbb{A}(x)\nabla u, \nabla u\rangle \, dx \\
&\quad -2\int_{B_1} \langle \mathbb{A}(x)\nabla u, \mathbb{A}(x)\nabla u\rangle \, dx + \mathbb{E}_{\mathrm{r}},
\end{aligned} \tag{4.1}$$

where we have denoted

$$\mathbb{E}_{\mathrm{r}} := \sum_{i,j,k,l} \int_{B_1} x_j(\partial_{x_i} a_{ij}) a_{kl} u_{x_k} u_{x_l} \, dx - 2\sum_{i,j,k,l} \int_{B_1} u_{x_i} a_{ij} u_{x_k} x_l (\partial_{x_j} a_{kl}) \, dx + \sum_{i,j,m,n} \int_{B_1} a_{ij} x_j u_{x_m} u_{x_n} (\partial_{x_i} a_{mn}) \, dx.$$

Remark 4.2. *Note that all the terms of $\mathbb{E}_{\mathrm{r}}$ contain derivates of the entries of $\mathbb{A}$ and does not contain any terms from the Hessian matrix $[D^2u]$. In particular, (4.1) already captures the cancellations of the form $\langle x, [D^2u]\nabla u\rangle = \langle \nabla u, [D^2u]x\rangle$ which is crucially used in the case of harmonic functions to prove Proposition 3.2.*

Proof of Lemma 4.1. Simple calculations give

$$\begin{aligned}
\operatorname{div}(\mathbb{A}\nabla u \langle x, \mathbb{A}\nabla u\rangle) &= \operatorname{div}(\mathbb{A}\nabla u) \langle x, \mathbb{A}\nabla u\rangle + \langle \mathbb{A}\nabla u, \mathbb{A}\nabla u\rangle + \langle \mathbb{A}\nabla u, [D(\mathbb{A}\nabla u)]x\rangle, \\
\operatorname{div}(\mathbb{A}x \langle \mathbb{A}\nabla u, \nabla u\rangle) &= \operatorname{div}(\mathbb{A}x) \langle \mathbb{A}\nabla u, \nabla u\rangle + \langle \mathbb{A}x, [D(\mathbb{A}\nabla u)]\nabla u\rangle + \langle \mathbb{A}x, [D^2u]\mathbb{A}\nabla u\rangle,
\end{aligned} \tag{4.2}$$

where $D(\mathbb{A}\nabla u)$ is a matrix of the form

$$(D(\mathbb{A}\nabla u))_{ij} = \sum_{k=1}^n (\partial_{x_i} a_{jk}) u_{x_k} + (D^2u\mathbb{A})_{ij} =: (D(A)(\nabla u))_{ij} + ([D^2u]\mathbb{A})_{ij}, \tag{4.3}$$

and $([D^2u]\mathbb{A})$ is the matrix product of $[D^2u] \cdot \mathbb{A}$. Using (4.3) in (4.2) and noting that $\operatorname{div}(\mathbb{A}\nabla u) = 0$, we get

$$\begin{aligned}
\operatorname{div}(\mathbb{A}\nabla u \langle x, \mathbb{A}\nabla u\rangle) &= \langle \mathbb{A}\nabla u, \mathbb{A}\nabla u\rangle + \langle \mathbb{A}\nabla u, [D(\mathbb{A})(\nabla u)]x\rangle + \langle \mathbb{A}\nabla u, [D^2u]\mathbb{A}x\rangle, \\
\operatorname{div}(\mathbb{A}x \langle \mathbb{A}\nabla u, \nabla u\rangle) &= \operatorname{div}(\mathbb{A}x) \langle \mathbb{A}\nabla u, \nabla u\rangle + \langle \mathbb{A}x, [D(\mathbb{A})(\nabla u)]\nabla u\rangle + 2\langle \mathbb{A}x, [D^2u]\mathbb{A}\nabla u\rangle,
\end{aligned} \tag{4.4}$$

