

NONLINEAR SCHRÖDINGER-POISSON SYSTEMS IN DIMENSION TWO: THE ZERO MASS CASE

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ABSTRACT. We provide an existence result for a Schrödinger-Poisson system in gradient form, set in the whole plane, in the case of zero mass. Since the setting is limiting for the Sobolev embedding, we admit nonlinearities with subcritical or critical growth in the sense of Trudinger-Moser. In particular, the absence of the mass term requires a nonstandard functional framework, based on homogeneous Sobolev spaces. These features, combined with the logarithmic behaviour of the kernel of the Poisson equation, make the analysis delicate, since standard variational tools cannot be applied. The system is solved by considering the corresponding logarithmic Choquard equation. We prove the existence of a mountain pass-type solution via a careful analysis on specific Cerami sequences, whose boundedness is achieved by exploiting an appropriate functional, obtained by evaluating the energy functional on particular paths.

1. INTRODUCTION

We aim at investigating existence of positive solutions of the planar Schrödinger-Poisson system in gradient form given by

$$(1.1) \quad \begin{cases} -\Delta u = \Phi f(u) & \text{in } \mathbb{R}^2, \\ -\Delta \Phi = 2\pi F(u) & \text{in } \mathbb{R}^2, \end{cases}$$

where f is a positive continuous nonlinearity with subcritical or critical growth in the sense of Trudinger-Moser, and $F(t) := \int_0^t f(s) ds$. The main goal is to face the combined difficulties of working in the limiting setting of the Sobolev embeddings and the fact that in the first equation of (1.1) the mass term is missing. This makes the problem challenging, not only for the variational approach, but also in the choice of a nonstandard functional framework.

Schrödinger-Poisson systems of the form

$$(1.2) \quad \begin{cases} -\Delta u + V(x)u = \Phi f(u) & \text{in } \mathbb{R}^N, \\ -\Delta \Phi = F(u) & \text{in } \mathbb{R}^N, \end{cases}$$

with a potential V which is usually positive, are of great importance in several fields of physics, since they serve as models for the interaction of two identically charged particles in electromagnetism, as well as for the self-interaction of the wave function with its own gravitational field in quantum mechanics; they appear also in the Hartree theory for crystals, and in astrophysics in the study of selfgravitating boson stars; for the physics background we refer to [4, 34] and to the references therein. From a mathematical

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point of view, they are interesting since they can be analysed by variational techniques. Indeed, one may (formally) solve the Poisson equation in (1.1) by means of the Riesz kernel

$$K_N(x) := \begin{cases} \frac{C_N}{|x|^{N-2}} & \text{if } N \geq 3, \\ \frac{1}{2\pi} \ln \frac{1}{|x|} & \text{if } N = 2, \end{cases}$$

where C_N is an explicit positive constant, and consider

$$\Phi_u(x) := (K_N * F(u))(x) = \int_{\mathbb{R}^N} K_N(x-y)F(u(y)) dy.$$

Substituting it in the first equation of (1.2), one may rewrite the system as a Choquard equation, that is an integro-differential equation of Schrödinger type with a convolutive right-hand side:

$$(1.3) \quad -\Delta u(x) + V(x)u = (K_N * F(u)) f(u) \quad \text{in } \mathbb{R}^N.$$

Besides the evident advantage of the reduction to a single equation, since the system (1.2) is of gradient type, if $N \geq 3$ the equation (1.3) is variational in the Sobolev space $H^1(\mathbb{R}^N)$ thanks to the Hardy-Littlewood-Sobolev inequality (Proposition 2.5 below), provided suitable polynomial growth conditions on f are fulfilled. In this respect, there are a huge number of works about Choquard-type equations, especially from the last decades, and we refer to the seminal works [15, 16, 35, 36] and the references therein. The planar case $N = 2$ is more delicate because of the interplay between the logarithmic behaviour of the Riesz kernel, and the exponential maximal growth of the nonlinearities, due to the Pohožaev-Trudinger-Moser inequality in the full space proved by Ruf [42], see also [8]. The first attempt of considering this case is to be referred to Stubbe [45] and later on formalised by Cingolani and Weth [19, 20]: they set the problem in a constraint space which takes into account in the seminorm a contribution of the logarithmic kernel. This analysis, which is peculiar for the case of a linear coupling in the system, namely $f(u) = u$ in (1.2), was then extended for the general case of a nonlinearity with critical exponential growth in [14]. Taking indeed into account the behaviour at 0 of the nonlinearity, and by means of a log-weighted version of the Pohožaev-Trudinger inequality, a proper functional setting was found, in which the functional associated to (1.3) turns out to be well-defined. We also point out that the sharp version of such inequality has been recently obtained in [46]. The approach in [14] was then generalised for Choquard equations with weights in [9] and for quasilinear Schrödinger-Poisson systems in [7]. A different approach was recently proposed in [33]: here, instead, the underlying functional space remains $H^1(\mathbb{R}^2)$, while the logarithmic kernel is uniformly approximated by polynomial kernels. For further developments of this method we refer to [11–13], also in quasilinear fractional contexts.

All the above works deal with Choquard equations of the form (1.3), where V is a nontrivial potential. The special case of an identically zero potential, the so-called “zero-mass case”, emerges in some physical context, e.g. in the nonabelian gauge theory of particle physics, such as the study of the Yang-Mills equation, see [27]. The main difference with the “mass-case” lies in the natural framework in which such problems are studied, namely the homogeneous Sobolev spaces $D_0^{1,2}(\mathbb{R}^N)$ defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect of the L^2 -norm of the gradient. In the higher-dimensional case $N \geq 3$, in light of the critical Sobolev embedding into $L^{2^*}(\mathbb{R}^N)$ with $2^* = \frac{2N}{N-2}$, such setting is appropriate for both Schrödinger and Choquard equations of the kind

$$(1.4) \quad -\Delta u = (K_N * F(u)) f(u) \quad \text{in } \mathbb{R}^N,$$

as shown e.g. in [1–3, 5]. However, if $N = 2$, not only any embedding into Lebesgue spaces is out of reach, but also one cannot distinguish in $D_0^{1,2}(\mathbb{R}^2)$ between functions which just differ by constants. In this framework only few results are available for Schrödinger equations and Choquard equations of the kind (1.4) with $K_2 = \frac{1}{2\pi} \ln \frac{1}{|\cdot|}$. On the one hand, one may try to tweak the operator, so that the corresponding natural space recovers good embedding properties in Lebesgue and Orlicz spaces: this is the strategy used in [10, 21] for the Schrödinger case, and in [22, 40, 41] for the Choquard case with both polynomial and logarithmic kernels. On the other hand, for equations with zero mass driven by the pure Laplacian, the results available in the literature [17, 18, 47] cover just the linear case $f(u) = u$ in (1.4) (up to adding local nonlinearities) since the approach of [19, 45] is pursued. However, as remarked in [17],

“it is really interesting to observe how remarkable the impact of the logarithmic integral kernel $\ln |x|$ is, because it allows us to establish much richer and better existence results than those available for other elliptic equations, in spite of its sign-changing and unbounded properties”.

Indeed, differently from the Schrödinger equations with zero mass, the presence of the logarithmic kernel combined with $f(u) = u$ allows to recover $H^1(\mathbb{R}^2)$ as suitable functional framework, thanks to a careful splitting in positive and negative part of the logarithm (see (3.2) below).

Main goal of this paper is to extend the existence results in the zero-mass case of [17, 18, 47] in the direction of [14], that is aiming at considering the general case of a nonlinear function f , and covering both the cases of subcritical and critical growth in the sense of Ruf’s inequality. To this aim, several difficulties need to be faced: first, the unusual functional setting, which does not appear in the above cited works because of the linear behaviour of f ; then the analysis on Cerami sequences, which arise from the mountain pass geometry of the functional associated to (1.5), is largely affected by the possibly exponential growth of the nonlinear terms and, in particular in the critical case, it is very delicate; eventually also the final proof “à la Lions” of the existence theorem is pretty non-standard and rather technical. Finally, we derive from our results for the Choquard equation the corresponding for the Schrödinger-Poisson system (1.1) in a suitable functional setting in the spirit of [7, 40]. We stress that this step is often neglected in the literature by just considering it as “natural”: here it finds a rigorous justification.

Before stating our main results, let us specify the growth conditions we are assuming on the nonlinearity. Throughout the paper we suppose that $f \in C^1(\mathbb{R})$, $f(s) > 0$ as $s > 0$, while $f(s) = 0$ for $s \leq 0$; moreover it satisfies:

(f_1) $f(s) \asymp s^{p-1}$ as $s \rightarrow 0$ for some $p > 2$;

and either is subcritical or critical in the following sense:

(f_{sc}) for any $\alpha > 0$, $\lim_{s \rightarrow +\infty} f(s)/e^{\alpha s^2} = 0$ and for some $C > 0$, $f(s) \geq Cs^{p-1}$ as $s \rightarrow +\infty$;

(f_c) there exists $\alpha_0 > 0$ such that

$$\lim_{s \rightarrow +\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0 & \text{if } \alpha > \alpha_0, \\ +\infty & \text{in } \alpha < \alpha_0. \end{cases}$$

Under such conditions, and in particular in light of the behaviour near 0 given by (f_1), we are going to see that the functional setting in which it is convenient to look for solutions of the Choquard equation

associated to the system (1.1), namely

$$(1.5) \quad -\Delta u + (\ln |\cdot| * F(u)) f(u) = 0 \quad \text{in } \mathbb{R}^2,$$

is

$$D^{1,2}L_\omega^p(\mathbb{R}^2) := D_0^{1,2}(\mathbb{R}^2) \cap L^p(\omega \, dx),$$

where the weight function $\omega(x) := \ln(b + |x|)$ with $b > 1$. Note that this space, which corresponds to the intersection space $H^1 L_\omega^p(\mathbb{R}^2)$ detected in [14] for the study of the same Choquard equation with positive mass, gathers all important features of our problem: the absence of mass since we are dealing with the homogeneous Sobolev space $D^{1,2}(\mathbb{R}^2)$, the nonlinear behaviour of f , and the logarithmic kernel in the weight. We will see that this space enjoys good embedding properties, in particular exponential nonlinearities are allowed. We are therefore lead to the following definition:

Definition 1.1 (Solution of (1.5)). *We say that $u \in D^{1,2}L_\omega^p(\mathbb{R}^2)$ is a weak solution of (1.5) if*

$$\int_{\mathbb{R}^2} \nabla u \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} F(u(y)) \, dy \right) f(u(x)) \varphi(x) \, dx$$

for all $\varphi \in D^{1,2}L_\omega^p(\mathbb{R}^2)$.

Of course, in order to prove existence for (1.5), we need some further assumptions on f , which are gathered here:

(f_2) there exist $C > \tau > 0$ such that $\tau \leq \frac{F(s)f'(s)}{f(s)^2} \leq C$ for all $s > 0$;

(f_3) $\lim_{s \rightarrow +\infty} \frac{F(s)f'(s)}{f(s)^2} = 1$, or equivalently $\lim_{s \rightarrow +\infty} \frac{d}{ds} \frac{F(s)}{f(s)} = 0$;

(f_4) $\lim_{s \rightarrow +\infty} \frac{s^3 f(s) F(s)}{e^{2\alpha_0 s^2}} \geq \beta > \mathcal{V}$, where \mathcal{V} will be explicitly given in (4.34);

(f_5) $f'(s) \asymp s^{p-2}$ as $s \rightarrow 0$ and there exists $\alpha' > 0$ and $s' > 0$ such that $f'(s) \lesssim e^{\alpha' s^2}$ for $s > s'$.

We postpone to Section 2.3 a detailed list of consequences of our assumptions. Here we just emphasise that (f_2) implies an Ambrosetti-Rabinowitz condition and the monotonicity of f ; (f_3)-(f_4) will be used in the analysis of the boundedness of Cerami sequences when dealing with critical nonlinearities, the latter being related to the deFigueredo-Miyagaki-Ruf condition in [23] and used to prove a fine upper bound for the mountain pass level in Section 4; (f_5) is a mild condition about the growth at ∞ of the nonlinearity, which well agrees with both (f_{sc})-(f_c), and will be exploited in the conclusive compactness argument in Section 5.

Our main result reads as follows.

Theorem 1.2 (Existence for (1.5)). *Suppose (f_1), (f_2), (f_5) hold, and either*

i) f is subcritical as in (f_{sc}),

or

ii) f is critical as in (f_c) and (f_3), (f_4) are fulfilled.

Then there exists a positive solution to (1.5) in the sense of Definition 1.1.

Once we have found a weak solution of the logarithmic Choquard equation (1.5), we can go back to the original Schrödinger-Poisson system. First, we need a precise meaning of solution for (1.1).

The weighted Lebesgue space $L_s(\mathbb{R}^2)$, $s > 0$, is defined as

$$L_s(\mathbb{R}^2) := \left\{ u \in L_{loc}^1(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} \frac{|u(x)|}{1 + |x|^{2+2s}} dx < +\infty \right\}.$$

Definition 1.3. For $\mathfrak{f} \in \mathcal{S}'(\mathbb{R}^2)$ we say that a function $\varphi \in L_1(\mathbb{R}^2)$ is a solution of the linear Poisson equation $-\Delta\Phi = \mathfrak{f}$ in \mathbb{R}^2 if

$$\int_{\mathbb{R}^2} \Phi(-\Delta\varphi) = \langle \mathfrak{f}, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^2).$$

Definition 1.4 (Solution of (1.1)). We say that (u, Φ) is a weak solution of the Schrödinger-Poisson system (1.1) if

$$\int_{\mathbb{R}^2} \nabla u \cdot \nabla \varphi dx = \int_{\mathbb{R}^2} \Phi f(u) \varphi dx$$

for all $\varphi \in D^{1,2}L_\omega^p(\mathbb{R}^2)$, and Φ solves $-\Delta\Phi = 2\pi F(u)$ in \mathbb{R}^2 in the sense of Definition 1.3.

Theorem 1.5 (Existence for (1.1)). Under the conditions of Theorem 1.2, the Schrödinger-Poisson system (1.1) possesses a solution $(u, \Phi) \in D^{1,2}L_\omega^p(\mathbb{R}^2) \times L_s(\mathbb{R}^2)$ for all $s > 0$ such that u is positive and $\Phi = \Phi_u := \ln \frac{1}{|\cdot|} * F(u)$.

Remark 1.6. It is worth to point out that:

- (1) this work can be seen as an extension to the zero mass-case of the results in [14], to the general case of a nonlinearity with possibly exponential growth of the results in [17, 18, 47], and to the pure Laplacian case to those in [40];
- (2) it is sufficient to prove the existence of a nonnegative nontrivial solution of (1.5) in order to retrieve its positivity by the strong maximum principle for semilinear equations, see e.g. [38, Theorem 11.1].

Overview. In Section 2 we describe the functional framework in which it is convenient to set our problem, discuss our assumptions, and collect some useful results. The variational framework is then described in Section 3, where we show the mountain pass geometry for the energy functional, while the existence of special Cerami sequences, and their boundedness is detailed in Section 4; we stress that these arguments turn out to be a delicate matter. After some careful mountain pass estimates, the proof of the existence for the log-Choquard equation (1.5) is given in Section 5. Finally, in Section 6 we derive from it the existence result for the Schrödinger-Poisson system (1.1).

Notation. For $R > 0$ and $x_0 \in \mathbb{R}^N$ we denote by $B_R(x_0)$ the ball of radius R and center x_0 . Given a set $\Omega \subset \mathbb{R}^N$, we denote $\Omega^c := \mathbb{R}^N \setminus \Omega$, and its characteristic function by χ_Ω . The space of the infinitely differentiable functions which are compactly supported is denoted by $C_c^\infty(\mathbb{R}^N)$, $\mathcal{M}(\mathbb{R}^N)$ stands for the space of measurable functions in \mathbb{R}^N , while \mathcal{S} is the Schwartz space of rapidly decreasing functions and \mathcal{S}' the dual space of tempered distributions. For $p \in [1, +\infty]$ the Lebesgue space of p -integrable functions is denoted by $L^p(\mathbb{R}^N)$ with norm $\|\cdot\|_p$. For $q > 1$ we define its conjugate Hölder exponent as $q' := \frac{q}{q-1}$. The symbol \lesssim indicates that an inequality holds up to a multiplicative constant depending only on structural constants, while $f \asymp g$ means that $c_1 f \leq g \leq c_2 f$ for some $c_1, c_2 > 0$. Finally, $o_n(1)$ denotes a vanishing

real sequence as $n \rightarrow +\infty$. Hereafter, the letter C will be used to denote positive constants which are independent of relevant quantities and whose value may change from line to line.

