INHOMOGENEOUS WAVE KINETIC EQUATION AND ITS HIERARCHY IN POLYNOMIALLY WEIGHTED L^∞ SPACES

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Abstract. Inspired by ideas stemming from the analysis of the Boltzmann equation, in this paper we expand well-posedness theory of the spatially inhomogeneous 4-wave kinetic equation, and also analyze an infinite hierarchy of PDE associated with this nonlinear equation. More precisely, we show global in time well-posedness of the spatially inhomogeneous 4-wave kinetic equation for polynomially decaying initial data. For the associated infinite hierarchy, we construct global in time solutions using the solutions of the wave kinetic equation and the Hewitt-Savage theorem. Uniqueness of these solutions is proved by using a combinatorial board game argument tailored to this context, which allows us to control the factorial growth of the Dyson series.

Contents

1. Introduction

The goal of this paper is to establish global well-posedness results for certain partial differential equations that appear in the context of wave turbulence, so we start by briefly reviewing some mathematical results pertaining to wave turbulence.

1.1. **A framework of wave turbulence.** When faced with a dynamical system containing a large number of nonlinear interacting waves, instead of considering individual trajectories it is often helpful to study ensembles. Such a statistical mechanic treatment of the dynamics is what is usually referred to as wave turbulence, the main topic of which is focused on rigorously deriving and analyzing an effective equation for the non-equilibrium dynamics of the relevant microscopic system. While appearance of a wave kinetic equation goes back to works of Peierls in 1920 [41] and Hasselmann in early 1960s [27, 28], revived interest was inspired, in part, by the influential work [51] of Zakharov, L'vov and Falkovich in 1992 that uncovered a power-law type stationary solutions analogous to the Kolmogorov spectra of hydrodynamic turbulence. For more details, see for example Nazarenko [39] and Newell-Rumpf [40].

The attention of the mathematical community in wave turbulence has largely focused, so far, on the derivation of wave kinetic equations starting from dynamics governed by nonlinear dispersive equations. In a pioneering work [37], Lukkarinen and Spohn obtained a rigorous derivation of a linearized spatially homogeneous wave kinetic equation from a cubic nonlinear Scrödinger equation (NLS) at statistical equilibrium, by employing Feynmann diagrams expansions. For 3-wave systems, linearized wave turbulence convergence results were also obtained by Faou [22]. Regarding the out of the equilibrium case, starting from a cubic NLS with random data out of equilibrium, emergence of the 4-wave spatially homogeneous kinetic equation

$$\partial_t f = \mathcal{C}[f], \tag{1.1}$$

with the collision operator defined as in (1.8), has been studied in a sequence of papers [8, 13, 14, 15] that led to the works of Deng-Hani [16, 17], who obtained a derivation up to the kinetic time[1] and their recent work [18] where the derivation is obtained as long as the nonlinear wave kinetic equation (1.1) is well-posed. We would also like to note that starting from a stochastic Zakharov-Kuznetsov equation (which is a multidimensional generalization of Korteveg-de-Vries equation) with multiplicative noise, Staffilani-Tran [47] derived the spatially homogeneous 3-wave kinetic equation up to the kinetic time.

The spatially *inhomogeneous* wave kinetic equation

$$\partial_t f + v \cdot \nabla_x f = \mathcal{C}[f], \tag{1.2}$$

with operator $\mathcal{C}$ describing interaction of waves, appears in the physics literature [50, 51]. This type of equation is used for modeling ocean waves [44]. Moreover, Spohn [46] discusses the emergence of an inhomogeneous phonon Boltzmann equation and addresses its connection to nonlinear waves. The first rigorous derivation result regarding the spatially inhomogeneous wave kinetic equation was obtained by the first author of this paper, Collot and Germain [5], who derived a 3-wave kinetic equation from quadratic Scrödinger-type nonlinearities. On the other hand, starting from a stochastic Zakharov-Kuznetsov equation with multiplicative noise, Hannani-Rosenzweig-Staffilani-Tran [26] derived a spatially inhomogeneous 3-wave kinetic equation up to the kinetic time. Recently, Hani-Shatah-Zhu [29] derived several inhomogeneous and homogeneous wave kinetic equations from the simplified model of the Wick NLS whose main feature is the absence of all self-interactions in the correlation expansions of its solutions.