Subtracting twice the first equation in (4.4) from the second equation in (4.4) in order to cancel the terms containing $[D^2u]$ (since $\langle \mathbb{A}x, [D^2u]\mathbb{A}\nabla u\rangle = \langle \mathbb{A}\nabla u, [D^2u]\mathbb{A}x\rangle$), we get

$$\begin{aligned}
&2\operatorname{div}(\mathbb{A}\nabla u \langle x, \mathbb{A}\nabla u\rangle) - \operatorname{div}(\mathbb{A}x \langle \mathbb{A}\nabla u, \nabla u\rangle) \\
&\qquad = 2\langle \mathbb{A}\nabla u, \mathbb{A}\nabla u\rangle + 2\langle \mathbb{A}\nabla u, [D(\mathbb{A})(\nabla u)]x\rangle - \operatorname{div}(\mathbb{A}x) \langle \mathbb{A}\nabla u, \nabla u\rangle - \langle \mathbb{A}x, [D(\mathbb{A})(\nabla u)]\nabla u\rangle.
\end{aligned} \tag{4.5}$$

We see that $\operatorname{div}(\mathbb{A}x) = \sum_{i=1}^n a_{ii} + \sum_{i,j} x_j \partial_{x_i} a_{ij}$ which we substitute in (4.5) and integrate over B_1 to get

$$\begin{aligned}
2\int_{B_1} \operatorname{div}(\mathbb{A}\nabla u \langle x, \mathbb{A}\nabla u\rangle) \, dx &= \int_{B_1} \operatorname{div}(\mathbb{A}x \langle \mathbb{A}\nabla u, \nabla u\rangle) \, dx + 2\int_{B_1} \langle \mathbb{A}\nabla u, \mathbb{A}\nabla u\rangle \, dx \\
&\quad +2\int_{B_1} \langle \mathbb{A}\nabla u, [D(\mathbb{A})(\nabla u)]x\rangle \, dx - \int_{B_1} \langle \mathbb{A}x, [D(\mathbb{A})(\nabla u)]\nabla u\rangle \, dx \\
&\quad - \int_{B_1} \left(\sum_i^n a_{ii}\right) \langle \mathbb{A}\nabla u, \nabla u\rangle \, dx - \int_{B_1} \left(\sum_{i,j} x_j \partial_{x_i} a_{ij}\right) \langle \mathbb{A}\nabla u, \nabla u\rangle \, dx.
\end{aligned}$$

We can now apply the divergence theorem to complete the proof of the lemma. ∎

Corollary 4.3. *In the case $\mathbb{A}(x) \equiv \mathbb{A}$ is a constant matrix, then $\mathbb{E}_{\mathrm{r}} = 0$ and (4.1) becomes*

$$\begin{aligned}\int_{S_1} \langle \mathbb{A}\nabla u, \nabla u\rangle \langle \mathbb{A}x, x\rangle \, d\sigma &= 2\int_{S_1} \langle \mathbb{A}\nabla u, x\rangle^2 \, d\sigma + \int_{B_1} \left(\sum_i^n a_{ii}\right) \langle \mathbb{A}\nabla u, \nabla u\rangle \, dx \\ &\quad -2\int_{B_1} \langle \mathbb{A}\nabla u, \mathbb{A}\nabla u\rangle \, dx.\end{aligned} \tag{4.6}$$

Let us now prove a matrix inequality using the Schur complement of a block matrix that will be the higher dimensional replacement of the cancellation used in (c) of (2.3).

