CLASSIFICATION OF TWO AND THREE DIMENSIONAL COMPLETE GRADIENT YAMABE SOLITONS

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Abstract. In this paper, we completely classify nontrivial nonflat three dimensional complete shrinking and steady gradient Yamabe solitons without any assumptions. We also give examples of complete expanding gradient Yamabe solitons. Furthermore, we give a proof of the classification of nontrivial two dimensional complete gradient Yamabe solitons without any assumptions.

1. Introduction

As is well known, the geometric flow is one of the most powerful tools for understanding the structure of Riemannian manifolds. In particular, the Yamabe flow is one of the central fields of the theory and has developed rapidly (cf. [2], [3]). Gradient Yamabe solitons are self similar solutions of the Yamabe flow and expected to be a singularity model. Therefore, the classification problem is one of the most important ones. To classify gradient Yamabe solitons, there are many studies with curvature assumptions and locally conformally flat conditions (cf. [10], [8], [6] and [12]). These studies give affirmative partial answers to the Yamabe soliton version of the Perelman conjecture, that is, any nontrivial complete steady gradient Yamabe soliton is rotationally symmetric. Recently, the author solved the problem (cf. [13]).

In order to gain a deeper understanding of gradient Yamabe solitons, we consider complete gradient Yamabe solitons without assuming any curvature assumptions. In this paper, we completely classify nontrivial nonflat three dimensional complete steady and shrinking gradient Yamabe solitons **without any assumptions**. In particular, we show that positive scalar curvature is necessary for the Yamabe soliton version of the Perelman conjecture. We also give some examples of the complete expanding gradient Yamabe solitons. Furthermore, we also

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give a proof of the classification of nontrivial two dimensional complete gradient Yamabe solitons. For two dimensional solitons, similar results were given by Bernstein and Mettler (cf. [1]) using different arguments.

Remark 1.1. *As is well known, the original Perelman conjecture* [14] *is that any three dimensional complete noncompact κ-noncollapsed steady gradient Ricci soliton with positive curvature is rotationally symmetric, which was proven by S. Brendle* [4] *(see also* [5]*).*

2. Preliminary

An n dimensional Riemannian manifold (M^n, g) is called a gradient Yamabe soliton if there exists a smooth function F on M and a constant $\lambda \in \mathbb{R}$, such that $\nabla\nabla F = (R - \lambda)g$, where $\nabla\nabla F$ is the Hessian of F, and R is the scalar curvature of M. If F is constant, then M is called trivial. If $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, then the Yamabe soliton is called shrinking, steady, or expanding, respectively.

To classify complete gradient Yamabe solitons, we use Tashiro's theorem ([16], see also [12]).

Theorem 2.1 ([16])**.** *A complete Riemannian manifold (M^n, g) which satisfies that for any smooth functions F and φ on M, $\nabla\nabla F = \varphi g$ is either* (1) *compact and rotationally symmetric, or* (2) *rotationally symmetric and equal to the warped product $([0,\infty), dr^2) \times_{|\nabla F|} (\mathbb{S}^{n-1}, \bar{g}_S)$, where $\bar{g}_S$ is the round metric on $\mathbb{S}^{n-1}$, and F has one critical point at* 0*, or* (3) *the warped product $(\mathbb{R}, dr^2) \times_{|\nabla F|} (N^{n-1}, \bar{g})$, where F has no critical point.*

Remark 2.2. *The potential function F depends only on r, and $F'(r) > 0$, except at the critical point. It is well known that the manifold (M, g, F, φ) that satisfies the condition $\nabla\nabla F = \varphi g$ was studied by Cheeger and Colding* [9]*.*

The Riemannian curvature tensor is definded by

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z,$$

for $X, Y, Z \in \mathfrak{X}(M)$, where ∇ is the Levi-Civita connection of M. The Ricci tensor R_{ij} is defined by $R_{ij} = R_{ipjp}$, where, $R_{ijk\ell} = g(R(\partial_i, \partial_j)\partial_k, \partial_\ell)$.

3. Classification of three dimensional complete steady gradient Yamabe solitons

In this section, we classify three dimensional complete steady gradient Yamabe solitons without any assumptions. In particular, the assumption that the scalar curvature is positive is essential for the Yamabe soliton version of the Perelman conjecture.