2. FUNCTIONAL SPACE AND PRELIMINARY RESULTS

2.1. The space $D^{1,2}L_\omega^p(\mathbb{R}^2)$. Let us define the linear space

$$D^{1,2}(\mathbb{R}^2) := \left\{ u \in L_{\text{loc}}^1(\mathbb{R}^2) \mid \nabla u \in L^2(\mathbb{R}^2)^2 \right\}$$

with seminorm $\|\nabla \cdot\|_2$. Note that, by the unboundedness of the domain, this seminorm cannot control the L^2 -norm of the elements of $D^{1,2}(\mathbb{R}^2)$, and therefore $D^{1,2}(\mathbb{R}^2) \supsetneq H^1(\mathbb{R}^2)$. To retrieve a normed space, one need to introduce the relation $u \sim v \Leftrightarrow v = u + c$ with $c \in \mathbb{R}$, and define the quotient space $\dot{D}^{1,2}(\mathbb{R}^2) := \{[u] \mid u \in D^{1,2}\}$, which turns out to be a Hilbert space with norm $\|\nabla \cdot\|_2$ (see [26, Lemma II.6.2]). On the other hand, one may also introduce the space

$$D_0^{1,2}(\mathbb{R}^2) := \text{completion of } C_c^\infty(\mathbb{R}^2) \text{ w.r.t. } \|\nabla \cdot\|_2.$$

By [26, Theorem II.7.5] the two spaces are isomorphic: $D_0^{1,2}(\mathbb{R}^2) = \dot{D}^{1,2}(\mathbb{R}^2)$.

In order to find a suitable variational framework for the system (1.1), for $p > 2$ and $b > 1$ we define

$$L_\omega^p(\mathbb{R}^2) := L^p(\mathbb{R}^2, \omega \, dx) := \{u \in \mathcal{M}(\mathbb{R}^2) \mid \|u\|_{*,p} < +\infty\},$$

where the weight function ω is given by $\omega(x) := \ln(b + |x|)$, and

$$\|u\|_{*,p} := \left(\int_{\mathbb{R}^2} |u|^p \ln(b + |x|) \, dx \right)^{\frac{1}{p}},$$

and we consider the space

$$D^{1,2}L_\omega^p(\mathbb{R}^2) := D^{1,2}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2, \omega \, dx),$$

with norm

$$\|u\| := \left[\int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \left(\int_{\mathbb{R}^2} |u|^p \ln(b + |x|) \, dx \right)^{\frac{2}{p}} \right]^{\frac{1}{2}}.$$

Note that, by the choice of $b > 1$, in $D^{1,2}L_\omega^p(\mathbb{R}^2)$ it is possible to control the p -norm by the seminorm $\|u\|_{*,p}$. Indeed,

$$(2.1) \quad \|u\|_p^p = \int_{\mathbb{R}^2} |u(x)|^p \, dx \leq (\ln b)^{-1} \int_{\mathbb{R}^2} \ln(b + |x|) |u(x)|^p \, dx = (\ln b)^{-1} \|u\|_{*,p}^p.$$

Therefore,

$$D^{1,2}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2, \omega \, dx) = D^{1,2}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2) \cap L^p(\mathbb{R}^2, \omega \, dx),$$

and so, since $C_c^\infty(\mathbb{R}^2)$ is dense both in $D_0^{1,2}(\mathbb{R}^2)$ and $L^p(\mathbb{R}^2)$, we have the characterisation

$$\begin{aligned} D^{1,2}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2, \omega \, dx) &= D^{1,2}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2) \cap L^p(\mathbb{R}^2, \omega \, dx) \\ &= D_0^{1,2}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2) \cap L^p(\mathbb{R}^2, \omega \, dx) = D_0^{1,2}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2, \omega \, dx). \end{aligned}$$

Furthermore, by [30, Theorem 1.11] $D^{1,2}L_\omega^p(\mathbb{R}^2)$ is a reflexive Banach space, whose dual is given by

$$(2.2) \quad D^{-1,2}L_\omega^p(\mathbb{R}^2) := (D^{1,2}(\mathbb{R}^2) \cap L_\omega^p(\mathbb{R}^2))' = D^{-1,2}(\mathbb{R}^2)|_{D^{-1,2}L_\omega^p} + L^{p'}(\mathbb{R}^2, \omega \, dx)|_{D^{-1,2}L_\omega^p},$$

since $(L_\omega^p(\mathbb{R}^2))' = L^{p'}(\mathbb{R}^2, \omega dx)$, see [44, Theorem 14.9], and [26, Theorem II.8.1] for the representation of the space $D^{-1,2}(\mathbb{R}^2)$. Let us state now important embedding properties of our space.

Lemma 2.1. *The space $X := D^{1,2}L_\omega^p(\mathbb{R}^2)$ is compactly embedded in $L^q(\mathbb{R}^2)$ for all $q \geq p$.*

Proof. The embedding $X \hookrightarrow L^p(\mathbb{R}^2)$ is a consequence of (2.1). Let now $q > p$, then by Proposition 2.6 below (applied with $j = 0, m = 1, r = 2, N = 2, q = p$) we have

$$\|u\|_q \leq C \|u\|_p^{\frac{p}{q}} \|\nabla u\|_2^{\frac{q-p}{q}},$$

which implies $X \hookrightarrow L^q(\mathbb{R}^2)$, for every $q > p$.

Let us now prove the compactness of these embeddings by relying on the Riesz criterion, see [39, Theorem XIII.66], which needs the continuity of the translation in the Lebesgue norm and a uniform decay at infinity of the elements in X . To this aim, let $S \subset X$ be a bounded subset, then it is also bounded in $L^q(\mathbb{R}^2)$, $q \geq p$. Let $R > 0$, $u \in S$ and $q \geq p$. Then, by Hölder inequality,

$$\int_{\{|x| \geq R\}} |u|^q dx \leq \|u\|_{(q-1)p'}^{q-1} \int_{\{|x| \geq R\}} |u|^p dx \leq C \int_{\{|x| \geq R\}} |u|^p dx,$$

since $(q-1)p' \geq p$ and the continuity of the embedding shown before. Moreover,

$$\int_{\{|x| \geq R\}} |u|^p dx \leq \frac{\|u\|_{*,p}^p}{\ln(b+R)} \leq \frac{C}{\ln(b+R)},$$

since $u \in S$ bounded in X . Hence, for $q \geq p$, for any $\varepsilon > 0$ one can choose $R > 0$ large enough such that

$$(2.3) \quad \int_{B_R(0)^c} |u|^q dx \leq \varepsilon^q.$$

Let us now prove the continuity of the translation in $L^q(\mathbb{R}^2)$. Since $X \subset D_0^{1,2}(\mathbb{R}^2)$, by density we can work within $C_c^\infty(\mathbb{R}^2)$. Let $u \in C_c^\infty(\mathbb{R}^2)$ and $h \in \mathbb{R}^2$. Following [6, Proposition 9.3] and defining $\tau_h u := u(\cdot + h)$, by Jensen inequality we have

$$|u(x+h) - u(x)|^2 = \left| \int_0^1 h \cdot \nabla u(x+th) dt \right|^2 \leq |h|^2 \int_0^1 |\nabla u(x+th)|^2 dt.$$

Integrating on \mathbb{R}^2 and using the Fubini-Tonelli theorem,

$$\|\tau_h u - u\|_2^2 = \int_{\mathbb{R}^2} |u(x+h) - u(x)|^2 dx \leq |h|^2 \int_0^1 \int_{\mathbb{R}^2} |\nabla u(x+th)|^2 dx dt = |h|^2 \|\nabla u\|_2^2.$$

Hence,

$$\begin{aligned} \|\tau_h u - u\|_q^q &= \int_{\mathbb{R}^2} |u(x+h) - u(x)|^q dx \leq \left(\int_{\mathbb{R}^2} |u(x+h) - u(x)|^{2(q-1)} dx \right)^{\frac{1}{2}} \|\tau_h u - u\|_2 \\ &\lesssim \left(\|\tau_h u\|_{2(q-1)}^{q-1} + \|u\|_{2(q-1)}^{q-1} \right) |h| \|\nabla u\|_2 \\ &\lesssim 2 \|u\|_{2(q-1)}^{q-1} \|u\| |h| \lesssim \|u\|^q |h| \end{aligned}$$

by the continuous embedding showed above, since $p > 2$ implies that $2(q-1) > p$ for all $q \geq p$.

The above inequality, together with (2.3), completes the proof. \square

2.2. Useful theorems and inequalities. Since our nonlinearities are of exponential growth, we recall here some important functional inequalities that we will exploit to control them. The first result is a generalised Cao's inequality in $D^{1,2}(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ taken from [28, Theorem 1.1], which is a particular case of the very general result obtained therein. Then, we report a Pohožaev-Trudinger-Moser inequality with logarithmic weight in $D^{1,2}L_\omega^p(\mathbb{R}^2)$ from¹ [14].

Theorem 2.2 ([28], Theorem 1.1). *Let $q \geq p$ and $\lambda < 4\pi$. Then there exists a constant $C := C(p, q, \lambda) > 0$ such that for all $u \in D^{1,2}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ with $\|\nabla u\|_2 \leq 1$ there holds*

$$(2.4) \quad \int_{\mathbb{R}^2} e^{\lambda u^2} |u|^q dx \leq C \|u\|_p^p.$$

If $\lambda \geq 4\pi$, (2.4) remains true but the constant C is not uniform in u .

Theorem 2.3 ([14], Theorem 3.3). *Let g satisfy assumptions (f_1) and either (f_{sc}) or (f_c) , and $G(s) = \int_0^s g(t) dt$. Then, the space $D^{1,2}L_\omega^p(\mathbb{R}^2)$ embeds into the weighted Orlicz space $L_G(\mathbb{R}^2, \omega dx)$, namely*

$$\int_{\mathbb{R}^2} G(\alpha|u|) \ln(b + |x|) dx < +\infty$$

for any $u \in D^{1,2}L_\omega^p(\mathbb{R}^2)$ and any $\alpha > 0$. Furthermore, for any $\alpha \leq 1/\sqrt{p}$ one has

$$\sup_{\|u\|^2 \leq 1} \int_{\mathbb{R}^2} G(\alpha|u|) \ln(b + |x|) dx < +\infty.$$

Corollary 2.4 ([14], Corollary 3.4). *For any $\alpha > 0$ the functional*

$$u \mapsto \int_{\mathbb{R}^2} G(\alpha|u|) \ln(b + |x|) dx, \quad u \in D^{1,2}L_\omega^p(\mathbb{R}^2),$$

is continuous, where G is as in Theorem 2.3.

Throughout the paper, we will make great use of the following well-known results: the Hardy-Littlewood-Sobolev inequality (see [31, Theorem 4.3]), and the Gagliardo-Nirenberg inequality (see [37, Formula (2.2)]).

Proposition 2.5 (HLS inequality). *Let $N \geq 1$, $\mu \in (0, N)$, and $s, r > 1$ with $\frac{1}{s} + \frac{\mu}{N} + \frac{1}{r} = 2$. There exists a constant $C = C(N, \mu, s, r)$ such that for all $f \in L^s(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$ one has*

$$\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^\mu} * f \right) h dx \leq C \|f\|_s \|h\|_r.$$

Proposition 2.6 (GN inequality). *Let $N, m \in \mathbb{N}$ and $u \in L^p(\mathbb{R}^N)$ such that $D^m u \in L^r(\mathbb{R}^N)$, where $q, r \in [1, +\infty]$. Then, there exists a constant $C := C(N, m, j, p, r, \theta) > 0$ such that*

$$(2.5) \quad \|D^j u\|_q \leq C \|D^m u\|_r \|u\|_p,$$

where j satisfies

$$\frac{1}{q} = \frac{j}{N} + \theta \left(\frac{1}{r} - \frac{m}{N} \right) + \frac{1-\theta}{p}, \quad \text{for every } j \text{ such that } \frac{j}{m} \leq \theta \leq 1.$$

In particular:

¹Note that the space $H_\omega^1(\mathbb{R}^2)$ in [14] coincides with our $D^{1,2}L_\omega^p(\mathbb{R}^2)$.

- if $j = 0$, $rm < N$ and $p = +\infty$, then we need that either u vanishes at infinity or $u \in L^s(\mathbb{R}^N)$ for some $s \in (0, +\infty)$;
- if $r \in (1, +\infty)$ and $m - j - N/r$ is a nonnegative integer, then (2.5) holds only for θ satisfying $\frac{j}{m} \leq \theta < 1$.

We end this section by recalling an abstract result from [25, Proposition 3.1] (see also [48, Theorem 2.8] for the version with the Palais-Smale condition), which will be needed to prove the existence of bounded Cerami sequences.

Proposition 2.7. *Let \tilde{X} be a Banach space, M_0 be a closed subspace of a metric space M , and $\Gamma_0 \subset C(M_0, \tilde{X})$. Define*

$$\tilde{\Gamma} = \left\{ \gamma \in C(M, \tilde{X}) : \gamma|_{M_0} \in \Gamma_0 \right\}.$$

If $\Psi \in C^1(\tilde{X}, \mathbb{R})$ satisfies

$$\infty > c := \inf_{\gamma \in \tilde{\Gamma}} \sup_{u \in M} \Psi(\gamma(u)) > a := \sup_{\gamma_0 \in \Gamma_0} \sup_{u \in M_0} \Psi(\gamma_0(u)),$$

then, for every $\varepsilon \in (0, \frac{c-a}{a})$, $\delta > 0$, and $\gamma \in \tilde{\Gamma}$, with $\sup_{u \in M} \Psi(\gamma(u)) \leq c + \varepsilon$, there exists $u \in \tilde{X}$ such that

- $c - 2\varepsilon \leq \Psi(u) \leq c + 2\varepsilon$,
- $\text{dist}(u, \gamma(M)) \leq 2\delta$,
- $(1 + \|u\|_{\tilde{X}}) \|\Psi'(u)\|_{\tilde{X}'} \leq \frac{8\varepsilon}{\delta}$.

2.3. Consequences of the assumptions. To end this Section, let us point out some immediate consequences of (f_0) – (f_5) which will be of use in our analysis, together with some comments in this regard:

i) by (f_1) and (f_{sc}) , for any $r, \alpha > 0$ and $s_0 > 1$ there is $C > 0$ such that

$$(2.6) \quad 0 \leq F(s) \leq C \cdot \begin{cases} s^p & \text{for } s \leq s_0, \\ s^r e^{\alpha s^2} & \text{for } s > s_0 \end{cases}$$

while, if (f_{sc}) is replaced by (f_c) , the upper bound changes as follows: for any $r > 0$, $\alpha > \alpha_0$ and $s_0 > 1$ there is $C > 0$ such that

$$(2.7) \quad 0 \leq F(s) \leq C \cdot \begin{cases} s^p & \text{for } s \leq s_0, \\ s^r e^{\alpha s^2} & \text{for } s > s_0, \end{cases} \quad \text{and} \quad 0 \leq f(s) \leq C \cdot \begin{cases} s^{p-1} & \text{for } s \leq s_0, \\ s^r e^{\alpha s^2} & \text{for } s > s_0; \end{cases}$$

ii) by (f_1) and (f_{sc}) or (f_c) , there is $C > 0$ such that for any $s > 0$

$$(2.8) \quad F(s) \geq Cs^p;$$

iii) assumption (f_2) implies that f is monotone increasing. Moreover,

$$\frac{d}{dt} \frac{F(t)}{f(t)} = \frac{f^2(t) - F(t)f'(t)}{f^2(t)} \leq 1 - \tau,$$

from which one infers

$$(2.9) \quad F(t) \leq (1 - \tau)tf(t) \quad \text{for any } t \geq 0.$$

- iv) (f_5) is related to the well-known de Figueiredo-Miyagaki-Ruf condition [23] and is crucial in order to estimate the mountain pass level and gain compactness, see Lemma 4.5. We note here that such an assumption, which goes back to [24], avoids the prescription of a global lower-bound on F of the kind (2.8) but with C large enough: it is widely used in the literature but is not of practical verification. A condition similar to (f_5) appears also e.g. in [7, 14, 17, 40].
- v) Examples of nonlinearities which satisfy the assumptions of Theorem 1.2 are $F(s) = s^p$ and $F(s) = s^p e^s$ with $p > 2$ (concerning (f_{sc})), and $F(s) = s^p \chi_{\{s < 1\}}(s) + s^q e^{s^2} \chi_{\{s \geq 1\}}(s)$ with $q > -2$ (concerning (f_c)).