Despite exciting activity on the rigorous derivation of wave kinetic equations, to the best of our knowledge, the analysis of wave kinetic equations themselves has been carried out only in some instances. For example, we note that Escobedo-Velázquez [21] constructed solutions to the spatially homogeneous bosonic Nordheim equation that exhibit blow up in finite time. Germain-Ionescu-Tran [25] proved local well-posedness of the spatially homogeneous 4-wave kinetic equations in L^2_v and L^∞_v based spaces. Moreover, Menegaki [38] showed L^2-stability near equilibrium, and Collot-Dietert-Germain [12] showed stability and cascades of the Kolomogorov-Zakharov spectra, all in the cases of spatially homogeneous equations. On the other hand, for the spatially inhomogeneous equation, we are aware only of the work the first author of this paper [4], who recently obtained global well-posedness for the 4-wave kinetic equation in exponentially weighted $L^\infty_{x,v}$ spaces, employing classical tools of the kinetic theory of particles.

[1]The kinetic time is the relevant time scale for which one expects the system to exhibit a kinetic behavior.

The aim of this paper is to broaden the analysis of the inhomogeneous 4-wave kinetic equation and its associated infinite hierarchy of PDE - the inhomogeneous 4-wave kinetic hierarchy. We start by presenting an overview of the results of this paper. Notation and precise statements of results are then presented in subsections 1.3 and 1.4.

1.2. Results of this paper in a nutshell.

(1) Inspired by ideas stemming from the analysis of the Boltzmann equation, in this paper we expand well-posedness theory of the spatially inhomogeneous 4-wave kinetic equation

$$\partial_t f + v \cdot \nabla_x f = \mathcal{C}[f], \tag{1.3}$$

where the collision operator is defined in (1.8). In particular, we prove existence of a unique global in time solution to the corresponding integral equation in *polynomially weighted* $L^\infty_{x,v}$ spaces (see Definition 1.21 for the precise definition of the solution and Theorem 1.4 for the well-posedness result for the equation). This result is a consequence of a novel a priori bound (Proposition 2.1), which is motivated by the a priori bound obtained by Toscani [48] for solutions of the Boltzmann equation. However, in the current context of the wave kinetic equation (1.3), by exploiting higher order multilinear operators (1.10), we were able to rely on the integrals with higher order velocity weights (see Lemma A.9). Thanks to these additional weights, we did not need Carleman-like representation, as was the case for the Boltzmann equation in [48, 6]

(2) Furthermore, in this paper we also study the spatially inhomogeneous 4-wave kinetic hierarchy

$$\partial_t f^{(k)} + \sum_{j=1}^{k} v_k \cdot \nabla_{x_k} f^{(k)} = \mathfrak{C}^{(k+2)} f^{(k+2)}, \tag{1.4}$$

with $\mathfrak{C}^{(k+2)}$ given by (1.29). To the best of our knowledge, this is the first paper that analyzes a spatially inhomogeneous wave kinetic hierarchy. We note that homogeneous version of the 4-wave kinetic hierarchy has been studied by Rosenzweig-Staffilani in [43], who obtained a local in time existence and uniqueness of solutions in polynominally weighted L^∞_v spaces. Also, Deng-Hani [17] derived[2] spatially homogeneous 4-wave kinetic *hierarchy* based on a derivation of 4-wave kinetic equation and a propagation of chaos for this equation.

The main objects of this paper - the equation (1.3) and the hierarchy (1.4) - are connected by the fact that the hierarchy (1.4) admits a special class of factorized solutions

$$f^{(k)}(t, X_k, V_k) = \prod_{j=1}^{k} f(t, x_j, v_j), \tag{1.5}$$

with each factor f solving the 4-wave kinetic equation (1.3), where $X_k = (x_1, \ldots, x_k)$ and $V_k = (v_1, \ldots, v_k)$. We note that this factorization is analogous to the relationship between solutions of the Boltzmann equation and the Boltzmann hierarchy[3] [36, 33, 24, 42, 9] as well as the relationship between solutions of the NLS and the infinite hierarchy that appears in

[2]This kind of derivation is in contrast to derivations of infinite Boltzmann and Schrödinger hierarchies from particles systems, where corresponding nonlinear equations are obtained from infinite hierarchies. Instead, in [17], authors utilize the derivation of nonlinear wave kinetic equation in order to derive associated infinite hierarchy.