Theorem 3.1. *Let (M, g, F) be a nontrivial nonflat three dimensional complete steady gradient Yamabe soliton. Then, (M, g, F) is either*

(1) $([0,\infty), dr^2) \times_{|\nabla F|} (\mathbb{S}^2, \bar{g}_S)$ *with nonnegative scalar curvature, or*
(2) $(\mathbb{R}, dr^2) \times_{|\nabla F|} (\mathbb{R}^2, \bar{g}_{can})$ *with negative scalar curvature, or*
(3) $(\mathbb{R}, dr^2) \times_{|\nabla F|} (\mathbb{S}^2, \bar{g}_S)$. *In this case, there exists at least one point, such that, $R = 0$.*

Proof.

Case (1) of Theorem 2.1. Since M is compact, it is trivial (cf. [11]).

Case (3) of Theorem 2.1.

In this case, by a direct calculation, we can get formulas of the warped product manifold of the warping function $(0 <)|\nabla F| = F'(r)$. For $a, b, c, d = 2, 3$,

$$R_{1a1b} = -F'F'''\bar{g}_{ab}, \quad R_{1abc} = 0, \tag{3.1}$$
$$R_{abcd} = (F')^2\bar{R}_{abcd} + (F'F'')^2(\bar{g}_{ad}\bar{g}_{bc} - \bar{g}_{ac}\bar{g}_{bd}),$$

$$R_{11} = -2\frac{F'''}{F'}, \quad R_{1a} = 0, \tag{3.2}$$
$$R_{ab} = \bar{R}_{ab} - ((F'')^2 + F'F''')\bar{g}_{ab},$$

$$R = (F')^{-2}\bar{R} - 2\Big(\frac{F''}{F'}\Big)^2 - 4\frac{F'''}{F'}, \tag{3.3}$$

where the curvature tensors with bar are the curvature tensors of $(N, \bar{g})$. By (3.3), it is obvious that $\bar{R}$ is constant. Since $(N^2, \overline{g})$ is a 2 dimensional manifold,

$$\overline{R}_{abcd} = -\frac{\bar{R}}{2}(\bar{g}_{ad}\bar{g}_{bc} - \bar{g}_{ac}\bar{g}_{bd}). \tag{3.4}$$

By (3.3), one has

$$\rho^2\rho' + 2\rho'^2 + 4\rho\rho'' = \bar{R}, \tag{3.5}$$

where $\rho(r) = F'(r)$. By differentiating both sides of (3.5), we have

$$2\rho\rho'^2 + \rho^2\rho'' + 8\rho'\rho'' + 4\rho\rho''' = 0. \tag{3.6}$$

Assume that $R(= \rho') \geq 0$. If $\rho'' \leq 0$, then the positive function ρ is concave. Hence ρ is constant. By (3.5), $\bar{R} = 0$. By (3.4) and (3.1), M is flat. Therefore, $\rho'' > 0$ on some interval (r_1, r_2).

If $r_2 = +\infty$, since $\rho' \geq 0$, ρ goes to infinity as $r \nearrow +\infty$. Hence, the left hand side of (3.5) goes to infinity as $r \nearrow +\infty$, which cannot happen.

Assume that $r_2 < +\infty$, that is, $\rho'' > 0$ on (r_1, r_2) and $\rho'' = 0$ at r_2. By (3.6), we have $\rho''' \leq 0$ at r_2. Hence, one has $\rho'' \leq 0$ on (r_2, r_3) for some r_3. By the same argument, ρ' is monotone decreasing on $(r_2, +\infty)$. Without loss of generality, we can assume that the point r_2 is the first critical one of ρ'. Since $\rho' \geq 0$ and $\rho'' > 0$ on $(-\infty, r_2)$, there exists r_0 such that, $\rho''' = 0$. However, this is contradicted by the equation (3.6).

Therefore, we have $\rho' < 0$ on some open interval (r_1, r_2).

Assume that $r_2 = +\infty$. Since $\rho > 0$, one has $\rho' \nearrow 0$ and $\rho'' \searrow 0$ as $r \nearrow +\infty$. By (3.5), $\bar{R}$ must be 0. If $\rho' = 0$ at r_1, by (3.5), $\rho'' = 0$ at r_1. We also have $\rho''' = 0$ at r_1 by (3.6). Iterating the same argument shows that $\rho^{(k)} = 0$ at r_1 for any $k \geq 1$. Taylor expansion shows that ρ is constant, hence M is flat. Therefore, we have $R = \rho' < 0$ on $\mathbb{R}$.