3. THE VARIATIONAL FRAMEWORK

Formally, we can associate to the logarithmic Choquard equation (1.5) the energy functional $\mathcal{I} : D^{1,2}L_\omega^p(\mathbb{R}^2) \rightarrow \mathbb{R}$ given by

$$(3.1) \quad \mathcal{I}(u) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| F(u(x)) F(u(y)) dx dy.$$

The aim of this section is to show that \mathcal{I} is indeed well-defined and regular in the space $D^{1,2}L_\omega^p(\mathbb{R}^2)$ described in Section 2. First, we state an identity which will play a crucial rôle throughout the paper:

$$(3.2) \quad \ln|x-y| = \ln(b+|x-y|) - \ln\left(1 + \frac{b}{|x-y|}\right).$$

This splitting was first used by [45] with $b = 1$, and subsequently developed by [19]. In [17, 47] it was applied with $b > 1$, and this allows for the embedding $L^p(\mathbb{R}^2, \omega dx) \hookrightarrow L^p(\mathbb{R}^2)$ as shown in Section 2.

According to (3.2), and following the approach of [19], we define the bilinear forms

$$\begin{aligned} (u, v) &\mapsto A_1(u, v) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(b+|x-y|) u(x)v(y) dx dy, \\ (u, v) &\mapsto A_2(u, v) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln\left(1 + \frac{b}{|x-y|}\right) u(x)v(y) dx dy, \\ (u, v) &\mapsto A_0(u, v) := A_1(u, v) - A_2(u, v) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u(x)v(y) dx dy \end{aligned}$$

Since $b > 1$ one has

$$(3.3) \quad \ln(b+|x-y|) \leq \ln(b+|x|+|y|) \leq \ln(b+b|x|+b|y|) \leq \ln((b+|x|)(b+|y|)) = \ln(b+|x|) + \ln(b+|y|),$$

and we can therefore estimate the bilinear form A_1 by

$$(3.4) \quad \begin{aligned} |A_1(u, v)| &\leq \int_{\mathbb{R}^2} \ln(b+|x|) |u(x)| dx \int_{\mathbb{R}^2} |v(y)| dy + \int_{\mathbb{R}^2} |u(x)| dx \int_{\mathbb{R}^2} \ln(b+|y|) |v(y)| dy \\ &\leq \|u\|_{*,1} \|v\|_1 + \|u\|_1 \|v\|_{*,1} \end{aligned}$$

for every $u, v \in L^1(\mathbb{R}^2, \omega dx)$. Concerning A_2 , since $\ln(b+r) \leq r$ for every $r \geq 0$ (with the strict inequality if $r > 0$), then, by Hardy-Littlewood-Sobolev inequality (Proposition 2.5), there exists a positive constant C such that

$$(3.5) \quad A_2(u, v) \leq b \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} u(x)v(y) dx dy \lesssim \|u\|_{\frac{4}{3}} \|v\|_{\frac{4}{3}}$$

for every $u, v \in L^{\frac{4}{3}}(\mathbb{R}^2)$. For $F \in C(\mathbb{R})$, we also define the following functionals:

$$\begin{aligned} u &\mapsto I_1(u) := A_1(F(u), F(u)) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(b + |x - y|) F(u(x)) F(u(y)) \, dx \, dy, \\ u &\mapsto I_2(u) := A_2(F(u), F(u)) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \left(1 + \frac{b}{|x - y|} \right) F(u(x)) F(u(y)) \, dx \, dy, \\ u &\mapsto I_0(u) := A_0(F(u), F(u)) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x - y| F(u(x)) F(u(y)) \, dx \, dy. \end{aligned}$$

Note that, if $F(u) \geq 0$, then $I_1(u) \geq 0$ and $I_2(u) \geq 0$. With this notation, the energy functional (3.1) can be rewritten as

$$\mathcal{I}(u) = \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} I_0(u),$$

and we are going to prove that \mathcal{I} is well-defined in $X := D^{1,2} L_{\omega}^p(\mathbb{R}^2)$, see Lemma 3.2. For the rest of the paper we always use this notation to indicate our space.

Before going into details of the proof, we prepose an extension of Lemma [19, Lemma 2.1] to our framework. It will be crucial in order to transfer estimates from the bilinear form A_1 to the norm $\|\cdot\|_{*,p}$, since it will be mainly applied with $\varphi_n = F(u_n)$ in combination with the lower bound (2.8).

Lemma 3.1. *Let $p > 1$, $u \in L^p(\mathbb{R}^2) \setminus \{0\}$ and nonnegative sequences $\{u_n\}_n \subset L^p(\mathbb{R}^2)$ such that $u_n \rightarrow u$ a.e. in \mathbb{R}^2 as $n \rightarrow +\infty$, and $\{\varphi_n\}_n \subset L^1(\mathbb{R}^2)$. Let moreover $F \in C(\mathbb{R})$ with $F(t) > 0$ for $t > 0$.*

- (a) *If $A_1(F(u_n), \varphi_n) \leq C$ and $\|\varphi_n\|_1 \leq C$ for all $n \in \mathbb{N}$, then there exist $n_0 \in \mathbb{N}$ and $\bar{C} > 0$ such that $\|\varphi_n\|_{*,1} \leq \bar{C}$ for all $n > n_0$.*
- (b) *If $A_1(F(u_n), \varphi_n) \rightarrow 0$ and $\|\varphi_n\|_1 \rightarrow 0$ as $n \rightarrow +\infty$, then $\|\varphi_n\|_{*,1} \rightarrow 0$ as $n \rightarrow +\infty$.*

Proof. Since $u_n \rightarrow u$ a.e. in \mathbb{R}^2 and F is continuous, by Egorov's theorem there exist $n_0 \in \mathbb{N}$, $R > 0$, and $\delta > 0$, and a measurable set $A \subset B_R(0)$ with positive measure, such that $F(u_n(x)) \geq \delta$ for all $n \geq n_0$. For $x \in A$ and $y \in \mathbb{R}^2 \setminus B_{(1+b)R}(0)$ we have

$$b + |x - y| \geq b + |y| - |x| \geq b + \left(1 - \frac{1}{b+1}\right) |y| = b \left(1 + \frac{1}{b+1} |y|\right) \geq b(1 + |y|)^{\frac{1}{b+1}}$$

by Bernoulli's inequality. Hence,

$$\begin{aligned} A_1(F(u_n), \varphi_n) &\geq \int_{\mathbb{R}^2 \setminus B_{(b+1)R}(0)} \int_A \ln(b + |x - y|) F(u_n(x)) \varphi_n(y) \, dx \, dy \\ &\geq C \left(\int_A F(u_n(x)) \, dx \right) \left(\int_{\mathbb{R}^2 \setminus B_{(b+1)R}(0)} \ln \left(b(1 + |y|)^{\frac{1}{b+1}} \right) \varphi_n(y) \, dy \right) \\ &\geq C \frac{\delta |A|}{b+1} \int_{\mathbb{R}^2 \setminus B_{(b+1)R}(0)} \ln \left(b^{b+1} (1 + |y|) \right) \varphi_n(y) \, dy \\ &\geq C \frac{\delta |A|}{b+1} \int_{\mathbb{R}^2 \setminus B_{(b+1)R}(0)} \ln(b + |y|) \varphi_n(y) \, dy \\ &\geq C \frac{\delta |A|}{b+1} (\|\varphi_n\|_{*,1} - \ln(b + (b+1)R) \|\varphi_n\|_1), \end{aligned}$$

having used the fact that $b^{b+1} > b > 1$. This yields both (a) and (b), since then

$$\|\varphi_n\|_{*,1} \leq \frac{b+1}{\delta|A|C} A_1(F(u_n), \varphi_n) + \ln(b + (b+1)R) \|\varphi_n\|_1.$$

□

3.1. Regularity of the functional \mathcal{I} . Now, we move our attention to the well-posedness and regularity of the functional in our space $X = D^{1,2}L_\omega^p(\mathbb{R}^2)$.

Proposition 3.2. *Under (f_1) and (f_{sc}) or (f_c) , the functionals I_1, I_2, I_0 and \mathcal{I} are well-defined and of class C^1 on X , and*

$$\mathcal{I}'(u)[v] = \int_{\mathbb{R}^2} \nabla u \cdot \nabla v + 2A_0(F(u), f(u)v) = \int_{\mathbb{R}^2} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^2} (\ln|\cdot| * F(u)) f(u)v \, dx.$$

Proof. First, note that $\ln(b + |x|) \geq \ln b > 0$, since $b > 1$, implies that

$$(3.6) \quad \int_{\mathbb{R}^2} F(u(x)) \, dx \leq (\ln b)^{-1} \int_{\mathbb{R}^2} \ln(b + |x|) F(u(x)) \, dx < +\infty$$

by Theorem 2.3. Hence, from (3.4) it follows that

$$I_1(u) \leq 2\|F(u)\|_{*,1}\|F(u)\|_1 < +\infty.$$

On the other hand, combining (3.5) with (2.6) or (2.7) for $r > \frac{3}{4}p > \frac{3}{2} > 1$ and $\alpha > \alpha_0$, one has

$$(3.7) \quad \begin{aligned} I_2(u) &\leq b\|F(u)\|_{\frac{4}{3}}^2 \lesssim \|u\|_{\frac{4}{3}p}^{2p} + \left(\int_{\mathbb{R}^2} |u|^{\frac{4}{3}r} e^{\frac{4}{3}\alpha|u|^2} \, dx \right)^{\frac{3}{2}} \\ &\lesssim \|u\|_{\frac{4}{3}p}^{2p} + \|\nabla u\|_2^{2r} \left(\int_{\mathbb{R}^2} \left(\frac{u}{\|\nabla u\|_2} \right)^{\frac{4}{3}r} e^{\frac{4}{3}\alpha\|u\|^2 \left(\frac{u}{\|\nabla u\|_2} \right)^2} \, dx \right)^{\frac{3}{2}} \\ &\lesssim \|u\|_{\frac{4}{3}p}^{\frac{3}{2}p} \|\nabla u\|_2^{\frac{p}{2}} + C(u) \|\nabla u\|_2^{2r - \frac{3}{2}p} \|u\|_{\frac{4}{3}p}^{\frac{3}{2}p} < +\infty, \end{aligned}$$

by Proposition 2.6 and Theorem 2.2. Consequently, \mathcal{I} is well-defined in X .

Let now $\{u_n\}_n \subset X$ be a sequence such that $u_n \rightarrow u$ in X , that is

$$(3.8) \quad \|u_n - u\|^2 = \|\nabla u_n - \nabla u\|_2^2 + \|u_n - u\|_{*,p}^2 \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

We have

$$\begin{aligned} |I_1(u_n) - I_1(u)| &= \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(b + |x - y|) (F(u_n(x))F(u_n(y)) - F(u(x))F(u(y))) \, dx \, dy \right| \\ &\leq \int_{\mathbb{R}^2} \ln(b + |x|) F(u_n(x)) \, dx \int_{\mathbb{R}^2} |F(u_n(y)) - F(u(y))| \, dy \\ &\quad + \int_{\mathbb{R}^2} \ln(b + |y|) |F(u_n(y)) - F(u(y))| \, dy \int_{\mathbb{R}^2} F(u_n(x)) \, dx \\ &\quad + \int_{\mathbb{R}^2} \ln(b + |x|) |F(u_n(x)) - F(u(x))| \, dx \int_{\mathbb{R}^2} F(u(y)) \, dy \\ &\quad + \int_{\mathbb{R}^2} \ln(b + |y|) F(u(y)) \, dy \int_{\mathbb{R}^2} |F(u_n(x)) - F(u(x))| \, dx, \end{aligned}$$

and all four terms tend to 0 as $n \rightarrow \infty$, since the functionals $\int_{\mathbb{R}^2} F(u) dx$, $\int_{\mathbb{R}^2} F(u) \ln(b + |x|) dx$ on X are continuous thanks to Corollary 2.4.

For any $u \in X$, the first Gâteaux derivative of I_1 at u along $v \in X$ is given by

$$I_1'(u)[v] = 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(b + |x - y|) F(u(x)) f(u(y)) v(y) dx dy.$$

By (3.3), we have

$$\begin{aligned} \frac{1}{2} |I_1'(u)[v]| &\leq \int_{\mathbb{R}^2} \ln(b + |x|) F(u(x)) dx \int_{\mathbb{R}^2} f(u(y)) |v(y)| dy \\ &\quad + \int_{\mathbb{R}^2} \ln(b + |y|) f(u(y)) |v(y)| dy \int_{\mathbb{R}^2} F(u(x)) dx \\ &\leq \|F(u)\|_{*,1} \|f(u)\|_{\frac{p}{p-1}} \|v\|_p + \|F(u)\|_1 \|f(u)\|_{*,\frac{p}{p-1}} \|v\|_{*,p} < +\infty \end{aligned}$$

by Theorems 2.2 and 2.3. Now, let again $\{u_n\}_n \subset X$ and $u \in X$ be as in (3.8). We have

$$\begin{aligned} \frac{1}{2} |I_1'(u_n)[v] - I_1'(u)[v]| &\leq \int_{\mathbb{R}^2} \ln(b + |x|) F(u(x)) dx \int_{\mathbb{R}^2} |f(u_n(y)) - f(u(y))| |v(y)| dy \\ &\quad + \int_{\mathbb{R}^2} F(u(x)) dx \int_{\mathbb{R}^2} \ln(b + |y|) |f(u_n(y)) - f(u(y))| |v(y)| dy \\ &\quad + \int_{\mathbb{R}^2} \ln(b + |x|) |F(u_n(x)) - F(u(x))| dx \int_{\mathbb{R}^2} f(u_n(y)) |v(y)| dy \\ &\quad + \int_{\mathbb{R}^2} |F(u_n(x)) - F(u(x))| dx \int_{\mathbb{R}^2} \ln(b + |y|) f(u_n(y)) |v(y)| dy. \end{aligned}$$

Recall now (3.6), and analogously

$$(3.9) \quad \int_{\mathbb{R}^2} |F(u_n(x)) - F(u(x))| dx \leq (\ln b)^{-1} \int_{\mathbb{R}^2} \ln(b + |x|) |F(u_n(x)) - F(u(x))| dx = o_n(1)$$

by Corollary 2.4. Moreover, by Hölder's inequality

$$\int_{\mathbb{R}^2} f(u_n(y)) |v(y)| dy \leq C(b) \int_{\mathbb{R}^2} \ln(b + |y|) f(u_n(y)) |v(y)| dy \lesssim \|f(u_n)\|_{*,\frac{p}{p-1}} \|v\|_{*,p} \leq C(u) \|v\|,$$

and

$$\begin{aligned} \int_{\mathbb{R}^2} |f(u_n(y)) - f(u(y))| |v(y)| dy &\leq (\ln b)^{-1} \int_{\mathbb{R}^2} \ln(b + |y|) |f(u_n(y)) - f(u(y))| |v(y)| dy \\ &\lesssim \|f(u_n) - f(u)\|_{*,\frac{p}{p-1}} \|v\|_{*,p} = o_n(1) \|v\|, \end{aligned}$$

since f is continuous. Combining the above inequalities, one infers

$$\frac{1}{2} |I_1'(u_n)[v] - I_1'(u)[v]| \leq C(u) \|v\| o_n(1),$$

namely $I_1' \in C(X)$. Let us now focus on I_2 . For $\{u_n\}_n \subset X$ and $u \in X$ for which (3.8) hold, by (3.5) one has

$$|I_2(u_n) - I_2(u)| \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \left(1 + \frac{b}{|x - y|} \right) F(u_n(x)) |F(u_n(y)) - F(u(y))| dx dy$$

$$\begin{aligned}
& + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \left(1 + \frac{b}{|x-y|} \right) F(u(y)) |F(u_n(x)) - F(u(x))| \, dx \, dy \\
& \lesssim \|F(u_n) - F(u)\|_{\frac{4}{3}} \left(\|F(u_n)\|_{\frac{4}{3}} + \|F(u)\|_{\frac{4}{3}} \right),
\end{aligned}$$

and tends to 0 as $n \rightarrow +\infty$, since $\|F(u_n) - F(u)\|_{\frac{4}{3}} = o_n(1)$ as in (3.9), $\|F(u)\|_{\frac{4}{3}} < +\infty$ as in (3.6), and $\|F(u_n)\|_{\frac{4}{3}} \leq C$ by continuity.