[3]which is an infinite hierarchy of coupled linear equations appearing in a rigorous derivation of the Boltzmann equation from many particle systems

the derivation of the NLS from quantum many particle systems, referred to as the Gross-Pitaevskii hierarchy [19, 20, 10, 11, 45, 34, 3]. In contrast to known derivations of the Boltzmann and Gross-Pitaevskii hierarchies from many particle systems, the derivation of the inhomogeneous wave kinetic hierarchy (1.4) is an open problem.

We now summarize main results of this paper pertaining to the spatially inhomogeneous wave kinetic hierarchy (1.4).

(a) Since in this paper we first establish the existence of global in time solutions to the wave kinetic equation (1.3), we can use these solutions to construct a global in time solution of the wave kinetic hierarchy (1.4) for a special class of initial data, the so-called admissible[4] initial data (for the precise statement, see Definition 1.7). This construction is implemented in a similar fashion as in our recent work [6] on the well-posedness for the Boltzmann hierarchy. More precisely,

Step 1: Since initial data $F_0 = (f_0^{(k)})_{k=1}^\infty$ of the wave kinetic hierarchy (1.4) is assumed to be admissible, by the Hewitt-Savage theorem [30], there is a unique Borel probability measure π such that F_0 can be represented as a convex combination of tensorized states with respect to π over the set of probability densities $\mathcal{P}$ i.e.

$$f_0^{(k)} = \int_{\mathcal{P}} h_0^{\otimes k} d\pi(h_0).$$

It can be shown that the measure π is supported on a set of probability densities of a space-velocity polynomial decay (see Proposition 3.2 below, which was proved in [6]).

Step 2: For each h_0 in the support of the measure π, by the well-posedness result of this paper for the wave kinetic equation (see Theorem 1.4), there exists a global in time solution $h(t)$ to the wave kinetic equation (1.3) of the same polynomial decay as the initial data. Finally, equipped with these solutions of the wave kinetic equation, we construct a solution $F = (f^{(k)})_{k=1}^\infty$ of the wave kinetic hierarchy (1.4) as follows:

$$f^{(k)}(t) := \int_{\mathcal{P}} h(t)^{\otimes k} d\pi(h_0), \quad k \in \mathbb{N}, \tag{1.6}$$

and prove that it belongs to the class of polynomially weighted $L^\infty_{x,v}$ solutions of (1.4).

Remark 1.1. *This two-step proof of existence of solutions to the inhomogeneous wave kinetic hierarchy* (1.4) *is different than the proof of existence of solutions to the homogeneous wave kinetic hierarchy by Rosenzweig-Staffilani in* [43]. *Our approach utilizes the global in time existence of solutions to the corresponding nonlinear equation. Indeed, as it can be seen in* (1.6)*, our solution to the hierarchy is built from corresponding solutions to the nonlinear equation* (1.3)*. On the other hand, the work* [43] *employs representing a solution of the infinite hierarchy by iterating Duhamel formulas. In our case, such an iteration is not needed for proving existence of solutions for the infinite hierarchy* (1.4)*.*

(b) We also prove uniqueness of solutions to the wave kinetic hierarchy (1.4). While existence of solutions to the wave kinetic hierarchy is established in this paper for admissible initial data, our uniqueness proof does not use admissibility. More precisely,

[4]In instances of particle systems, admissible data can be thought of as the marginals of a probability density.

we prove uniqueness of a mild solution (see Definition 1.6) to the wave kinetic hierarchy (1.4) corresponding to initial data in polynomially weighted $L^\infty_{x,v}$ spaces. The main ingredients of the uniqueness proof are:

(i) an a priori estimate (see Proposition 3.5) for the wave kinetic hierarchy (1.4). We prove this estimate by adapting to the context of the infinite hierarchy ideas used to obtain the a priori bound (see Proposition 2.1) for the wave kinetic equation (1.3);

(ii) a new combinatorial board game argument, inspired by:
- the board game argument introduced by Klainerman and Machedon [35] in the context of the Gross-Pitaevskii hierarchy corresponding to the cubic nonlinear Schrödinger equation,
- the adaptation of T. Chen and the third author of this paper [10] of the board game for the Gross-Pitaevskii hierarchy corresponding to the quntic NLS, and
- our recent use [6] of board game arguments in the context of the Boltzmann hierarchy in $L^\infty_{x,v}$-based spaces[5].