Lemma 4.4. *Let $\mathbb{P}$ be the matrix similar to $\mathbb{A}$ in polar coordinates which we write as*

$$\mathbb{P} = \left(\begin{array}{c|c} p_{11} & P_{12} \\ \hline P_{21} & \mathbb{P}_{22}\end{array}\right),$$

where p_{11} is a scalar, $P_{12} = P_{21}^\top$ is $(n-1)$ vector and $\mathbb{P}_{22}$ is $(n-1)\times(n-1)$ matrix. Then the following inequality holds:

$$\lambda p_{11}|\xi|^2 \le p_{11} \langle \mathbb{P}_{22}\xi, \xi\rangle - \langle P_{12}, \xi\rangle^2 \qquad \textit{for every } \xi \in \mathbb{R}^{n-1}\setminus\{0\}. \tag{4.7}$$

Proof. We have

$$\underbrace{\begin{pmatrix} 1 & 0 \\ -P_{21}p_{11}^{-1} & I\end{pmatrix}}_{:=\mathbb{S}} \underbrace{\begin{pmatrix} p_{11} & P_{12} \\ P_{21} & P_{22}\end{pmatrix}}_{=\mathbb{P}} \underbrace{\begin{pmatrix} 1 & -p_{11}^{-1}P_{12} \\ 0 & I\end{pmatrix}}_{=\mathbb{S}^\top} = \underbrace{\begin{pmatrix} p_{11} & 0 \\ 0 & P_{22} - P_{21}p_{11}^{-1}P_{12}\end{pmatrix}}_{:=\mathbb{M}}.$$

From this, we see that if we consider the $\mathbb{R}^n$ vector $\begin{pmatrix}0\\ \xi\end{pmatrix}$, then using (1.1), we have

$$\begin{aligned}\lambda|\xi|^2 \le \lambda\frac{\langle P_{12},\xi\rangle^2}{p_{11}^2} + \lambda|\xi|^2 &\le \left\langle \mathbb{P}\begin{pmatrix} -p_{11}^{-1}\langle P_{12},\xi\rangle \\ \xi\end{pmatrix}, \begin{pmatrix} -p_{11}^{-1}\langle P_{12},\xi\rangle \\ \xi\end{pmatrix}\right\rangle \\ &= \left\langle \mathbb{S}\mathbb{P}\mathbb{S}^\top \begin{pmatrix}0\\ \xi\end{pmatrix}, \begin{pmatrix}0\\ \xi\end{pmatrix}\right\rangle = \left\langle \mathbb{M}\begin{pmatrix}0\\ \xi\end{pmatrix}, \begin{pmatrix}0\\ \xi\end{pmatrix}\right\rangle \\ &= \left\langle \begin{pmatrix} p_{11} & 0 \\ 0 & P_{22} - P_{21}p_{11}^{-1}P_{12}\end{pmatrix}\begin{pmatrix}0\\ \xi\end{pmatrix}, \begin{pmatrix}0\\ \xi\end{pmatrix}\right\rangle \\ &= \left\langle (\mathbb{P}_{22} - P_{21}p_{11}^{-1}P_{12})\xi, \xi\right\rangle\end{aligned}$$

In particular, we have

$$\lambda p_{11}|\xi|^2 \le p_{11}\langle \mathbb{P}_{22}\xi,\xi\rangle - \langle P_{12},\xi\rangle^2 \qquad \text{for every } \xi\in\mathbb{R}^{n-1},$$

which completes the proof of the lemma. □

We are now ready to prove Theorem 1.3 in the case $\mathbb{A}(x) \equiv \mathbb{A}$ is a constant matrix:

Proof of Theorem 1.3. We make a note about the notation used, we will use $\mathbb{A}$ and ∇u to denote quantities in Cartesian coordinates and $\mathbb{P}$ and $\bar{\nabla} u$ to denote analogue quantities in Polar coordinates. It is easy to see that the matrices $\mathbb{A}$ and $\mathbb{P}$ are similar to each other via the spherical orthogonal change of variables, see the proof of Theorem 2.1 for this in $\mathbb{R}^2$.