Computing the first variation of I_2 at $u \in X$ along $v \in X$ we get

$$I'_2(u)[v] = 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \left(1 + \frac{b}{|x-y|} \right) F(u(x)) f(u(y)) v(y) \, dx \, dy$$

and so

$$\begin{aligned}
\frac{1}{2} |I'_2(u)[v]| & \leq b \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} F(u(x)) f(u(y)) |v(y)| \, dx \, dy \\
& \lesssim \|F(u)\|_{\frac{4}{3}} \|f(u)\|_{\frac{4}{3}p'} \|v\|_p \leq \|F(u)\|_{\frac{4}{3}} \|f(u)\|_{\frac{4}{3}p'} \|v\| < +\infty
\end{aligned}$$

by Theorem 2.2. Analogously, for $\{u_n\}_n \subset X$ and $u \in X$ as in (3.8),

$$\frac{1}{2} |I'_2(u_n)[v] - I'_2(u)[v]| \lesssim \left(\|F(u_n)\|_{\frac{4}{3}} \|f(u_n) - f(u)\|_{\frac{4}{3}p'} + \|F(u_n) - F(u)\|_{\frac{4}{3}} \|f(u)\|_{\frac{4}{3}p'} \right) \|v\|$$

which again tends to 0 by the above arguments. As a result, both I_1 and I_2 are of class C^1 on X : consequently, also $I_0 = I_1 - I_2$ and \mathcal{I} have the same regularity. \square

4. ANALYSIS OF CERAMI SEQUENCES

Usually, a mountain pass geometry of the functional (see Lemma 4.1 below) directly provides the existence of a Cerami sequence, namely a sequence $\{u_n\}_n \subset X$ such that

$$(4.1) \quad \mathcal{I}(u_n) \rightarrow c_{mp}, \quad (1 + \|u_n\|) \|\mathcal{I}'(u_n)\|_{X'} \rightarrow 0,$$

which, by some compactness argument which exploits first the boundedness of such sequence in X , yields the existence of a weak solution. In our case, however, the proof of the boundedness of the Cerami sequence is not standard, and we need to improve the properties that such a sequence has. The abstract result contained in Proposition 2.7 allows us to find a Cerami sequence with the additional property that $\mathcal{J}(u_n) \rightarrow 0$ as $n \rightarrow +\infty$ (see Lemma 4.2 below), where the functional $\mathcal{J} : X \rightarrow \mathbb{R}$ is given by

$$(4.2) \quad u \mapsto \mathcal{J}(u) := 2\|\nabla u\|_2^2 - 2I_0(u) + 2A_0(F(u), f(u)u) - \frac{1}{2}\|F(u)\|_1^2.$$

The boundedness of $\{u_n\}_n$ in X will follow then by combining (4.1) and $\mathcal{J}(u_n) \rightarrow 0$. This strategy, which was first employed in [43] in the high-dimensional case, was implemented in the planar case by [17, 25] in the case $f(u) = u$. Here we need to extend it to the case of a general nonlinearity.

Let us first define

$$\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0 \text{ and } \mathcal{I}(\gamma(1)) < 0\},$$

and the mountain pass level

$$(4.3) \quad c_{mp} := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \mathcal{I}(\gamma(t)).$$

Lemma 4.1. *Assume (f_1) and (f_{sc}) or (f_c) hold. Then, the set Γ is nonempty and $0 < c_{mp} < +\infty$.*

Proof. We start with a control from below of I_1 : since $\ln(b+r) \geq \ln b$, for $r \geq 0$ and $b > 1$, it follows by (2.8) that

$$(4.4) \quad I_1(u) = \int_{\mathbb{R}^2} \ln(b+|x-y|) F(u(x)) F(u(y)) dx dy \geq \ln b \left(\int_{\mathbb{R}^2} F(u(x)) dx \right)^2 \geq C \ln b \|u\|_p^{2p}.$$

Concerning I_2 let us refine the upper bound proved in (3.7). If f satisfies assumption (f_{sc}) or (f_c) , then by (2.6) or (2.7), respectively, one has

$$\begin{aligned} I_2(u) &\lesssim \|u\|_{\frac{4}{3}p}^{2p} + \|\nabla u\|_2^{2r} \left(\int_{\mathbb{R}^2} \left(\frac{u}{\|\nabla u\|_2} \right)^{\frac{4}{3}r} e^{\frac{4}{3}\alpha\|u\|^2 \left(\frac{u}{\|\nabla u\|_2} \right)^2} dx \right)^{\frac{3}{2}} \\ &\lesssim \|u\|_{\frac{3}{2}p}^{\frac{3}{2}p} \|\nabla u\|_2^{\frac{p}{2}} + \|\nabla u\|_2^{2r-\frac{3}{2}p} \|u\|_{\frac{3}{2}p}^{\frac{3}{2}p}, \end{aligned}$$

having used Proposition 2.6 on the first term and Theorem 2.2 for the second, with the choice $r > \frac{3}{4}p > \frac{3}{2} > 1$, $\alpha > \alpha_0$ close to α_0 (with a little abuse of notation, for $\alpha_0 = 0$ if (f_{sc}) is assumed) and having required that $\frac{4}{3}\alpha\|u\|^2 < 4\pi$. As a result, for any $u \in X$ with $\|u\| < \sqrt{3\pi\alpha^{-1}}$, and choosing now $r = p$, we get

$$(4.5) \quad I_2(u) \lesssim \|u\|_{\frac{3}{2}p}^{\frac{3}{2}p} \|\nabla u\|_2^{\frac{p}{2}} < +\infty.$$

Hence, combining (4.4) and (4.5), by Young's inequality with ν and ν' to be chosen later, we get

$$(4.6) \quad \mathcal{I}(u) = \frac{1}{2} \|\nabla u\|_2^2 + I_1(u) - I_2(u) \geq \left(\frac{1}{2} \|\nabla u\|_2^2 - \frac{C}{\nu'} \|\nabla u\|_2^{\frac{p}{2}\nu'} \right) + \left(\ln b \|u\|_p^{2p} - \frac{C}{\nu} \|u\|_p^{\frac{3}{2}p\nu} \right).$$

Choosing $\nu \in \left(\frac{4}{3}, \frac{4}{(4-p)_+} \right)$, which is nonempty since $p > 1$, namely

$$\begin{cases} \nu > \frac{4}{3}, & \text{if } p \geq 4, \\ \nu \in \left(\frac{4}{3}, \frac{4}{4-p} \right), & \text{if } p \in (2, 4), \end{cases}$$

we easily infer from (4.6) that 0 is a local minimum for \mathcal{I} .

Let us now evaluate the functional \mathcal{I} along the fiber set $\{t^2 u(t \cdot) : u \in X, t > 0\}$. For a fixed $u \in X$ we have

$$\begin{aligned} \mathcal{I}(t^2 u(tx)) &= \frac{t^4}{2} \int_{\mathbb{R}^2} |\nabla u(x)|^2 dx + \frac{t^{-4}}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| F(t^2 u(x)) F(t^2 u(y)) dx dy \\ &\quad - \frac{t^{-4} \ln t}{2} \left(\int_{\mathbb{R}^2} F(t^2 u(x)) dx \right)^2. \end{aligned}$$

For $u \in C_c^\infty(B_{1/4}(0))$, one then gets

$$\begin{aligned} \mathcal{I}(t^2 u(tx)) &\leq \frac{t^4}{2} \int_{\mathbb{R}^2} |\nabla u(x)|^2 dx - \frac{t^{-4}}{2} (\ln 2 + \ln t) \left(\int_{\mathbb{R}^2} F(t^2 u(x)) dx \right)^2 \\ &\leq \frac{t^4}{2} \int_{\mathbb{R}^2} |\nabla u(x)|^2 dx - \frac{\ln 2 + \ln t}{2} t^{4(p-1)} \left(\int_{\mathbb{R}^2} |u|^p dx \right)^2 \rightarrow -\infty \end{aligned}$$

as $t \rightarrow +\infty$. Hence $\sup_{t>0} \mathcal{I}(t^2 u(tx)) < +\infty$, and there exists $t_* = t_*(u) > 0$ such that $\mathcal{I}(t_*^2 u(t_* x)) = \max_{t>0} \mathcal{I}(t^2 u(tx))$. Now, the function $\gamma(t) = (\tilde{t}t)^2 u(\tilde{t}t \cdot)$, with $\tilde{t} \gg t_*$ has the properties that $\gamma \in C([0, 1], X)$, $\gamma(0) = 0$, and $\mathcal{I}(\gamma(1)) < 0$. As a result, $\gamma \in \Gamma$, namely $\Gamma \neq \emptyset$ and $c_{mp} < +\infty$. Since \mathcal{I} has a local minimum in 0 by (4.6), there exist a constant $a_0 > 0$ and $\rho > 0$ such that

$$\mathcal{I}(u) \geq a_0 \quad \text{if } u \in S_\rho(0) := \left\{ u \in X : \|\nabla u\|_2^2 + \|u\|_p^p = \rho \right\}.$$

Let $\gamma \in \Gamma$, then $\|\nabla(\gamma(1))\|_2^2 + \|\gamma(1)\|_2^2 > \rho$, and by the mean value theorem there exists $\bar{t} \in [0, 1]$ such that $\|\nabla(\gamma(\bar{t}))\|_2^2 + \|\gamma(\bar{t})\|_2^2 = \rho$. This means that $\gamma(\bar{t}) = 0$, hence $\mathcal{I}(\gamma(\bar{t})) \geq a_0$. Therefore,

$$\sup_{t \in [0, 1]} \mathcal{I}(\gamma(t)) \geq \mathcal{I}(\gamma(\bar{t})) \geq a_0 > 0.$$

Taking the infimum on Γ , we can conclude that $c_{mp} > 0$. \square

With the help of the abstract result Proposition 2.7, we are in a position to prove the existence of a specific Cerami sequence.

Lemma 4.2. *Assume (f_1) and (f_{sc}) or (f_c) hold. Then there exists a Cerami sequence $\{u_n\}_n \subset X$ at the mountain pass level c_{mp} defined in (4.3), such that*

$$(4.7) \quad \mathcal{J}(u_n) \rightarrow 0.$$

Proof. Let $\tilde{X} := \mathbb{R} \times X$ be the Banach space endowed with the norm $\|(s, v)\|_{\tilde{X}} := (|s|^2 + \|v\|^2)^{\frac{1}{2}}$. Consider the continuous map $\rho : \tilde{X} \rightarrow X$ defined as

$$\rho(s, v)[x] := e^{2s} v(e^s x), \quad s \in \mathbb{R}, \quad v \in X, \quad x \in \mathbb{R}^2$$

and

$$\Psi := \mathcal{I} \circ \rho : \tilde{X} \rightarrow \mathbb{R}.$$

We compute

$$\begin{aligned} \Psi(s, v) &= \mathcal{I}(\rho(s, v)) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \rho(s, v)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x - y| F(\rho(s, v)[x]) F(\rho(s, v)[y]) dx dy \\ &= \frac{e^{6s}}{2} \int_{\mathbb{R}^2} |\nabla v(e^s x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x - y| F(e^{2s} v(e^s x)) F(e^{2s} v(e^s y)) dx dy \\ &= \frac{e^{4s}}{2} \int_{\mathbb{R}^2} |\nabla v|^2 dx' + \frac{e^{-4s}}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x' - y'| F(e^{2s} v(x')) F(e^{2s} v(y')) dx' dy' \\ &\quad - s \frac{e^{-4s}}{2} \left(\int_{\mathbb{R}^2} F(e^{2s} v(x')) dx' \right)^2. \end{aligned}$$

By Lemma 3.2, Ψ is of class C^1 on \tilde{X} , therefore we can compute the partial derivatives of Ψ . For $s \in \mathbb{R}$ and $v \in X$ one has

$$\begin{aligned} \partial_s \Psi(s, v) &= 2e^{4s} \int_{\mathbb{R}^2} |\nabla v|^2 dx - 2e^{-4s} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x - y| F(e^{2s} v(x)) F(e^{2s} v(y)) dx dy \\ &\quad + 2e^{-4s} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x - y| F(e^{2s} v(x)) f(e^{2s} v(y)) e^{2s} v(y) dx dy \end{aligned}$$

$$\begin{aligned}
& + 2se^{-4s} \left(\int_{\mathbb{R}^2} F(e^{2s}v) \right)^2 - \frac{e^{-4s}}{2} \left(\int_{\mathbb{R}^2} F(e^{2s}v) \right)^2 - 2se^{-4s} \left(\int_{\mathbb{R}^2} F(e^{2s}v) \right) \left(\int_{\mathbb{R}^2} f(e^{2s}v)e^{2s}v \right) \\
& = 2e^{4s} \int_{\mathbb{R}^2} |\nabla v|^2 dx - 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| F(e^{2s}v(e^s x)) F(e^{2s}v(e^s y)) dx dy \\
& \quad + 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| F(e^{2s}v(e^s x)) f(e^{2s}v(e^s y)) e^{2s}v(e^s y) dx dy - \frac{1}{2} \left(\int_{\mathbb{R}^2} F(e^{2s}v(e^s x)) dx \right)^2 \\
& = \mathcal{J}(\rho(s, v)),
\end{aligned}$$

where \mathcal{J} is defined in (4.2). On the other hand, for $w \in X$, one has

$$\begin{aligned}
\partial_v \Psi(s, v)[w] & = \partial_v \mathcal{I}(\rho(s, v))[w] \\
& = e^{4s} \int_{\mathbb{R}^2} \nabla v(x) \cdot \nabla w(x) dx - se^{-4s} \left(\int_{\mathbb{R}^2} F(e^{2s}v(x)) dx \right) \left(\int_{\mathbb{R}^2} f(e^{2s}v(x))w(x) dx \right) \\
& \quad + e^{-4s} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| F(e^{2s}v(x)) f(e^{2s}v(y)) e^{2s}w(y) dx dy \\
& = \int_{\mathbb{R}^2} \nabla(e^{2s}v(e^s x')) \cdot \nabla(e^{2s}w(e^s x')) dx' \\
& \quad - s \left(\int_{\mathbb{R}^2} F(e^{2s}v(e^s x')) dx' \right) \left(\int_{\mathbb{R}^2} f(e^{2s}v(e^s x'))w(e^s x') dx' \right) \\
& \quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x'-y'| F(e^{2s}v(e^s x')) f(e^{2s}v(e^s y')) e^{2s}w(e^s y') dx' dy' \\
& \quad + s \left(\int_{\mathbb{R}^2} F(e^{2s}v(e^s x')) dx' \right) \left(\int_{\mathbb{R}^2} f(e^{2s}v(e^s x'))w(e^s x') dx' \right) \\
& = \int_{\mathbb{R}^2} \nabla \rho(s, v) \cdot \nabla \rho(s, w) dx' \\
& \quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x'-y'| F(\rho(s, v)[x']) f(\rho(s, v)[y']) \rho(s, w)[y'] dx' dy' \\
& = \mathcal{I}'(\rho(s, v))[\rho(s, w)].
\end{aligned}$$

Hence, the first variation of Ψ at $(s, v) \in \tilde{X}$ along $(h, w) \in \tilde{X}$ is given by

$$(4.8) \quad \Psi'(s, v)(h, w) = \mathcal{I}'(\rho(s, v))[\rho(s, w)] + \mathcal{J}(\rho(s, v))h.$$

We are now going to apply Proposition 2.7 to the functional Ψ . To this end, let $\tilde{\Gamma} := \{\tilde{\gamma} \in C([0, 1], \tilde{X}) : \tilde{\gamma}(0) = (0, 0), \Psi(\tilde{\gamma}(1)) < 0\}$ and

$$(4.9) \quad \tilde{c} := \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \max_{t \in [0, 1]} \Psi(\tilde{\gamma}(t)).$$

With these choices, it follows that

$$(\rho \circ \tilde{\gamma})(0) = \rho(\tilde{\gamma}(0)) = \rho(0, 0) = 0$$

and

$$\mathcal{I}((\rho \circ \tilde{\gamma})(1)) = (\mathcal{I} \circ \rho)(\tilde{\gamma}(1)) = \Psi(\tilde{\gamma}(1)) < 0,$$

that is, $\Gamma = \{\rho \circ \tilde{\gamma} : \tilde{\gamma} \in \tilde{\Gamma}\}$ and the values (4.3) and (4.9) coincide. Let now $\{\gamma_n\} \subset \Gamma$ be a sequence of paths such that

$$\sup_{t \in [0,1]} \mathcal{I}(\gamma_n(t)) \leq c_{mp} + \frac{1}{n^2}.$$

Defining $\tilde{\gamma}_n(t) := (0, \gamma_n(t))$, which belongs to $\tilde{\Gamma}$, we have

$$\sup_{t \in [0,1]} \Psi(\tilde{\gamma}_n(t)) = \sup_{t \in [0,1]} \mathcal{I}(\gamma_n(t)) \leq c_{mp} + \frac{1}{n^2}.$$

Hence, Proposition 2.7 applied with $M = [0, 1]$ and $M_0 = \{0, 1\}$ yields the existence of a sequence $(s_n, v_n) \in \tilde{X}$ such that