Assume that $r_2 < +\infty$, that is, $\rho' = 0$ at r_2.
Case 1 $\bar{R} > 0$. In this case,

$$\rho'' = \frac{\bar{R}}{4\rho} > 0,$$

at r_2, hence $\rho' > 0$ on (r_2, r_3), for some r_3. If $\rho' = 0$ at r_3, then $\rho'' > 0$ at r_3. Hence, ρ is monotone increasing on $(r_2, +\infty)$. If $\rho' = 0$ at r_1, then $\rho'' > 0$ at r_1, which cannot happen. Therefore, $\rho' < 0$ on $(-\infty, r_2)$.
Case 2 $\bar{R} = 0$. In this case, $\rho'' = 0$ at r_2. By (3.6), one has $\rho''' = 0$ at r_2. Iterating the same argument shows that $\rho^{(k)}(r_2) = 0$ for $k \geq 1$. By Taylor expansion, we have $\rho(r) = \rho(r_2)$, that is, ρ is constant and it is flat.
Case 3 $\bar{R} < 0$. In this case, $\rho'' < 0$ at r_2. Hence, $\rho' < 0$ on (r_1, r_3) for some r_3. If $r_3 = +\infty$, by the same argument as above, one has $\bar{R} = 0$, which is a contradiction. Hence $r_3 < +\infty$. Iterating the same argument shows that the positive function ρ is monotone decreasing on $(r_1, +\infty)$, which cannot happen.

Case (2) of Theorem 2.1. If $\rho' \leq 0$ on $[0, +\infty)$, since $\rho > 0$, $\rho' \nearrow 0$ and $\rho'' \searrow 0$ as $r \nearrow +\infty$. By (3.5), we have a contradiction. Since $\rho > 0$ on $(0, +\infty)$, we only have to consider that $\rho' > 0$ on an interval $(0, r_1)$ for some r_1. If $\rho' = 0$ at r_1, then we have $\rho'' > 0$ at r_1. Hence one has $\rho' > 0$ on (r_1, r_2) for some r_2. Iterating the same argument shows that ρ is monotone increasing on $(0, +\infty)$.

$\square$

4. Classification of three dimensional complete shrinking gradient Yamabe solitons

The author showed that any nontrivial nonflat complete shrinking gradient Yamabe solitons with $R > \lambda$ is rotationally symmetric and gave some examples with $R = \lambda$ for higher dimensional manifolds (cf. [13]). In this section, we completely classify nontrivial nonflat three dimensional complete shrinking gradient Yamabe solitons without any assumptions. In particular, we show that nontrivial nonflat three dimensional complete shrinking gradient Yamabe solitons are rotationally symmetric.

Theorem 4.1. *Let (M, g, F, λ) be a nontrivial nonflat three dimensional complete shrinking gradient Yamabe soliton. Then, (M, g, F, λ) is either*

(1) $([0, \infty), dr^2) \times_{|\nabla F|} (\mathbb{S}^2, \bar{g}_S)$, *or*
(2) $(\mathbb{R}, dr^2) \times_{|\nabla F|} (\mathbb{S}^2, \bar{g}_S)$.

Proof. Case (1) of Theorem 2.1. Since M is compact, it is trivial (cf. [11]).
Case (3) of Theorem 2.1.

By (3.3), one has

$$\rho^2\rho' + \lambda\rho^2 + 2\rho'^2 + 4\rho\rho'' - \bar{R} = 0. \tag{4.1}$$

Assume that $R \geq 0$, that is, $\rho' \geq -\lambda$. By (4.1), we have

$$\rho'' \leq \frac{\bar{R} - 2\rho'^2}{4\rho}.$$

If $\bar{R} \leq 0$, then the positive function ρ is concave. Hence, ρ is constant. By (4.1), we have $\bar{R} = \lambda\rho^2 > 0$, which is a contradiction. Therefore, $\bar{R}$ is positive.

We consider that $\rho' < -\lambda$ on some interval (r_1, r_2). If $r_2 = +\infty$, then $\rho = 0$ at some point, which cannot happen because $\rho > 0$. Hence, $r_2 < +\infty$, that is, $\rho' = -\lambda$ at r_2. By (4.1), we have $\rho'' = \frac{\bar{R} - 2\lambda^2}{4\rho}$ at r_2.