The proof follows in several steps:

Step 1: Using integration by parts and noting that $-\operatorname{div} \mathbb{A}(x)\nabla u = 0$, we have

$$\int_{B_1} \langle \mathbb{A}(x)\nabla u, \nabla u\rangle \, dx = \int_{S_1} (u-k)\langle \mathbb{A}(x)\nabla u, \vec{n}\rangle \, d\sigma,$$

where we take $k = \fint_{S_1} u\, d\sigma$. This can be further estimated after changing to polar coordinates to get

$$\begin{aligned}\int_{S_1} (u-k)\langle \mathbb{A}(x)\nabla u, \vec{n}\rangle \, d\sigma &\le \frac{\varepsilon}{2}\int_{S_1} (u-k)^2\, d\sigma + \frac{1}{2\varepsilon}\int_{S_1} \left\langle \mathbb{P}(x)\bar{\nabla} u, \vec{e_1}\right\rangle^2 \, d\sigma \\ &\overset{(a)}{\le} \frac{\varepsilon}{2(n-1)}\int_{S_1} |u_T|^2\, d\sigma + \frac{1}{2\varepsilon}\int_{S_1} \left\langle \mathbb{P}(x)\bar{\nabla} u, \vec{e_1}\right\rangle^2 \, d\sigma,\end{aligned} \tag{4.8}$$

where to obtain (a), we made use of Proposition 3.1.

Step 2: We will apply (4.7) with $\xi = u_T$ to get

$$\begin{aligned} p_{11}\lambda|u_T|^2 &\leq p_{11}\left\langle \mathbb{P}_{22}u_T, u_T\right\rangle - \left\langle P_{12}, u_T\right\rangle^2 \\ &= p_{11}\left\langle \mathbb{P}_{22}u_T, u_T\right\rangle - \left\langle P_{12}, u_T\right\rangle^2 + \left\langle \mathbb{P}\bar{\nabla}u, \vec{e_1}\right\rangle^2 - \left\langle \mathbb{P}\bar{\nabla}u, \vec{e_1}\right\rangle^2 \\ &= p_{11}\left\langle \mathbb{P}\bar{\nabla}u, \bar{\nabla}u\right\rangle - \left\langle \mathbb{P}\bar{\nabla}u, \vec{e_1}\right\rangle^2 \end{aligned} \tag{4.9}$$

Using (4.9) along with $\lambda \leq p_{11} \leq \Lambda$, we estimate $\displaystyle\int_{S_1} |u_T|^2\, d\sigma$ in (4.8) to get

$$\int_{B_1} \left\langle \mathbb{P}(x)\bar{\nabla}u, \bar{\nabla}u\right\rangle\, dx \leq \frac{\varepsilon}{2(n-1)\lambda}\int_{S_1}\left\langle \mathbb{P}\bar{\nabla}u, \bar{\nabla}u\right\rangle\, d\sigma + \left(\frac{1}{2\varepsilon} - \frac{\varepsilon}{2(n-1)\lambda\Lambda}\right)\int_{S_1}\left\langle \mathbb{P}(x)\bar{\nabla}u, \vec{e_1}\right\rangle^2\, d\sigma. \tag{4.10}$$

Step 3: We recall Pohožaev identity from (4.6) noting that $\displaystyle\sum_i^n a_{ii} = \sum_i^n p_{ii}$ to get

$$\begin{aligned} \int_{S_1}\left\langle \mathbb{P}\bar{\nabla}u, \bar{\nabla}u\right\rangle \left\langle \mathbb{P}\vec{e_1}, \vec{e_1}\right\rangle\, d\sigma &= 2\int_{S_1}\left\langle \mathbb{P}\bar{\nabla}u, \vec{e_1}\right\rangle^2\, d\sigma + \int_{B_1}\left(\sum_i^n p_{ii}\right)\left\langle \mathbb{P}\bar{\nabla}u, \bar{\nabla}u\right\rangle\, dx \\ &\quad -2\int_{B_1}\left\langle \mathbb{P}\nabla u, \mathbb{P}\nabla u\right\rangle\, dx + \mathbb{E}_{\mathrm{r}}\,. \end{aligned} \tag{4.11}$$