- (a) $\Psi(s_n, v_n) \rightarrow \tilde{c}$,
- (b) $\text{dist}((s_n, v_n), \{0\} \times \gamma_n([0, 1])) \rightarrow 0$,
- (c) $(1 + \|(s_n, v_n)\|_{\tilde{X}}) \|\Psi'(s_n, v_n)\|_{\tilde{X}'} \rightarrow 0$,

as $n \rightarrow +\infty$. We observe that (b) implies

$$(4.10) \quad s_n \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Defining now $u_n := \rho(s_n, v_n)$, by (a) we get

$$\mathcal{I}(u_n) = \mathcal{I}(\rho(s_n, v_n)) = \Psi(s_n, v_n) \rightarrow c_{mp}, \text{ as } n \rightarrow +\infty$$

while, taking $h = 1$ and $w = 0$ in (4.8), from (c) we also infer

$$\mathcal{J}(u_n) = \mathcal{J}(\rho(s_n, v_n)) \rightarrow 0$$

as $n \rightarrow +\infty$. To obtain the last required property, observe that for a given $v \in X$, defining

$$w_n = e^{-2s_n} v(e^{-s_n} \cdot),$$

(4.10) allows to show that

$$\begin{aligned} \|w_n\|^2 &= \|\nabla w_n\|_2^2 + \|w_n\|_{*,p}^2 \\ &= e^{-6s_n} \int_{\mathbb{R}^2} |\nabla v(e^{-s_n} x)|^2 dx + e^{-2s_n} \left(\int_{\mathbb{R}^2} \ln(b + |x|) |v(e^{-s_n} x)|^p dx \right)^{\frac{1}{p}} \\ (4.11) \quad &= e^{-4s_n} \int_{\mathbb{R}^2} |\nabla v(x')|^2 dx' + e^{-2s_n \left(\frac{p-1}{p}\right)} \left(\int_{\mathbb{R}^2} \ln(b + e^{s_n} |x'|) |v(x')|^p dx' \right)^{\frac{1}{p}} \\ &= (1 + o_n(1)) \int_{\mathbb{R}^2} |\nabla v(x')|^2 dx' + (1 + o_n(1)) \left(\int_{\mathbb{R}^2} \ln(b + e^{s_n} |x'|) |v(x')|^p dx' \right)^{\frac{1}{p}} \\ &= (1 + o_n(1)) \|v\|^2 + o_n(1) \|v\|_p^p, \end{aligned}$$

as $n \rightarrow +\infty$, where in the last step we used the fact that

$$\ln(b + e^{-s_n} |x|) \leq \ln(b + |x|) + \ln \left(1 + \frac{(1 - e^{-s_n})|x|}{b + |x|} \right) \leq \ln(b + |x|) + \ln(1 + (1 - e^{-s_n})).$$

Analogously one can show that $\|v_n\| = (1 + o_n(1))\|u_n\|$, therefore, on the one hand, by (4.8) with $h = 0$ and (4.10), one infers

$$\begin{aligned}
 (4.12) \quad (1 + \|(s_n, v_n)\|_{\tilde{X}}) |\Psi'(s_n, v_n)(0, w_n)| &= (1 + (|s_n|^2 + \|v_n\|^2)^{\frac{1}{2}}) |\mathcal{I}'(\rho(s_n, v_n))\rho(s_n, w_n)| \\
 &= (1 + o_n(1) + \|v_n\|) |\mathcal{I}'(u_n)v| \\
 &= (1 + o_n(1) + (1 + o_n(1))\|u_n\|) |\mathcal{I}'(u_n)v|,
 \end{aligned}$$

while, on the other hand, by (4.11),

$$\begin{aligned}
 (4.13) \quad (1 + \|(s_n, v_n)\|_{\tilde{X}}) |\Psi'(s_n, v_n)(0, w_n)| &\leq (1 + \|(s_n, v_n)\|_{\tilde{X}}) \|\Psi'(s_n, v_n)\|_{\tilde{X}'} \|w_n\| \\
 &= o_n(1)(1 + o_n(1))\|v\|.
 \end{aligned}$$

Combining together (4.12)-(4.13) we deduce

$$(1 + o_n(1) + (1 + o_n(1))\|u_n\|) \|\mathcal{I}'(u_n)\|_{X'} \rightarrow 0$$

as $n \rightarrow +\infty$, which readily implies

$$(1 + \|u_n\|) \|\mathcal{I}'(u_n)\|_{X'} \rightarrow 0$$

as $n \rightarrow +\infty$. □

The extra property (4.7) obtained in Proposition 4.2 is crucial to prove the boundedness of a Cerami sequence in X , as shown next.

Lemma 4.3. *Assume (f_1) , (f_2) and (f_{sc}) or (f_c) hold. Let $\{u_n\}_n \subset X$ be a sequence such that*

$$(4.14) \quad \mathcal{I}(u_n) \rightarrow c_{mp}, \quad (1 + \|u_n\|) \|\mathcal{I}'(u_n)\|_{X'} \rightarrow 0, \quad \mathcal{J}(u_n) \rightarrow 0.$$

Then, $\{u_n\}_n$ is bounded in X . Moreover, there exist $C_1, C_2 > 0$ such that

$$(4.15) \quad |I_1(u_n)| = \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(b + |x - y|) F(u_n(x)) F(u_n(y)) \, dx \, dy \right| \leq C_1,$$

$$(4.16) \quad |A_1(F(u_n), f(u_n)u_n)| = \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(b + |x - y|) F(u_n(x)) f(u_n(y)) u_n(y) \, dx \, dy \right| \leq C_2.$$

Proof. We first show that $\{\nabla u_n\}_n$ is bounded in $L^2(\mathbb{R}^2)$ by combining the information on \mathcal{I} and its derivative, following the strategy of [14, Lemma 6.1]. To this aim we introduce the sequence

$$v_n := \begin{cases} \frac{F(u_n)}{f(u_n)} & \text{if } u_n > 0, \\ (1 - \tau)u_n & \text{if } u_n < 0, \end{cases}$$

for which, by (2.9), $|v_n| \leq (1 - \tau)|u_n|$ hold, hence $\{v_n\}_n \subset L^p_\omega(\mathbb{R}^2)$. Moreover, a simple computation shows that

$$\nabla \left(\frac{F(u_n)}{f(u_n)} \right) = \left(1 - \frac{F(u_n)f'(u_n)}{(f(u_n))^2} \right) \nabla u_n,$$

therefore (f_2) implies $|\nabla v_n| \leq C|\nabla u_n|$, from which $\{v_n\}_n \subset D^{1,2}(\mathbb{R}^2)$ and in turn $\{v_n\}_n \subset X$. Therefore, they may be used as test functions for $\mathcal{I}'(u_n) \in X'$, obtaining

$$\begin{aligned}
 (4.17) \quad & \int_{\{u_n \geq 0\}} |\nabla u_n|^2 \left(1 - \frac{F(u_n)f'(u_n)}{(f(u_n))^2} \right) + (1 - \tau) \int_{\{u_n < 0\}} |\nabla u_n|^2 \\
 & + \int_{\{u_n \geq 0\}} (\ln |\cdot| * F(u_n)) F(u_n) + (1 - \tau) \int_{\{u_n < 0\}} (\ln |\cdot| * F(u_n)) f(u_n) u_n \\
 & = |\mathcal{I}'(u_n)[v_n]| \leq \|\mathcal{I}'(u_n)\|_{X'} \|v_n\| \lesssim \|\mathcal{I}'(u_n)\|_{X'} \|u_n\| = o_n(1).
 \end{aligned}$$

Since $f \equiv 0$ on \mathbb{R}^- , the last term in the left-hand side is zero. Combining this with $\mathcal{I}(u_n) \rightarrow c_{mp}$, one infers

$$\begin{aligned}
 \|\nabla u_n\|_2^2 - 2c_{mp} + o_n(1) &= \int_{\mathbb{R}^2} \left(\ln \frac{1}{|\cdot|} * F(u_n) \right) F(u_n) \\
 &= \int_{\{u_n \geq 0\}} |\nabla u_n|^2 \left(1 - \frac{F(u_n)f'(u_n)}{(f(u_n))^2} \right) + (1 - \tau) \int_{\{u_n < 0\}} |\nabla u_n|^2 + o_n(1) \\
 &\leq (1 - \tau) \|\nabla u_n\|_2^2 + o_n(1),
 \end{aligned}$$

from which it is easy to obtain

$$(4.18) \quad \|\nabla u_n\|_2^2 \leq \frac{2c_{mp}}{\tau} + o_n(1).$$

This, together with the first two conditions in (4.14), yields

$$(4.19) \quad |I_0(u_n)| = \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x - y|) F(u_n(x)) F(u_n(y)) \, dx \, dy \right| \leq C_1$$

$$(4.20) \quad |A_0(F(u_n), f(u_n)u_n)| = \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x - y|) F(u_n(x)) f(u_n(y)) u_n(y) \, dx \, dy \right| \leq C_2.$$

Using (4.18), (4.19), and (4.20), the condition $\mathcal{J}(u_n) \rightarrow 0$ directly implies

$$(4.21) \quad \|u_n\|_p^{2p} \leq C \left(\int_{\mathbb{R}^2} F(u_n) \right)^2 \leq C,$$

where the first inequality is due to (2.8). In light of (4.21) and (4.18), $\{I_2(u_n)\}_n$ is bounded thanks to (4.5). Recalling the decomposition $I_0 = I_1 - I_2$, this and (4.19) imply (4.15). The bound (4.16) follows by similar arguments using A_0, A_1, A_2 . The uniform boundedness of $\|u_n\|_{*,p}$ is then a consequence of Lemma 3.1(a), applied with $\varphi_n = F(u_n)$, and (2.8). \square

Remark 4.4. Thanks to Lemma 4.3, from now on we can always suppose that Cerami sequences at level c_{mp} verifying (4.14) are nonnegative. Indeed, $u_n^- := \min\{u_n, 0\} \in X$ and $u_n^- \leq 0$ and thus, recalling that $f \equiv 0$ on \mathbb{R}^- by assumption, one has

$$\begin{aligned}
 \|\nabla u_n^-\|_2^2 &= \|\nabla u_n^-\|_2^2 + \int_{\mathbb{R}^2} (\ln |\cdot| * F(u_n)) f(u_n) u_n^- \\
 &= \mathcal{I}'(u_n)[u_n^-] \leq \|\mathcal{I}'(u_n)\|_{X'} \|u_n^-\| = o_n(1)
 \end{aligned}$$

since $\|u_n^-\| \leq \|u_n\| \leq C$ by Lemma 4.3. This implies that $u_n^- \rightarrow 0$ in X as $n \rightarrow +\infty$ and therefore that $\{u_n^+\}_n$ is a Cerami sequence of \mathcal{I} at level c , which henceforth we will simply denote by $\{u_n\}_n$.

4.1. Mountain pass level estimate in the critical case. Under assumption (f_{sc}) the boundedness of the Cerami sequences is enough to proceed with the main existence argument, see Section 5, since uniform estimates of the nonlinear terms may be deduced by (2.6) by choosing a suitably small exponent α . This of course is not the case when we are dealing with critical exponential nonlinearities, and we need to prove that under (f_c) , and in particular taking into account (f_4) , the critical mountain pass level is below a noncompactness threshold. To this aim, let us introduce the usual Moser sequence

$$\tilde{w}_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\ln n} & \text{for } 0 \leq |x| \leq \frac{\rho}{n}, \\ \frac{\ln(\rho/|x|)}{\sqrt{\ln n}} & \text{for } \frac{\rho}{n} < |x| < \rho, \\ 0 & \text{for } |x| \geq \rho. \end{cases}$$

It is easy to see that $\|\nabla \tilde{w}_n\|_2 = 1$ for all $n \in \mathbb{N}$, and that

$$\begin{aligned} \|\tilde{w}_n\|_{*,p}^p &= \int_{B_\rho} \ln(b + |x|) |\tilde{w}_n|^p dx \\ &= (2\pi)^{1-\frac{p}{2}} (\ln n)^{\frac{p}{2}} \int_0^{\rho/n} \ln(b+r) r dr + \frac{(2\pi)^{1-\frac{p}{2}}}{(\ln n)^{\frac{p}{2}}} \int_{\rho/n}^\rho \ln^p\left(\frac{\rho}{r}\right) \ln(b+r) r dr \\ &\leq (2\pi)^{1-\frac{p}{2}} (\ln n)^{\frac{p}{2}} \frac{\rho^2 \ln(b + \rho/n)}{2n^2} + (2\pi)^{1-\frac{p}{2}} \frac{\ln(b + \rho)}{(\ln n)^{\frac{p}{2}}} \int_{\rho/n}^\rho \ln^p\left(\frac{\rho}{r}\right) r dr. \end{aligned}$$

The last term in the previous expression can be estimated as follows. On the one hand, for $k \in \mathbb{N}$,

$$\int \ln^k\left(\frac{\rho}{r}\right) r dr = \frac{r^2}{2} \sum_{j=0}^k (\ln(\frac{\rho}{r}))^{k-j} \frac{k(k-1) \cdots (k-j+1)}{2^j}.$$

On the other hand, since p may be an integer or not, a rough estimate reads as follows

$$\int_{\rho/n}^\rho \ln^p\left(\frac{\rho}{r}\right) r dr \leq \int_{\rho/n}^\rho \left\{ \ln^{[p]}(\frac{\rho}{r}) + \ln^{[p]+1}(\frac{\rho}{r}) \right\} r dr = \frac{\rho^2 [p]!}{2^{[p]+1}} \left(1 + \frac{[p]+1}{2} \right) + o_n(1),$$

so that eventually

$$1 \leq \|\tilde{w}_n\|^2 \leq 1 + \delta_n + o_n\left(\frac{1}{\ln n}\right),$$

where

$$(4.22) \quad \delta_n = \rho^{\frac{4}{p}} \frac{(2\pi)^{\frac{2}{p}-1}}{\ln n} \ln^{\frac{2}{p}}(b + \rho) \left[\frac{[p]!}{2^{[p]+1}} \left(1 + \frac{[p]+1}{2} \right) \right]^{\frac{2}{p}} + o_n\left(\frac{1}{\ln n}\right).$$

Hence we normalise the Moser sequence $\{\tilde{w}_n\}_n$ by defining

$$(4.23) \quad w_n := \frac{\tilde{w}_n}{\sqrt{1 + \delta_n}}, \quad n \in \mathbb{N}.$$

Lemma 4.5. *Under (f_1) , (f_c) , (f_2) , (f_4) , one has*

$$(4.24) \quad c_{mp} < \frac{2\pi}{\alpha_0}.$$

Proof. The proof follows the same arguments of [14, Lemma 5.2]. We claim that there exists n such that

$$(4.25) \quad \max_{t \geq 0} \mathcal{I}(tw_n) < \frac{2\pi}{\alpha_0}.$$

Let us argue by contradiction and suppose this is not the case, so that for all n let $t_n > 0$ be such that

$$(4.26) \quad \max_{t \geq 0} \mathcal{I}(tw_n) = \mathcal{I}(t_n w_n) \geq \frac{2\pi}{\alpha_0}.$$

Then t_n satisfies

$$\frac{d}{dt} \Big|_{t=t_n} \mathcal{I}(tw_n) = 0$$

and

$$(4.27) \quad t_n^2 \geq \int_{\mathbb{R}^2} \left[\ln \frac{1}{|\cdot|} * F(t_n w_n) \right] t_n w_n f(t_n w_n) dx,$$

$$(4.28) \quad t_n^2 \geq \frac{4\pi}{\alpha_0} + \int_{\mathbb{R}^2} \left[\ln \frac{1}{|\cdot|} * F(t_n w_n) \right] F(t_n w_n) dx.$$

Note that in (4.27) we have an inequality instead of the equality since in the energy functional it appears only $\|\nabla w_n\|_2^2 \leq \|w_n\|^2 = 1$. From now on let us suppose $\rho \leq 1/2$. This will simplify a few estimates, since for any $(x, y) \in \text{supp } w_n \times \text{supp } w_n$ we have $|x - y| > 1$, and in turn $\ln(1/|x - y|) > 0$. Let us now proceed in three steps.

Step 1. The following holds: $\limsup_{n \rightarrow +\infty} t_n^2 \geq 4\pi/\alpha_0$.

Let us assume by contradiction that $\limsup_n t_n^2 < 4\pi/\alpha_0$: this implies that, up to a subsequence, there exists a positive constant δ_0 such that $t_n^2 \leq 4\pi/\alpha_0 - \delta_0$ for n large enough. Since $\rho \leq \frac{1}{2}$, for any $|x| < \rho$, the set $\{y : |x - y| > 1, |y| < \rho\}$ is empty. Recalling that the functions w_n are compactly supported in B_ρ , we have

$$\int_{\mathbb{R}^2} \left[\ln \frac{1}{|\cdot|} * F(t_n w_n) \right] F(t_n w_n) dx = \int_{B_\rho} \int_{|x-y| \leq 1} \ln \frac{1}{|x-y|} F(t_n w_n(x)) F(t_n w_n(y)) dx dy \geq 0$$

and thus a contradiction with (4.28).

Step 2. The following holds: $\liminf_{n \rightarrow +\infty} t_n^2 \leq 4\pi/\alpha_0$.