If $\bar{R} \leq 0$, then $\rho'' \leq 0$. Hence, $\rho' \leq -\lambda$ on (r_1, r_3) for some r_3. Iterating the same argument shows that $\rho' \leq -\lambda$ on $(r_1, +\infty)$, which cannot happen. Therefore, $\bar{R}$ is positive.

□

5. Classification of two dimensional complete gradient Yamabe solitons

In this section, we completely classify nontrivial two dimensional complete gradient Yamabe solitons.

Theorem 5.1. *Let (M^2, g, F, λ) be a nontrivial two dimensional complete gradient Yamabe soliton. Then (M^2, g, F, λ) is one of the following.*

Steady case:

Hamilton's cigar soliton.

Shrinking case:

There exists no complete shrinking gradient Yamabe solitons.

Expanding case:

(1) $(\mathbb{R}\times N^1, g = dr^2 + F'(r)^2 d\theta^2, F, \lambda)$. *The Gaussian curvature K of $\mathbb{R}\times N^1$ satisfies $\frac{\lambda}{2} < K < 0$, and is flat near infinity.*
(2) $(\mathbb{R}\times N^1, g = dr^2 + F'(r)^2 d\theta^2, F, \lambda)$. *The Gaussian curvature K of $\mathbb{R}\times N^1$ satisfies $K < \frac{\lambda}{2}$, and is hyperbolic such that $K = \frac{\lambda}{2}$ near infinity.*
(3) $([0, +\infty)\times \mathbb{S}^1, g = dr^2 + F'(r)^2 d\theta^2, F, \lambda)$. *The Gaussian curvature K of $[0, +\infty)\times \mathbb{S}^1$ satisfies $\frac{\lambda}{2} < K < 0$, and is flat near infinity.*
(4) $([0, +\infty)\times \mathbb{S}^1, g = dr^2 + F'(r)^2 d\theta^2, F, \lambda)$. *The Gaussian curvature is positive, and flat near infinity.*
(5) $([0, +\infty)\times \mathbb{S}^1, g = dr^2 + F'(r)^2 d\theta^2, F)$. *The Gaussian curvature K of $[0, +\infty)\times \mathbb{S}^1$ satisfies $K < \frac{\lambda}{2}$, and is hyperbolic such that $K = \frac{\lambda}{2}$ near infinity.*
(6) $([0, +\infty)\times \mathbb{S}^1, g = dr^2 + (\lambda r)^2 d\theta^2, -\frac{\lambda}{2}r^2 + C)$ *for some $C \in \mathbb{R}$ with flat Gaussian curvature.*

Proof. It is well known that if M is compact, then it is trivial (cf. [11]). Therefore, we only have to consider (2) and (3) of Theorem 2.1. By the soliton equation, one has

$$F''(r) = R - \lambda.$$

By an elementary fact of the curvature of warped product, one has

$$R = -2\frac{F'''}{F'}.$$

Combining these equations, we have

$$2\rho'' + \rho\rho' + \lambda\rho = 0, \tag{5.1}$$

where, we denote that $\rho(r) := F'(r)$. By differentiating both sides of (5.1), we have

$$2\rho''' + \rho'^2 + \rho\rho'' + \lambda\rho' = 0. \tag{5.2}$$

By differentiating again one has

$$2\rho^{(4)} + 3\rho'\rho'' + \rho\rho''' + \lambda\rho'' = 0. \tag{5.3}$$

The equation (5.1) is an autonomous second order equation and can be made into a first order equation by using ρ as a new independent variable. If $\rho' = G(\rho)$, then $\rho'' = \dot{G}G$, and one has

$$2\dot{G}G + G\rho + \lambda\rho = 0. \tag{5.4}$$

Case (3) of Theorem 2.1.

Case 1 $\lambda = 0$.

In this case, it is easy to see that the solution of (5.1) is

$$\rho(r) = 2c_1^2 \tanh\left\{\frac{1}{2}(c_1^2 r + c_1^2 c_2)\right\},$$

where c_1 and c_2 are constants. However, it contradicts $\rho(r) > 0$ on $\mathbb{R}$.

Case 2 $\lambda > 0$.

Assume that $\rho' = 0$ at some point, that is $G = 0$ at some point $\rho_0 (> 0)$, by (5.4), we have

$$\lambda\rho_0 = 0,$$

which is a contradiction. Therefore, we have $\rho' > 0$ or $\rho' < 0$ on $\mathbb{R}$.