Step 4: Let us restrict $\varepsilon \leq \sqrt{(n-1)\lambda\Lambda}$ which is equivalent to $\left(\frac{1}{2\varepsilon} - \frac{\varepsilon}{2(n-1)\lambda\Lambda}\right) \geq 0$. Combining (4.11) and (4.10) gives

$$\begin{aligned} \int_{B_1}\left\langle \mathbb{P}(x)\bar{\nabla}u, \bar{\nabla}u\right\rangle\, dx &\leq \frac{\varepsilon}{2(n-1)\lambda}\int_{S_1}\left\langle \mathbb{P}\bar{\nabla}u, \bar{\nabla}u\right\rangle\, d\sigma + \left(\frac{1}{2\varepsilon} - \frac{\varepsilon}{2(n-1)\lambda\Lambda}\right)\times \\ &\times\frac{1}{2}\left(\int_{S_1}\left\langle \mathbb{P}\bar{\nabla}u, \bar{\nabla}u\right\rangle\left\langle \mathbb{P}\vec{e_1}, \vec{e_1}\right\rangle\, d\sigma - \int_{B_1}\left(\sum_i^n p_{ii}\right)\left\langle \mathbb{P}\bar{\nabla}u, \bar{\nabla}u\right\rangle\, dx + 2\int_{B_1}\left\langle \mathbb{P}\bar{\nabla}u, \mathbb{P}\bar{\nabla}u\right\rangle - \mathbb{E}_{\mathrm{r}}\right). \end{aligned}$$

In particular, after simplification, we get Simplifying, we get

$$\begin{aligned} &\left(\frac{1}{2\varepsilon} - \frac{\varepsilon}{2(n-1)\lambda\Lambda}\right)\frac{1}{2}\mathbb{E}_{\mathrm{r}} + \int_{B_1}\left(1 - \left(\frac{1}{2\varepsilon} - \frac{\varepsilon}{2(n-1)\lambda\Lambda}\right)\frac{(2\Lambda - (\sum_i^n p_{ii}))}{2}\right)\left\langle \mathbb{P}(x)\bar{\nabla}u, \bar{\nabla}u\right\rangle\, dx \leq \\ &\qquad\left(\frac{\varepsilon}{2(n-1)\lambda} + \left(\frac{1}{2\varepsilon} - \frac{\varepsilon}{2(n-1)\lambda\Lambda}\right)\frac{\Lambda}{2}\right)\int_{S_1}\left\langle \mathbb{P}\bar{\nabla}u, \bar{\nabla}u\right\rangle\, d\sigma. \end{aligned} \tag{4.12}$$

Step 5: Ignoring the contribution from $\mathbb{E}_{\mathrm{r}}$ momentarily (see Remark 4.5), we proceed as follows: Denoting $T := \displaystyle\sum_i^n p_{ii}$, we see that $n\lambda \leq T \leq n\Lambda$. So, let us maximize (4.12) over $\varepsilon \leq \sqrt{(n-1)\lambda\Lambda}$ and $n\lambda \leq T \leq n\Lambda$ to get

$$\int_{B_1}\max_{T\in[n\lambda, n\Lambda]}\max_{\varepsilon\in[0,\sqrt{(n-1)\lambda\Lambda}]}\frac{\left(1 - \left(\frac{1}{2\varepsilon} - \frac{\varepsilon}{2(n-1)\lambda\Lambda}\right)\frac{(2\Lambda - T)}{2}\right)}{\left(\frac{\varepsilon}{2(n-1)\lambda} + \left(\frac{1}{2\varepsilon} - \frac{\varepsilon}{2(n-1)\lambda\Lambda}\right)\frac{\Lambda}{2}\right)}\left\langle \mathbb{P}\bar{\nabla}u, \bar{\nabla}u\right\rangle\, dx \leq \int_{S_1}\left\langle \mathbb{P}\bar{\nabla}u, \bar{\nabla}u\right\rangle\, d\sigma.$$

This maximum is achieved when $T = n\Lambda$ and

$$\varepsilon = \frac{(2-n)\Lambda + \sqrt{(n-2)^2\Lambda^2 + 4(n-1)\lambda\Lambda}}{2} = \frac{\Lambda}{2}\left(-(n-2) + \sqrt{(n-2)^2 + \frac{4(n-1)\lambda}{\Lambda}}\right).$$

Furthermore, it is easy to see that $\varepsilon \in [0, \sqrt{(n-1)\lambda\Lambda}]$.