Let us suppose by contradiction that $\liminf_{n \rightarrow +\infty} t_n^2 > 4\pi/\alpha_0$. Hence, up to a subsequence, there exists a constant $\delta_0 > 0$ such that

$$t_n^2 \geq \frac{4\pi}{\alpha_0} + \delta_0$$

as $n \rightarrow +\infty$. Let us estimate from below the right hand side of (4.27) (taking into account the possible negative sign of the logarithmic function):

$$\begin{aligned}
 (4.29) \quad & \int_{\mathbb{R}^2} \left[\ln \frac{1}{|\cdot|} * F(t_n w_n) \right] t_n w_n f(t_n w_n) \, dx \\
 &= \int_{\{|x| \leq \frac{\rho}{n}, |y| \leq \frac{\rho}{n}\}} \ln \frac{1}{|x-y|} F(t_n w_n(x)) t_n w_n f(t_n w_n(y)) \, dx \, dy \\
 &\quad + \int_{\mathbb{R}^2 \times \mathbb{R}^2 \setminus \{|x| \leq \frac{\rho}{n}, |y| \leq \frac{\rho}{n}\}} \ln \frac{1}{|x-y|} F(t_n w_n(x)) t_n w_n f(t_n w_n(y)) \, dx \, dy \\
 &=: T_1 + T_2.
 \end{aligned}$$

Thanks to (f_4) we have for any $\varepsilon > 0$ (here we choose $\varepsilon = \beta/2$),

$$sf(s)F(s) \geq \frac{\beta - \varepsilon}{s^2} e^{2\alpha_0 s^2} = \frac{\beta}{2s^2} e^{2\alpha_0 s^2}, \quad \text{for all } s \geq s_\varepsilon = s_\beta.$$

By the very definition of w_n (see (4.23)) and since $|x-y| < 2\rho/n < 1$, for n large enough we can estimate I_1 as follows

$$\begin{aligned}
 T_1 &= \int_{B_{\rho/n}} t_n w_n f(t_n w_n) \, dy \int_{B_{\rho/n}} \ln \frac{1}{|x-y|} F(t_n w_n) \, dx \\
 &= \int_{B_{\rho/n}} t_n \frac{\sqrt{\ln n}}{\sqrt{2\pi(1+\delta_n)}} f\left(t_n \frac{\sqrt{\ln n}}{\sqrt{2\pi(1+\delta_n)}}\right) \, dy \int_{B_{\rho/n}} \ln \frac{1}{|x-y|} F\left(t_n \frac{\sqrt{\ln n}}{\sqrt{2\pi(1+\delta_n)}}\right) \, dx \\
 &\geq 2\pi\beta \frac{e^{\alpha_0 t_n^2 [\pi(1+\delta_n)]^{-1} \ln n}}{2\alpha_0 t_n^2 [\pi(1+\delta_n)]^{-1} \ln n} \int_{B_{\rho/n}} dy \int_{B_{\rho/n}} \ln \frac{1}{|x-y|} \, dx.
 \end{aligned}$$

Since

$$\int_{B_{\rho/n}} dy \int_{B_{\rho/n}} \ln \frac{1}{|x-y|} \, dx \geq |B_{\rho/n}|^2 \ln \frac{n}{2\rho} = \pi^2 \left(\frac{\rho}{n}\right)^4 \ln \frac{n}{2\rho}$$

we obtain

$$(4.30) \quad T_1 \geq \pi^3 \rho^4 \beta \frac{e^{(\alpha_0 t_n^2 [\pi(1+\delta_n)]^{-1} - 4) \ln n}}{\alpha_0 t_n^2 [\pi(1+\delta_n)]^{-1} \ln n} \ln \frac{n}{2\rho} \geq \frac{\pi^3 \rho^4 \beta}{\alpha_0 t_n^2} e^{\left(\frac{\alpha_0}{\pi} \frac{t_n^2}{1+\delta_n} - 4\right) \ln n}$$

for any $n \geq n(\rho, \beta)$. Note that since $\rho \leq 1/2$ we have

$$T_2 \geq 0.$$

Now, combining (4.27), (4.29) and (4.30) yields

$$(4.31) \quad t_n^4 \geq \frac{\pi^3 \rho^4 \beta}{\alpha_0} e^{\left(\frac{\alpha_0}{\pi} \frac{t_n^2}{1+\delta_n} - 4\right) \ln n}$$

which is a contradiction, either if $t_n \rightarrow +\infty$ or t_n stays bounded with $t_n^2 \geq \frac{4\pi}{\alpha_0} + \delta_0$. The proof of Step 2 is then completed. Observe that, as a consequence of Step 1 and Step 2

$$t_n^2 \rightarrow \frac{4\pi}{\alpha_0} \quad \text{as } n \rightarrow +\infty.$$

Moreover, as a byproduct of (4.31), we also have

$$e^{\left(\frac{\alpha_0}{\pi} \frac{t_n^2}{1+\delta_n} - 4\right) \ln n} \leq C,$$

for some $C > 0$, that is

$$(4.32) \quad \frac{t_n^2}{1+\delta_n} \leq \frac{4\pi}{\alpha_0} + \frac{C}{\ln n} = \frac{4\pi}{\alpha_0} + O\left(\frac{1}{\ln n}\right).$$

Step 3. We are now in a position of getting a contradiction and determine the quantity \mathcal{V} which appears in condition (f_4) . We have proved that $t_n^2 \rightarrow 4\pi/\alpha_0$. Moreover, we also know that $t_n^2 \geq 4\pi/\alpha_0$ by (4.28). By (4.31), recalling the definition (4.22) of δ_n , we have

$$\begin{aligned} \left(\frac{4\pi}{\alpha_0}\right)^2 + o_n(1) &\geq t_n^4 \geq \frac{\pi^3 \rho^4 \beta}{\alpha_0} e^{4\left(\frac{\alpha_0}{4\pi} \frac{t_n^2}{1+\delta_n} - 1\right) \ln n} \geq \frac{\pi^3 \rho^4 \beta}{\alpha_0} e^{-4\frac{\delta_n}{1+\delta_n} \ln n} \\ &\geq \frac{\pi^3 \rho^4 \beta}{\alpha_0} e^{-\rho^{\frac{4}{p}} (2\pi)^{\frac{2}{p}-1} \ln^{\frac{2}{p}}(b+\rho) \left[\frac{[p]!}{2^{[p]+1}} \left(1 + \frac{[p]+1}{2}\right)\right]^{\frac{2}{p}} + o_n(1)}. \end{aligned}$$

Passing to the limit, we obtain

$$(4.33) \quad \frac{16\pi^2}{\alpha_0^2} \geq \frac{\pi^3 \rho^4 \beta}{\alpha_0} e^{-\rho^{\frac{4}{p}} (2\pi)^{\frac{2}{p}-1} \ln^{\frac{2}{p}}(b+\rho) \left[\frac{[p]!}{2^{[p]+1}} \left(1 + \frac{[p]+1}{2}\right)\right]^{\frac{2}{p}}}.$$

Now set in assumption (f_4)

$$(4.34) \quad \mathcal{V} := \inf_{|x| \leq 1/2} \frac{16}{\alpha_0 \pi} |x|^{-4} e^{|x| \frac{4}{p} (2\pi)^{\frac{2}{p}-1} \ln^{\frac{2}{p}}(b+|x|) \left[\frac{[p]!}{2^{[p]+1}} \left(1 + \frac{[p]+1}{2}\right)\right]^{\frac{2}{p}}},$$

a quantity which is actually a minimum, since the right-hand function is continuous and unbounded as $|x| \rightarrow 0$. Finally, since $\beta > \mathcal{V}$, we can fix $\rho \in (0, 1/2]$ such that

$$\beta > \frac{16}{\alpha_0 \pi} \rho^{-4} e^{\rho^{\frac{4}{p}} (2\pi)^{\frac{2}{p}-1} \ln^{\frac{2}{p}}(b+\rho) \left[\frac{[p]!}{2^{[p]+1}} \left(1 + \frac{[p]+1}{2}\right)\right]^{\frac{2}{p}}}$$

to get

$$\frac{\pi^3 \rho^4 \beta}{\alpha_0} e^{-\rho^{\frac{4}{p}} (2\pi)^{\frac{2}{p}-1} \ln^{\frac{2}{p}}(b+\rho) \left[\frac{[p]!}{2^{[p]+1}} \left(1 + \frac{[p]+1}{2}\right)\right]^{\frac{2}{p}}} > \frac{4\pi}{\alpha_0},$$

which contradicts (4.33) and, therefore, (4.26). This shows that (4.25) holds, and in turn (4.24). \square

To avoid trivial solutions, in showing existence we will need to prove a result *à la* Lions (see Section 5 below). To this end, in the spirit of [14, Lemma 6.3], we need to improve the integrability for $F(u_n)$, where $\{u_n\}_n$ is the Cerami sequence given by Lemma 4.2, since this will enable us to uniformly control the terms appearing from an application of the Hardy-Littlewood-Sobolev inequality. Here the mountain pass level estimate given by Lemma 4.5 plays a crucial rôle. Unlike [14, Lemma 6.3], we cannot apply Ruf's version of the Trudinger-Moser inequality in $H^1(\mathbb{R}^2)$ because of the lack of the mass term in (1.5); however, since our space $X \subset L^p(\mathbb{R}^2)$, we will make use of the refinement of Cao's inequality in Theorem 2.2.

Lemma 4.6. *Assume $(f_1) - (f_4)$ and (f_c) . Let $\{u_n\}_n \subset X$ be a nonnegative Cerami sequence for \mathcal{I} at level $c_{mp} < \frac{2\pi}{\alpha_0}$, which is bounded in X . Then, for any $\gamma \in \left[1, \frac{2\pi}{\alpha_0 c_{mp}}\right)$ the following uniform bound holds*

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2} (F(u_n))^\gamma dx < +\infty.$$

Proof. Since $\{u_n\}_n$ is bounded in the reflexive space X , there exists $u \in X$ such that $u_n \rightharpoonup u$ in X , which implies and that, up to a subsequence, the convergence is strong in $L^q(\mathbb{R}^2)$, for $q \geq p$, and a.e. in \mathbb{R}^2 thanks to Lemma 2.1. Let us define the function

$$(4.35) \quad G(t) := \int_0^t \sqrt{\frac{F(s)f'(s)}{f(s)^2}} ds,$$

which is well-defined and C^1 thanks to (f_2) , and let $v_n := G(u_n)$. Then, by (f_2) one has

$$\|\nabla v_n\|_2^2 = \int_{\mathbb{R}^2} |\nabla u_n|^2 \left(\frac{F(u_n)f'(u_n)}{f(u_n)^2} \right) \leq C \|\nabla u_n\|_2^2 \leq C$$

and, by (f_1) and Hölder's inequality,

$$|G(t)|^p \leq t^{\frac{p}{2}} \left(\int_0^t \frac{F(s)f'(s)}{f(s)^2} ds \right)^{\frac{p}{2}} \leq t^{\frac{p}{2}} \left(-\frac{F(t)}{f(t)} + \lim_{s \rightarrow 0^+} \frac{F(s)}{f(s)} + t \right)^{\frac{p}{2}} \leq t^p \left(-\frac{F(t)}{tf(t)} + 1 \right)^{\frac{p}{2}} \leq t^p,$$

which yields at once $\|v_n\|_p \leq \|u_n\|_p$ and $\|v_n\|_{*,p} \leq \|u_n\|_{*,p}$. Note also that $G(t) > \sqrt{\tau}t$, hence $v_n > \sqrt{\tau}u_n$ a.e. in \mathbb{R}^2 . Recalling now that $u_n \geq 0$, and combining (4.17) with $\mathcal{I}(u_n) \rightarrow c_{mp}$, we infer

$$\begin{aligned} 2c_{mp} + o_n(1) &= \|\nabla u_n\|_2^2 + \int_{\mathbb{R}^2} (\ln |\cdot| * F(u_n)) F(u_n) \\ &= \|\nabla u_n\|_2^2 - \int_{\mathbb{R}^2} |\nabla u_n|^2 \left(1 - \frac{F(u_n)f'(u_n)}{f(u_n)^2} \right) \\ &= \int_{\mathbb{R}^2} |\nabla u_n|^2 \left(\frac{F(u_n)f'(u_n)}{f(u_n)^2} \right) = \|\nabla v_n\|_2^2. \end{aligned}$$

Since $c_{mp} < \frac{2\pi}{\alpha_0}$ by assumption, we deduce

$$(4.36) \quad \|\nabla v_n\|_2^2 = 2c_{mp} + o_n(1) < \frac{4\pi}{\alpha_0}.$$

Moreover, by (f_3) , as in [14, Lemma 6.3], for all $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$u_n \leq t_\varepsilon + \frac{v_n}{1-\varepsilon} \quad \text{for all } x \in \mathbb{R}^2.$$

From all this, together with (2.7) with $\alpha > \alpha_0$, we may estimate as follows:

$$\begin{aligned} \int_{\mathbb{R}^2} |F(u_n)|^\gamma &\leq C_\varepsilon \int_{\{u_n < t_\varepsilon\}} |u_n|^{p\gamma} + C_\varepsilon \int_{\{u_n \geq t_\varepsilon\}} \left(t_\varepsilon + \frac{v_n}{1-\varepsilon} \right)^{\gamma r} e^{\gamma\alpha(1+\varepsilon)(t_\varepsilon + \frac{v_n}{1-\varepsilon})^2} dx \\ &\leq C_\varepsilon \int_{\mathbb{R}^2} |u_n|^{p\gamma} + C_\varepsilon \int_{\{u_n \geq t_\varepsilon\}} e^{\gamma\alpha(\frac{1+\varepsilon}{1-\varepsilon})^2 \|\nabla v_n\|_2^2 \left(\frac{v_n}{\|\nabla v_n\|_2} \right)^2} dx, \end{aligned}$$

where in the last step we used the inequalities $s^r \leq C_\varepsilon e^{\varepsilon s^2}$ for any $s \geq t_\varepsilon$ and

$$\left(t_\varepsilon + \frac{v_n}{1-\varepsilon}\right)^2 \leq C_\varepsilon t_\varepsilon^2 + (1+\varepsilon) \left(\frac{v_n}{1-\varepsilon}\right)^2,$$

see [14, eq. (6.9)]. Choosing now ε small enough such that $t_\varepsilon \geq \tau^{-1/2}$, in the set $\{u_n \geq t_\varepsilon\}$ one has $v_n \geq 1$, and therefore, defining $w_n := \frac{v_n}{\|\nabla v_n\|_2}$, by means of the (strict) fine upper bound in (4.36), Theorem 2.2 with $q = p$, $\gamma \in [1, \frac{2\pi}{\alpha_0 c_{mp}})$, and choosing α close to α_0 , one has

$$\begin{aligned} \int_{\mathbb{R}^2} |F(u_n)|^\gamma &\leq C_\varepsilon \int_{\mathbb{R}^2} |u_n|^{p\gamma} + C_\varepsilon \int_{\mathbb{R}^2} |v_n|^p e^{\gamma\alpha(\frac{1+\varepsilon}{1-\varepsilon})^2 \|\nabla v_n\|_2^2 \left(\frac{v_n}{\|\nabla v_n\|_2}\right)^2} dx \\ (4.37) \quad &\leq C_\varepsilon \int_{\mathbb{R}^2} |u_n|^{p\gamma} + C_\varepsilon \|\nabla v_n\|_2^p \int_{\mathbb{R}^2} w_n^p e^{\gamma\alpha(\frac{1+\varepsilon}{1-\varepsilon})^2 \|\nabla v_n\|_2^2 w_n^2} dx \\ &\leq C_\varepsilon \|u_n\|_{p\gamma}^{p\gamma} + C_\varepsilon \|v_n\|_p^p \\ &\leq C_\varepsilon \|u_n\|_{p\gamma}^{p\gamma} + C_\varepsilon \|u_n\|_p^p \leq C, \end{aligned}$$

since $\{u_n\}_n$ is bounded in X and by Lemma 2.1. \square

From the proof of Lemma 4.6, and in particular combining (4.37) with the compact embedding provided by Lemma 2.1, we can say actually more:

Corollary 4.7. *Assume $(f_1) - (f_4)$ and (f_c) hold. Let $\{u_n\}_n \subset X$ be a nonnegative Cerami sequence which is bounded in X , and such that $u_n \rightarrow 0$ in $L^s(\mathbb{R}^2)$ for all $s > p$. Then, for any $\gamma \in \left(1, \frac{2\pi}{\alpha_0 c_{mp}}\right)$*

$$\|F(u_n)\|_\gamma \rightarrow 0 \quad \text{and} \quad \|f(u_n)u_n\|_\gamma \rightarrow 0.$$

Proof. Reasoning in the same way as for (4.37) in Lemma 4.6, for some $r > 0$ and $\alpha > \alpha_0$ we get

$$\int_{\mathbb{R}^2} |F(u_n)|^\gamma dx \leq C_\varepsilon \int_{\mathbb{R}^2} |u_n|^{p\gamma} + C_\varepsilon \int_{\mathbb{R}^2} |v_n|^r e^{\gamma\alpha(\frac{1+\varepsilon}{1-\varepsilon})^2 \|\nabla v_n\|_2^2 \left(\frac{v_n}{\|\nabla v_n\|_2}\right)^2} dx.$$

The first term on the right-hand side goes to 0 as $n \rightarrow +\infty$ since $p\gamma > p$, while for the second term we proceed by using Hölder's inequality and Theorem 2.2: indeed, choosing $\sigma \in (1, \sigma_0)$, with σ_0 close to 1 such that the exponent is less than 4π , r big enough such that $\frac{r}{2}\sigma > p$, and α close to α_0 , we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} |F(u_n)|^\gamma dx &\leq C_\varepsilon \|u_n\|_{p\gamma}^{p\gamma} + C_\varepsilon \left(\int_{\mathbb{R}^2} |v_n|^{\frac{r}{2}\sigma'} dx \right)^{\frac{1}{\sigma'}} \left(\int_{\mathbb{R}^2} |v_n|^{\frac{r}{2}\sigma} e^{\sigma\gamma\alpha(\frac{1+\varepsilon}{1-\varepsilon})^2 \|\nabla v_n\|_2^2 \left(\frac{v_n}{\|\nabla v_n\|_2}\right)^2} dx \right)^{\frac{1}{\sigma}} \\ &\leq C_\varepsilon \|u_n\|_{p\gamma}^{p\gamma} + C_\varepsilon \|u_n\|_{\frac{r}{2}\sigma'}^{\frac{r}{2}\sigma'} C \|v_n\|_{\frac{r}{2}\sigma}^{\frac{r}{2}\sigma}, \end{aligned}$$

which tends to 0 as $n \rightarrow +\infty$ since $p\gamma > s$.