(I) If $\rho' > 0$, by (5.1), one has $\rho'' < 0$. Therefore, ρ is a positive concave function, which cannot happen.

(II) Assume that $\rho' < 0$. If $\rho'' = 0$ at some point r_0, then by (5.1), one has $\rho'(r_0) = -\lambda$. By (5.2), we have $\rho'''(r_0) = 0$. By (5.3), one has $\rho^{(4)}(r_0) = 0$. Iterating the same argument shows that $\rho^{(k)}(r_0) = 0$ for $k \geq 2$. By Taylor expansion, we have

$$\rho(r) = -\lambda r + \lambda r_0 + \rho(r_0),$$

on $\mathbb{R}$. This contradicts $\rho(r) > 0$. Hence we have $\rho'' > 0$ or $\rho'' < 0$ on $\mathbb{R}$.

(II)-(a) Assume that $\rho'' > 0$. By (5.1), we have $\rho' < -\lambda$, which cannot happen, because $\rho > 0$.

(II)-(b) If $\rho'' < 0$, then ρ is a positive concave function, which cannot happen.

Case 3 $\lambda < 0$.

Assume that $\rho' = 0$ at some point. By (5.4), we have a contradiction. Therefore, we have $\rho' > 0$ or $\rho' < 0$ on $\mathbb{R}$.

(I) Assume that $\rho' > 0$. If $\rho'' = 0$ at some point r_1, the same argument as in (II) of Case 2 deduces a contradiction. Hence we have $\rho'' > 0$ or $\rho'' < 0$ on $\mathbb{R}$.

(I)-(a) Assume that $\rho'' > 0$. By (5.1), we have $\rho' < -\lambda$. Therefore, we have $\lambda < R < 0$. If $\rho' \nearrow c^2(< -\lambda)$ as $r \nearrow +\infty$, then one has $\rho'' \searrow 0$ as $r \nearrow +\infty$. Since ρ is positive and monotone increasing, by (5.1), we have a contradiction. Hence, we obtain that $\rho' \nearrow -\lambda$ as $r \nearrow +\infty$. Therefore, the curvature is flat near infinity.

(I)-(b) Assume that $\rho'' < 0$. Since ρ is a positive concave function, it cannot happen.

(II) Assume that $\rho' < 0$. By (5.1), $\rho'' > 0$. Since ρ is positive, $\rho' \searrow 0$ as $r \nearrow +\infty$. Therefore, $R < \lambda$ and it is hyperbolic such that $R = \lambda$ near infinity.

Case (2) of Theorem 2.1.

Note that $\rho(0) = 0$ and $\rho > 0$ on $(0, +\infty)$.

Case 1 $\lambda = 0$**.**
In this case, it is easy to see that the solution of (5.1) is

$$\rho(r) = 2c_1^2 \tanh\left\{\frac{1}{2}(c_1^2 r + c_1^2 c_2)\right\},$$

where c_1 and c_2 are constants. Since $\rho(0) = 0$, one has

$$c_1 = 0 \text{ or } c_2 = 0.$$

If $c_1 = 0$, then one has $\rho(r) \equiv 0$, which is a contradiction.
If $c_2 = 0$, then one has

$$F(r) = 4\log\left(\cosh\left(\frac{c_1^2}{2} r\right)\right).$$

It is Hamilton's cigar soliton.

Case 2 $\lambda > 0$**.**

Assume that $\rho' = 0$ at some point, that is $G = 0$ at some point, then by (5.4), we have $\rho = 0$, which is a contradiction. Therefore, we have $\rho' > 0$ or $\rho' < 0$ on $(0, +\infty)$.

(I) Assume that $\rho' > 0$ on $(0, +\infty)$. By (5.1), one has $\rho'' < 0$ on $(0, +\infty)$. If ρ is bounded from above, then one has $\rho' \searrow 0$ and $\rho'' \nearrow 0$ as $r \nearrow +\infty$, which contradicts (5.1). Therefore, $\rho \nearrow +\infty$ as $r \nearrow +\infty$. Since ρ' is positive and monotone decreasing, $\rho'' \searrow 0$ as $r \nearrow +\infty$. By (5.1), we have a contradiction.