Substituting the values of ε and T into (4.12) gives

$$\left(\sqrt{(n-2)^2 + \frac{4(n-1)\lambda}{\Lambda}}\right)\int_{B_1}\left\langle \mathbb{P}\bar{\nabla}u, \bar{\nabla}u\right\rangle\, dx \leq \int_{S_1}\left\langle \mathbb{P}\bar{\nabla}u, \bar{\nabla}u\right\rangle\, d\sigma,$$

which completes the proof of the theorem. $\square$

Remark 4.5. *If we were to keep track of $\mathbb{E}_{\mathrm{r}}$, then we would get*

$$\mathbb{E}_{\mathrm{r}}\,\frac{\frac{1}{2}\left(\frac{1}{2\varepsilon}-\frac{\varepsilon}{2(n-1)\lambda\Lambda}\right)}{\left(\frac{\varepsilon}{2(n-1)\lambda}+\left(\frac{1}{2\varepsilon}-\frac{\varepsilon}{2(n-1)\lambda\Lambda}\right)\frac{\Lambda}{2}\right)}+\int_{B_1}\frac{\left(1-\left(\frac{1}{2\varepsilon}-\frac{\varepsilon}{2(n-1)\lambda\Lambda}\right)\frac{(2\Lambda-T)}{2}\right)}{\left(\frac{\varepsilon}{2(n-1)\lambda}+\left(\frac{1}{2\varepsilon}-\frac{\varepsilon}{2(n-1)\lambda\Lambda}\right)\frac{\Lambda}{2}\right)}\left\langle\mathbb{P}\bar{\nabla}u,\bar{\nabla}u\right\rangle\,dx\le\int_{S_1}\left\langle\mathbb{P}\bar{\nabla}u,\bar{\nabla}u\right\rangle\,d\sigma.$$

which after substituting the values of ε and T, becomes

$$\frac{(n-2)}{\Lambda\left(\sqrt{(n-2)^2+\frac{4(n-1)\lambda}{\Lambda}}\right)}\,\mathbb{E}_{\mathrm{r}}+\left(\sqrt{(n-2)^2+\frac{4(n-1)\lambda}{\Lambda}}\right)\int_{B_1}\left\langle\mathbb{P}\bar{\nabla}u,\bar{\nabla}u\right\rangle\,dx\le\int_{S_1}\left\langle\mathbb{P}\bar{\nabla}u,\bar{\nabla}u\right\rangle\,d\sigma.$$

In particular, for general $\mathbb{A}(x)$ matrices, Theorem 1.3 holds provided $\mathbb{E}_{\mathrm{r}}\ge 0$.

Question 4.6. *Does the following inequality hold for some $\mathbf{C}_o>0$?*

$$\mathbf{C}_o\,\mathbb{E}_{\mathrm{r}}\ge\left(\sqrt{(n-2)^2+\frac{4(n-1)\lambda}{\Lambda}}\right)\int_{B_1}\left\langle\mathbb{P}\bar{\nabla}u,\bar{\nabla}u\right\rangle\,dx-\int_{S_1}\left\langle\mathbb{P}\bar{\nabla}u,\bar{\nabla}u\right\rangle\,d\sigma.$$

If the above inequality holds, then in the case $\mathbb{E}_{\mathrm{r}}\le 0$, we automatically get the required estimate, thus completing the proof.

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