Concerning $\|f(u_n)u_n\|_\gamma$, we can repeat the same computations as above, using now (f_1) and (f_c) , hence

$$\int_{\mathbb{R}^2} |f(u_n)u_n|^\gamma dx \rightarrow 0$$

as $n \rightarrow +\infty$, concluding the proof. \square

5. EXISTENCE IN THE CRITICAL AND SUBCRITICAL CASE: PROOF OF THEOREM 1.2

By Lemma 4.1 we know that \mathcal{I} satisfies the mountain pass geometry, and this yields the existence of a Cerami sequence $\{u_n\}_n \subset X$ for \mathcal{I} at level c_{mp} defined in (4.3), satisfying (4.14) by Lemma 4.2, which can be assumed nonnegative by Remark 4.4. By Lemma 4.3 such a sequence is bounded in X . Suppose by contradiction that $\{u_n\}_n$ is vanishing, namely

$$(5.1) \quad \liminf_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^2} \int_{B_2(y)} |u_n|^p dx = 0.$$

Since $\{u_n\}_n$ is bounded, then $u_n \rightarrow 0$ in $L^s(\mathbb{R}^2)$ as $n \rightarrow +\infty$ for every $s \in (p, +\infty)$ by [32, Lemma I.1]. By exploiting the inequality $\ln\left(1 + \frac{b}{t}\right) \lesssim t^q$ for $q \in (0, 1]$, and the Hardy-Littlewood-Sobolev inequality, if (f_c) is assumed one may estimate

$$(5.2) \quad \begin{aligned} I_2(u_n) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln\left(1 + \frac{b}{|x-y|}\right) F(u_n(x))F(u_n(y)) dx dy \\ &\lesssim \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{F(u_n(x))F(u_n(y))}{|x-y|^{\frac{4(\gamma-1)}{\gamma}}} dx dy \lesssim \|F(u_n)\|_\gamma^2 \rightarrow 0 \end{aligned}$$

as $n \rightarrow +\infty$, which holds by Corollary 4.7 for $\gamma \in \left(1, \min\left\{\frac{4}{3}, \frac{2\pi}{\alpha_0 c_{mp}}\right\}\right]$. On the other hand, when (f_{sc}) holds, by combining (2.6) with $r = p/2$ and Theorem 2.2 we get

$$\begin{aligned} \int_{\mathbb{R}^2} |F(u_n)|^\gamma &\lesssim \int_{\mathbb{R}^2} |u_n|^{\gamma p} + \int_{\mathbb{R}^2} |u_n|^{\gamma \frac{p}{2}} e^{\alpha|u_n|^2} \\ &\leq \|u_n\|_{\gamma p} + \|u_n\|_{\gamma \frac{p}{2}}^{\gamma \frac{p}{2}} \left(\int_{\mathbb{R}^2} |u_n|^{\gamma p} e^{2\alpha u_n^2} \right)^{\frac{1}{2}} \\ &\leq \|u_n\|_{\gamma p} + \|u_n\|_{\gamma \frac{p}{2}}^{\gamma \frac{p}{2}} \|\nabla u_n\|_2^{(\gamma-1)p} \|u_n\|_p^p \rightarrow 0, \end{aligned}$$

since the last two terms are bounded in n , and therefore again $I_2(u_n) \rightarrow 0$. Analogously, in both cases (f_c) -(f_{sc}), one may show that

$$(5.3) \quad A_2(F(u_n), f(u_n)u_n) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln\left(1 + \frac{b}{|x-y|}\right) F(u_n(x))f(u_n(y))u_n(y) dx dy \rightarrow 0$$

as $n \rightarrow +\infty$. Hence,

$$(5.4) \quad \begin{aligned} 2c_{mp} + o_n(1) &= 2\mathcal{I}(u_n) - \mathcal{I}'(u_n)[u_n] = I_0(u_n) - A_0(F(u_n), f(u_n)u_n) \\ &= 2I_1(u_n) - A_1(F(u_n), f(u_n)u_n) - 2I_2(u_n) + A_2(F(u_n), f(u_n)u_n) \\ &= \int_{\mathbb{R}^2} (\ln(b + |\cdot|) * F(u_n)) (F(u_n) - f(u_n)u_n) dx + o_n(1) < 0 \end{aligned}$$

for large n by (2.9), a contradiction. This implies that the vanishing (5.1) does not occur. Consequently, there exist $\delta > 0$ and a sequence $\{y_n\}_n \subset \mathbb{R}^2$ such that (up to a subsequence)

$$\int_{B_1(y_n)} |u_n|^p dx > \delta.$$

Defining $\tilde{u}_n := u_n(\cdot + y_n)$, it is easy to see that

$$(5.5) \quad \int_{B_1(0)} |\tilde{u}_n|^p dx > \delta,$$

and $\|\nabla \tilde{u}_n\|_2 + \|\tilde{u}_n\|_p = \|\nabla u_n\|_2 + \|u_n\|_p \leq C$. Therefore $\{\tilde{u}_n\}_n$ is a bounded sequence in $D^{1,2}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$. Moreover,

$$\|\tilde{u}_n\|_{*,p}^p = \int_{\mathbb{R}^2} \ln(b + |x - y_n|) |u_n(x)|^p dx \leq \|u_n\|_{*,p}^p + \ln(b + |y_n|) \|u_n\|_p^p$$

by (3.3), thus $\tilde{u}_n \in X$ for all $n \in \mathbb{N}$. Since $I_i(\tilde{u}_n) = I_i(u_n)$ for all $i \in \{0, 1, 2\}$, we also deduce that $\{\tilde{u}_n\}_n$ satisfies

$$(5.6) \quad \mathcal{I}(\tilde{u}_n) \rightarrow c_{mp}, \quad \mathcal{I}'(\tilde{u}_n)[\tilde{u}_n] \rightarrow 0, \quad \mathcal{J}(\tilde{u}_n) \rightarrow 0.$$

Note that second condition in (5.6) is weaker than the corresponding in Lemma 4.3. Furthermore, since $|I_0(\tilde{u}_n)| \leq C$ as in (5.4) and $I_2(\tilde{u}_n) \leq C$ as in (5.2), we conclude that $I_1(\tilde{u}_n) \leq C$. Applying Lemma 3.1(a) with $\varphi_n = F(\tilde{u}_n)$ we get

$$(5.7) \quad \|F(\tilde{u}_n)\|_{*,1} \leq C \quad \text{and thus} \quad \|\tilde{u}_n\|_{*,p} \leq C$$

by (2.8). Since $\{\tilde{u}_n\}_n$ is hence bounded in X , there exists $\tilde{u} \in X$ such that $\tilde{u}_n \rightharpoonup \tilde{u}$ in X and $\tilde{u}_n \rightarrow \tilde{u}$ in $L^q(\mathbb{R}^2)$ for all $q \geq p$, as well as a.e. in \mathbb{R}^2 by Lemma 2.1. By (5.5) it is easy to see that

$$(5.8) \quad \delta < \|\tilde{u}_n\|_{L^p(B_R(0))}^p \rightarrow \|\tilde{u}\|_{L^p(B_R(0))}^p,$$

hence $\tilde{u} \neq 0$. We next show that $\mathcal{I}'(\tilde{u}_n) \rightarrow 0$ in X' which, together with (5.6), makes \tilde{u}_n verify the same properties as u_n in Lemma 4.3. Indeed, $\mathcal{I}(\tilde{u}_n)[\varphi] = \mathcal{I}(u_n)[\varphi(\cdot - y_n)]$ for all $\varphi \in X$ and

$$(5.9) \quad \|\varphi(\cdot - y_n)\|^2 = \|\nabla \varphi\|_2^2 + \int_{\mathbb{R}^2} \ln(b + |x + y_n|) |\varphi(x)|^p dx \leq \|\varphi\|^2 + \ln(b + |y_n|) \|\varphi\|_p^p.$$

Moreover, if $|y_n| \leq b$, then $\ln(b + |y_n|) \leq \ln(2b)$, while if $|y_n| > b$ one can use (5.5) to show that

$$(5.10) \quad \begin{aligned} \|u_n\|_{*,p}^p &= \int_{\mathbb{R}^2} \ln(b + |x + y_n|) |\tilde{u}_n(x)|^p dx \\ &\geq \int_{B_1(0)} \ln(b + |x + y_n|) |\tilde{u}_n(x)|^p dx \geq \ln |y_n| \int_{B_1(0)} |\tilde{u}_n|^p \\ &\geq \delta \ln |y_n| \geq \delta \ln(b + |y_n|) \frac{\ln b}{\ln(2b)}. \end{aligned}$$

In the third step we used the simple inequality $|x + y_n| \geq |y_n| - |x_n| \geq |y_n| - 1 \geq |y_n| - b$, while the last step follows from $\frac{\ln |y_n|}{\ln(b + |y_n|)} > \frac{\ln b}{\ln(2b)}$, which holds by monotonicity. Eventually from (5.9)-(5.10) we get

$$\|\varphi(\cdot - y_n)\| \leq \left[\|\varphi\|^2 + \left(\frac{1}{\delta} \frac{\ln(2b)}{\ln b} \|u_n\|_{*,p}^p + \ln(2b) \right) \|\varphi\|_p^p \right]^{\frac{1}{2}},$$

which implies that

$$\begin{aligned} |\mathcal{I}'(\tilde{u}_n)[\varphi]| &\leq \|\mathcal{I}'(u_n)\|_{X'} \left[\|\varphi\|^2 + \ln(2b) \left(\frac{1}{\delta \ln b} \|u_n\|_{*,p}^p + 1 \right) \|\varphi\|_p^p \right]^{\frac{1}{2}} \\ &\leq C \|\mathcal{I}'(u_n)\|_{X'} \left[\|\varphi\|^2 + \|\varphi\|_p^p \right] \rightarrow 0, \end{aligned}$$

as $n \rightarrow +\infty$, by recalling that $\|u_n\|_{*,p}$ is uniformly bounded and that $\|\mathcal{I}'(u_n)\|_{X'} \rightarrow 0$ by Lemma 4.3. Since $\mathcal{I}'(\tilde{u}_n) \rightarrow 0$ in X' and $\mathcal{I}'(\tilde{u}_n)[\tilde{u}_n] \rightarrow 0$, we have then

$$(5.11) \quad o_n(1) = \mathcal{I}'(\tilde{u}_n)[\tilde{u}_n - \tilde{u}] = \int_{\mathbb{R}^2} \nabla \tilde{u}_n \nabla (\tilde{u}_n - \tilde{u}) + A_0(F(\tilde{u}_n), f(\tilde{u}_n)(\tilde{u}_n - \tilde{u})).$$

First,

$$(5.12) \quad \int_{\mathbb{R}^2} \nabla \tilde{u}_n \nabla (\tilde{u}_n - \tilde{u}) = \int_{\mathbb{R}^2} |\nabla (\tilde{u}_n - \tilde{u})|^2 + \int_{\mathbb{R}^2} \nabla \tilde{u} \nabla (\tilde{u}_n - \tilde{u}) = \|\tilde{u}_n - \tilde{u}\|_{D^{1,2}(\mathbb{R}^2)}^2 + o_n(1)$$

since $\tilde{u}_n \rightharpoonup \tilde{u}$ in $D^{1,2}(\mathbb{R}^2)$. Moreover, as in (5.2), one infers

$$(5.13) \quad A_2(F(\tilde{u}_n), f(\tilde{u}_n)(\tilde{u}_n - \tilde{u})) \lesssim \|F(\tilde{u}_n)\|_\gamma \|f(\tilde{u}_n)(\tilde{u}_n - \tilde{u})\|_\gamma \rightarrow 0.$$

Therefore, combining (5.11), (5.12), and (5.13) we infer

$$(5.14) \quad \begin{aligned} o_n(1) &= \|\tilde{u}_n - \tilde{u}\|_{D^{1,2}}^2 + A_1(F(\tilde{u}_n), f(\tilde{u}_n)(\tilde{u}_n - \tilde{u})) \\ &= \|\tilde{u}_n - \tilde{u}\|_{D^{1,2}}^2 + A_1(F(\tilde{u}_n), \tilde{f}(\tilde{u}_n - \tilde{u})(\tilde{u}_n - \tilde{u})) \\ &\quad + A_1\left(F(\tilde{u}_n), \left(\tilde{f}(\tilde{u}_n) - \tilde{f}(\tilde{u}_n - \tilde{u})\right)(\tilde{u}_n - \tilde{u})\right) \end{aligned}$$

as $n \rightarrow +\infty$, where we have defined

$$(5.15) \quad \tilde{f}(t) = \begin{cases} f(t) & \text{for } t \geq 0, \\ -f(-t) & \text{for } t < 0, \end{cases}$$

since $u_n \geq 0$. Note that if we prove that

$$(5.16) \quad A_1(F(\tilde{u}_n), \tilde{f}(\tilde{u}_n - \tilde{u})(\tilde{u}_n - \tilde{u})) \rightarrow 0,$$

one could apply Lemma 3.1(b) and obtain $\|\tilde{f}(\tilde{u}_n - \tilde{u})(\tilde{u}_n - \tilde{u})\|_{*,1} \rightarrow 0$, which in turn implies that $\|\tilde{u}_n - \tilde{u}\|_{*,p} \rightarrow 0$ as $n \rightarrow +\infty$ by (f_{sc}) or (f_c) . Together with $\tilde{u}_n \rightarrow \tilde{u}$ in $D^{1,2}(\mathbb{R}^2)$, this would conclude the proof. Since the first two terms in the right-hand side of (5.14) are positive, we are then lead to show that

$$(5.17) \quad A_1\left(F(\tilde{u}_n), \left(\tilde{f}(\tilde{u}_n) - \tilde{f}(\tilde{u}_n - \tilde{u})\right)(\tilde{u}_n - \tilde{u})\right) \rightarrow 0$$

as $n \rightarrow +\infty$. Using (3.3), we obtain

$$(5.18) \quad \begin{aligned} &\left| \int_{B_R(0)} (\ln(b + |\cdot|) * F(\tilde{u}_n)) \left[\left(\tilde{f}(\tilde{u}_n) - \tilde{f}(\tilde{u}_n - \tilde{u}) \right) (\tilde{u}_n - \tilde{u}) \right] dx \right| \\ &\leq \|F(\tilde{u}_n)\|_{*,1} \int_{\mathbb{R}^2} \left| \tilde{f}(\tilde{u}_n) - \tilde{f}(\tilde{u}_n - \tilde{u}) \right| |\tilde{u}_n - \tilde{u}| \\ &\quad + \|F(\tilde{u}_n)\|_1 \int_{\mathbb{R}^2} \ln(b + |x|) \left| \tilde{f}(\tilde{u}_n) - \tilde{f}(\tilde{u}_n - \tilde{u}) \right| |\tilde{u}_n - \tilde{u}| dx. \end{aligned}$$