(II) Assume that $\rho' < 0$. If $\rho'' = 0$ at some point r_0, then by (5.1), one has $\rho'(r_0) = -\lambda$. By (5.2), we have $\rho'''(r_0) = 0$. By (5.3), one has $\rho^{(4)}(r_0) = 0$. Iterating the same argument shows that $\rho^{(k)}(r_0) = 0$ for $k \geq 2$. By Taylor expansion, we have

$$\rho(r) = -\lambda r + \lambda r_0 + \rho(r_0),$$

on $(0, +\infty)$. This contradicts $\rho(r) > 0$. Hence we have $\rho'' > 0$ or $\rho'' < 0$ on $(0, +\infty)$.

(II)-(a) Assume that $\rho'' > 0$. By (5.1), we have $\rho' < -\lambda$, which cannot happen, because $\rho > 0$.

(II)-(b) If $\rho'' < 0$, then ρ is a positive monotone decreasing concave function, which cannot happen.

Case 3 $\lambda < 0$.
Assume that $\rho' = 0$ at some point, that is $G = 0$ at some point, then by (5.4), we have $\rho = 0$, which is a contradiction. Therefore, we have $\rho' > 0$ or $\rho' < 0$ on $(0, +\infty)$.

(I) Assume that $\rho' > 0$. If $\rho'' = 0$ at some point. By the same argument as in (II) of Case 2, we have $\rho(r) = -\lambda r$. Hence, one has $R = 0$ and $F(r) = -\frac{\lambda}{2}r^2 + C$, for some C. Therefore, we only have to consider $\rho'' > 0$ or $\rho'' < 0$ on $(0, +\infty)$.

(I)-(a) Assume that $\rho'' > 0$. By (5.1), we have $\rho' < -\lambda$. Therefore, we have $\lambda < R < 0$. If $\rho' \nearrow c^2 (< -\lambda)$ as $r \nearrow +\infty$, then one has $\rho'' \searrow 0$ as $r \nearrow +\infty$. Since ρ is positive and monotone increasing, by (5.1), we get a contradiction. Hence, we obtain that $\rho' \nearrow -\lambda$ as $r \nearrow +\infty$. Therefore, the curvature is flat near infinity.

(I)-(b) Assume that $\rho'' < 0$. By (5.1), we have $\rho' > -\lambda$, that is, $R > 0$. Note that $\rho(0) = 0$. By integrating both sides of $\rho' > -\lambda$, one has $-\lambda r < \rho(r)$. Therefore, $\rho \nearrow +\infty$ as $r \nearrow +\infty$. Assume that $\rho' \searrow c^2 (> -\lambda)$ as $r \nearrow +\infty$. Since $\rho'' \searrow 0$ as $r \nearrow +\infty$. By (5.1), we have a contradiction. Therefore, we have $\rho' \searrow -\lambda$ as $r \nearrow +\infty$, and the curvature is flat near infinity.

(II) Assume that $\rho' < 0$. By (5.1), $\rho'' > 0$ on $(0, +\infty)$. Since ρ is positive, $\rho' \searrow 0$ as $r \nearrow +\infty$. Therefore, $R < \lambda$ and it is hyperbolic such that $R = \lambda$ near infinity.

$\square$

Remark 5.2. *For two dimensional solitons, Theorem 5.1 solves many problems in* [7].

6. Appendix

For n dimensional complete expanding gradient Yamabe solitons, an elementary argument provides interesting examples that don't appear in the steady and shrinking cases.

Example. Let $(N^{n-1}, \bar{g})$ be an $(n-1)$ dimensional complete Riemannian manifold with constant negative scalar curvature $\bar{R}$. Then, for any $\alpha \in \mathbb{R}$, $(M, g, F, \lambda) = (\mathbb{R} \times N^{n-1}, dr^2 + \frac{\bar{R}}{\lambda}\bar{g}, \sqrt{\frac{\bar{R}}{\lambda}}r + \alpha, \lambda)$ is an n dimensional complete expanding gradient Yamabe soliton with $R = \lambda$.

In particular, $(M^3, g, F, \lambda) = (\mathbb{R} \times \mathbb{H}^2, dr^2 + \frac{\bar{R}}{\lambda}g_{\mathbb{H}}, \sqrt{\frac{\bar{R}}{\lambda}}r + \alpha, \lambda)$ is a 3 dimensional complete expanding gradient Yamabe soliton with $R = \lambda$, where $(\mathbb{H}^2, g_{\mathbb{H}})$ be a hyperbolic surface.

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