First, note that $\|F(\tilde{u}_n)\|_1 \lesssim \|F(\tilde{u}_n)\|_{*,1} \leq C$ by (5.7). If (f_{sc}) holds, then by (2.6) with $r = p$, one infers

$$\begin{aligned}
 \int_{\mathbb{R}^2} f(\tilde{u}_n) |\tilde{u}_n - \tilde{u}| &\lesssim \int_{\mathbb{R}^2} |\tilde{u}_n|^{p-1} |\tilde{u}_n - \tilde{u}| + \int_{\mathbb{R}^2} |\tilde{u}_n|^{p-1} |\tilde{u}_n - \tilde{u}| e^{\alpha \tilde{u}_n^2} \\
 (5.19) \quad &\leq \|\tilde{u}_n\|_p^{p-1} \|\tilde{u}_n - \tilde{u}\|_p + \left(\int_{\mathbb{R}^2} |\tilde{u}_n|^p e^{\alpha \frac{p}{p-1} \tilde{u}_n^2} \right)^{\frac{p-1}{p}} \|\tilde{u}_n - \tilde{u}\|_p \\
 &\lesssim \|\tilde{u}_n\|_p^{p-1} \|\tilde{u}_n - \tilde{u}\|_p = o_n(1),
 \end{aligned}$$

by Theorem 2.2 with $\alpha < 4\pi \frac{p-1}{p} (\sup_n \|\nabla \tilde{u}_n\|_2)^{-2}$. By similar estimate, one may also show that $\int_{\mathbb{R}^2} f(\tilde{u}_n) |\tilde{u}_n - \tilde{u}| = o_n(1)$. On the other hand, if f is critical, namely (f_c) holds, defining now $\tilde{v}_n = G(\tilde{u}_n)$ with G as in (4.35) and arguing as in Lemma 4.6, one finds

$$\begin{aligned}
 \int_{\mathbb{R}^2} f(\tilde{u}_n) |\tilde{u}_n - \tilde{u}| &\lesssim \int_{\mathbb{R}^2} |\tilde{u}_n|^{p-1} |\tilde{u}_n - \tilde{u}| + \int_{\mathbb{R}^2} \left(t_\varepsilon + \frac{\tilde{v}_n}{1-\varepsilon} \right)^{r-1} |\tilde{u}_n - \tilde{u}| e^{\alpha \left(t_\varepsilon + \frac{\tilde{v}_n}{1-\varepsilon} \right)^2} \\
 (5.20) \quad &\lesssim \|\tilde{u}_n\|_p^{p-1} \|\tilde{u}_n - \tilde{u}\|_p + \left(\int_{\mathbb{R}^2} \left(t_\varepsilon + \frac{\tilde{v}_n}{1-\varepsilon} \right)^{(r-1)\sigma'} e^{\alpha \sigma' \left(t_\varepsilon + \frac{\tilde{v}_n}{1-\varepsilon} \right)^2} \right)^{\frac{1}{\sigma'}} \|\tilde{u}_n - \tilde{u}\|_\sigma \\
 &= o_n(1),
 \end{aligned}$$

where $\sigma' > 1$ is close to 1 and $\alpha > \alpha_0$ close to α_0 , so that the last term is bounded, and by using that $\tilde{u}_n \rightarrow \tilde{u}$ in $L^q(\mathbb{R}^2)$ for $q \geq p$. Moreover, by (2.7) and the ε -Young inequality,

$$\begin{aligned}
 \int_{\mathbb{R}^2} |\tilde{f}(\tilde{u}_n - \tilde{u})| |\tilde{u}_n - \tilde{u}| &\lesssim \int_{\mathbb{R}^2} |\tilde{u}_n - \tilde{u}|^p + \int_{\mathbb{R}^2} |\tilde{u}_n - \tilde{u}|^r e^{\alpha |\tilde{u}_n - \tilde{u}|^2} \\
 &\leq o_n(1) + \left(\int_{\mathbb{R}^2} |\tilde{u}_n - \tilde{u}|^r e^{\varepsilon \alpha s \tilde{u}_n^2} \right)^{\frac{1}{s}} \left(\int_{\mathbb{R}^2} |\tilde{u}_n - \tilde{u}|^r e^{C_\varepsilon \alpha s' \tilde{u}^2} \right)^{\frac{1}{s'}} \\
 &\leq o_n(1) + \left(C \int_{\{\tilde{u}_n \leq 1\}} |\tilde{u}_n - \tilde{u}|^r + \int_{\{\tilde{u}_n > 1\}} |\tilde{u}_n - \tilde{u}|^r |\tilde{u}_n|^\kappa e^{\varepsilon \alpha s \tilde{u}_n^2} \right)^{\frac{1}{s}} \\
 (5.21) \quad &\times \left(C \int_{\{\tilde{u} \leq 1\}} |\tilde{u}_n - \tilde{u}|^r + \int_{\{\tilde{u} > 1\}} |\tilde{u}_n - \tilde{u}|^r |\tilde{u}|^\kappa e^{C_\varepsilon \alpha s' \tilde{u}^2} \right)^{\frac{1}{s'}} \\
 &\leq o_n(1) + \left(o_n(1) + \|\tilde{u}_n - \tilde{u}\|_{rq'}^{r/s} \left(\int_{\mathbb{R}^2} |\tilde{u}_n|^{\kappa q} e^{\varepsilon \alpha s q \tilde{u}_n^2} \right)^{\frac{1}{sq}} \right) \\
 &\times \left(o_n(1) + \|\tilde{u}_n - \tilde{u}\|_{rq'}^{r/s'} \left(\int_{\mathbb{R}^2} |\tilde{u}|^{\kappa q} e^{C_\varepsilon \alpha s' \tilde{u}^2} \right)^{\frac{1}{s'q}} \right).
 \end{aligned}$$

Choosing now the parameters $\alpha > \alpha_0$ close to α_0 and $\kappa, r, q, s > 1$ so that $rq' > p$, $\kappa q > p$ and $\varepsilon > 0$ such that $\varepsilon \alpha_0 q s \|\nabla \tilde{u}_n^2\| < 4\pi$, which is possible since $\{\tilde{u}_n\}_n$ is bounded in $D^{1,2}(\mathbb{R}^2)$, then one may apply Theorem 2.2 and, combining (5.21) with (5.19) or (5.20), get

$$\begin{aligned}
 (5.22) \quad \int_{\mathbb{R}^2} |\tilde{f}(\tilde{u}_n) - \tilde{f}(\tilde{u}_n - \tilde{u})| |\tilde{u}_n - \tilde{u}| &\lesssim o_n(1) + \left(o_n(1) + o_n(1) C \|\tilde{u}_n\|_p^{p/sq} \right) \left(o_n(1) + o_n(1) C \|\tilde{u}\|_p^{p/sq} \right) = o_n(1),
 \end{aligned}$$

since $\{\tilde{u}_n\}_n$ is also bounded in $L^p(\mathbb{R}^2)$. It remains to prove that also the last term in (5.18) vanishes. To this aim, since $f \in C^1(\mathbb{R})$ and so is \tilde{f} (cf. (5.15)), by Lagrange's theorem,

$$(5.23) \quad \tilde{f}(\tilde{u}_n) - \tilde{f}(\tilde{u}_n - \tilde{u}) = \tilde{f}'(w_n)\tilde{u} = f(|w_n|)\tilde{u}, \quad \text{where } w_n := \theta_n \tilde{u}_n + (1 - \theta_n)\tilde{u}$$

with $\theta_n : \mathbb{R}^2 \rightarrow [0, 1]$. Therefore, splitting $\mathbb{R}^2 = B_R(0) \cup B_R(0)^c$ for a fixed $R > 0$, we obtain

$$(5.24) \quad \begin{aligned} & \int_{\mathbb{R}^2} \ln(b + |x|) |\tilde{f}(\tilde{u}_n) - \tilde{f}(\tilde{u}_n - \tilde{u})| |\tilde{u}_n - \tilde{u}| dx \\ & \leq \ln(b + R) \int_{B_R(0)} |\tilde{f}(\tilde{u}_n) - \tilde{f}(\tilde{u}_n - \tilde{u})| |\tilde{u}_n - \tilde{u}| + \int_{B_R(0)^c} \ln(b + |x|) |f'(|w_n|)| \tilde{u} |\tilde{u}_n - \tilde{u}| dx \\ & \leq o_n(1) + \left(\int_{\mathbb{R}^2} \ln(b + |x|) |f'(|w_n|)|^{\frac{p}{p-2}} dx \right)^{\frac{p-2}{p}} \left(\int_{B_R(0)^c} \ln(b + |x|) |\tilde{u}|^p dx \right)^{\frac{1}{p}} (\|\tilde{u}_n\|_{*,p} + \|\tilde{u}\|_{*,p}) \end{aligned}$$

by (5.22). Since $\|\tilde{u}\|_{*,p} < +\infty$, for any chosen $\delta > 0$ one may find $R > 0$ large enough such that (by Lebesgue's dominated convergence)

$$(5.25) \quad \int_{B_R(0)^c} \ln(b + |x|) |\tilde{u}|^p dx < \delta.$$

Recalling (5.7), we are then left to show that

$$(5.26) \quad \int_{\mathbb{R}^2} \ln(b + |x|) |f'(|w_n|)|^{\frac{p}{p-2}} dx \leq C$$

for all n . By exploiting (5.23), (f_5) , again the ε -Young inequality, (5.7), and $u \in X$, one may estimate it as

$$(5.27) \quad \begin{aligned} & \int_{\mathbb{R}^2} \ln(b + |x|) |f'(|w_n|)|^{\frac{p}{p-2}} dx \\ & \lesssim \int_{\mathbb{R}^2} \ln(b + |x|) (|\tilde{u}_n|^p + |\tilde{u}|^p) dx + \int_{\mathbb{R}^2} \ln(b + |x|) (\tilde{u}_n + \tilde{u})^{\frac{pr}{p-2}} e^{\alpha' \frac{p}{p-2} (\tilde{u}_n + \tilde{u})^2} dx \\ & \lesssim C + \left(\int_{\mathbb{R}^2} \ln(b + |x|) \tilde{u}_n^p e^{\varepsilon \alpha' \frac{pq}{p-2} \tilde{u}_n^2} dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} \ln(b + |x|) \tilde{u}_n^p e^{C_\varepsilon \alpha' \frac{pq'}{p-2} \tilde{u}^2} dx \right)^{\frac{1}{q'}} \\ & \quad + \left(\int_{\mathbb{R}^2} \ln(b + |x|) \tilde{u}^p e^{\varepsilon \alpha' \frac{pq}{p-2} \tilde{u}^2} dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} \ln(b + |x|) \tilde{u}^p e^{C_\varepsilon \alpha' \frac{pq'}{p-2} \tilde{u}^2} dx \right)^{\frac{1}{q'}} \end{aligned}$$

by choosing $r = p - 2 > 0$, and we estimate the four terms separately. The first term is easily bounded independently of n if one chooses ε small enough: in fact, since $\sup_{n \in \mathbb{N}} \|\tilde{u}_n\| \leq \overline{C}$, one has

$$(5.28) \quad \int_{\mathbb{R}^2} \ln(b + |x|) \tilde{u}_n^p e^{\varepsilon \alpha' \frac{pq}{p-2} \tilde{u}_n^2} dx = \int_{\mathbb{R}^2} \ln(b + |x|) \tilde{u}_n^p e^{\varepsilon \alpha' \frac{pq}{p-2} \overline{C} \left(\frac{\tilde{u}_n}{\|\tilde{u}_n\|} \right)^2} dx \leq C,$$

by choosing ε so small that $\varepsilon \alpha' \frac{pq}{p-2} \overline{C} \leq (4\pi p)^{-1/2}$ by Theorem 2.3. In a similar way we may prove that the last term in (5.27), which is independent of n , is finite. Indeed,

$$(5.29) \quad \int_{\mathbb{R}^2} \ln(b + |x|) \tilde{u}^p e^{C_\varepsilon \alpha' \frac{pq'}{p-2} \tilde{u}^2} dx = \int_{\mathbb{R}^2} \ln(b + |x|) \tilde{u}^p e^{C_\varepsilon \alpha' \frac{pq'}{p-2} \|\tilde{u}\|^2 \left(\frac{\tilde{u}}{\|\tilde{u}\|} \right)^2} dx < +\infty$$

for all $\varepsilon > 0$, again by Theorem 2.3. Concerning now the second term, which mixes \tilde{u}_n and \tilde{u} , one may estimate as follows:

$$\begin{aligned} \int_{\mathbb{R}^2} \ln(b + |x|) \tilde{u}_n^p e^{C_\varepsilon \alpha' \frac{pq'}{p-2} \tilde{u}^2} dx &\leq \int_{\{\tilde{u} \leq \tilde{u}_n\}} \ln(b + |x|) \tilde{u}_n^p e^{C_\varepsilon \alpha' \frac{pq'}{p-2} \tilde{u}_n^2} dx \\ &\quad + \int_{\{\tilde{u} > \tilde{u}_n\}} \ln(b + |x|) \tilde{u}^p e^{C_\varepsilon \alpha' \frac{pq'}{p-2} \tilde{u}^2} dx \leq C \end{aligned}$$

by (5.28)-(5.29). One may argue similarly for the third term in (5.27). As a result, we obtained (5.26) which, together with (5.24)-(5.25), (5.22), (5.18), show that (5.17) holds, yielding (5.16). Hence, by (5.14), one finally infers that $\tilde{u}_n \rightarrow \tilde{u}$ in X . Since \mathcal{I} is a C^1 -functional, then \tilde{u} is a weak solution of (1.1), which is nontrivial thanks to (5.8). Since $\tilde{u}_n \geq 0$, by Remark 1.6 the solution \tilde{u} is positive in \mathbb{R}^2 .

6. BACK TO THE SYSTEM: PROOF OF THEOREM 1.5

Let $u \in X$ be the weak solution of the Choquard equation (1.5) given by Theorem 1.2 and define

$$\Phi_u(x) := \int_{\mathbb{R}^2} \ln\left(\frac{1}{|x-y|}\right) F(u(y)) dy.$$

Following the approach of [7], we aim at proving that Φ_u is a solution of the system (1.1) in the sense of Definition 1.4. First, we show that $\Phi_u \in L_s(\mathbb{R}^2)$, for all $s > 0$:

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{|\Phi_u(x)|}{1 + |x|^{2+2s}} dx &\leq \int_{\mathbb{R}^2} F(u(y)) \left(\int_{\mathbb{R}^2} \left| \ln \frac{1}{|x-y|} \right| \frac{1}{1 + |x|^{2+2s}} dx \right) dy \\ &\leq \int_{\mathbb{R}^2} F(u(y)) \left(\int_{\{|x-y|>1\}} \frac{\ln|x-y|}{1 + |x|^{2+2s}} dx + \int_{\{|x-y|\leq 1\}} \ln \frac{1}{|x-y|} dx \right) dy \\ &\leq \|F(u(y))\|_1 \int_{\mathbb{R}^2} \frac{\ln(1 + |x|)}{1 + |x|^{2+2s}} dx + \|F(u(y))\|_{*,1} \int_{\mathbb{R}^2} \frac{dx}{1 + |x|^{2+2s}} \\ &\quad + \|\ln(\cdot)\|_{L^1(B_1(0))} \int_{\mathbb{R}^2} F(u(y)) dy < +\infty \end{aligned}$$

for all $s > 0$, using (3.3) with $b = 1$, and Theorem 2.3 since $u \in X$. Define now the function

$$\tilde{w}_u(x) := \int_{\mathbb{R}^2} \ln\left(\frac{1 + |y|}{|x-y|}\right) F(u(y)) dy,$$

which we know by [29, Lemma 2.3] to be a solution in the sense of Definition 1.3 of $-\Delta \tilde{w}_u = \mathfrak{f}$ in \mathbb{R}^2 , where $\mathfrak{f} := F(u) \in L^1(\mathbb{R}^2)$, and compute the difference

$$\begin{aligned} \tilde{w}_u(x) - \Phi_u(x) &= \int_{\mathbb{R}^2} \left(\ln\left(\frac{1 + |y|}{|x-y|}\right) - \ln\left(\frac{1}{|x-y|}\right) \right) F(u(y)) dy \\ &= \int_{\mathbb{R}^2} \ln(1 + |y|) F(u(y)) dy < \|F(u)\|_{*,1} < +\infty, \end{aligned}$$

that is constant. This implies that Φ_u is a solution of (1.1) in the sense of Definition 1.4, by applying [29, Lemma 2.4], for which all such solutions of $-\Delta \Phi = \mathfrak{f}$ in \mathbb{R}^2 are of the form $\Phi = \tilde{w}_u + p$ with p polynomial of degree at most 1.

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