At the heart of board game arguments is a reorganization of the iterated Duhamel formulas (which contain a factorial number of terms) into an exponential number of equivalence classes (see Proposition 3.16 which achieves that[6] for hierarchy (1.4)). We note that this paper presents the first application of a board game argument in the context of a wave kinetic hierarchy.

We continue the introduction by precisely describing our results for the wave kinetic equation in Section 1.3 and results for the wave kinetic hierarchy in Section 1.4.

1.3. **Wave kinetic equation: notation and the main result.** The Cauchy problem for the spatially inhomogeneous 4-wave kinetic equation for a function $f : [0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ with initial data $f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, is given by

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mathcal{C}[f], \\ f(t=0) = f_0, \end{cases} \tag{1.7}$$

with the collisional operator defined as follows

$$\mathcal{C}[f] = \int_{\mathbb{R}^9} \delta(\Sigma)\delta(\Omega) f f_1 f_2 f_3 \left(\frac{1}{f} + \frac{1}{f_1} - \frac{1}{f_2} - \frac{1}{f_3} \right) dv_1\, dv_2\, dv_3, \tag{1.8}$$

and where the resonant manifolds are given by

$$\Sigma = v + v_1 - v_2 - v_3, \quad \Omega = |v|^2 + |v_1|^2 - |v_2|^2 - |v_3|^2. \tag{1.9}$$

We use the notation $f := f(v)$, $f_i := f(v_i)$, $i = 1, 2, 3$.

The collisional operator $\mathcal{C}[f]$ can be equivalently written as follows:

$$\mathcal{C}[f] = L_0(f, f, f) + L_1(f, f, f) - L_2(f, f, f) - L_3(f, f, f), \tag{1.10}$$

[5] We note that prior to [6] all implementations of the board game argument were done in L^2-based spaces.

[6] A special upper echelon form in this proposition can be understood as a representative of an equivalence class mentioned above.

where for functions $g, h, l : [0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ we denote

$$L_0(g,h,l) = \int_{\mathbb{R}^9} \delta(\Sigma)\delta(\Omega) g(v_1)h(v_2)l(v_3)\, dv_1\, dv_2\, dv_3, \tag{1.11}$$

$$L_1(g,h,l) = \int_{\mathbb{R}^9} \delta(\Sigma)\delta(\Omega) g(v)h(v_2)l(v_3)\, dv_1\, dv_2\, dv_3, \tag{1.12}$$

$$L_2(g,h,l) = \int_{\mathbb{R}^9} \delta(\Sigma)\delta(\Omega) g(v)h(v_1)l(v_3)\, dv_1\, dv_2\, dv_3, \tag{1.13}$$

$$L_3(g,h,l) = \int_{\mathbb{R}^9} \delta(\Sigma)\delta(\Omega) g(v)h(v_1)l(v_2)\, dv_1\, dv_2\, dv_3. \tag{1.14}$$

Notice that the operators L_0, L_1, L_2, L_3 are multilinear with respect to their arguments and monotone when the inputs are non-negative.

The collisional operator $\mathcal{C}[f]$ can also be written in weak formulation as follows [39, pp.122, eqn (8.18)]

$$\int_{\mathbb{R}^3} \mathcal{C}[f]\phi\, dv = \int_{\mathbb{R}^{12}} \delta(\Sigma)\delta(\Omega) f f_1 f_2 f_3 \left(\frac{1}{f} + \frac{1}{f_1} - \frac{1}{f_2} - \frac{1}{f_3}\right)(\phi + \phi_1 - \phi_2 - \phi_3)\, dv_1\, dv_2\, dv_3\, dv, \tag{1.15}$$

where ϕ is a test function appropriate for all the above integrations to make sense. Choosing $\phi \in \{1, v, |v|^2\}$ and using the resonant conditions, one can formally see that a solution f to (1.7) conserves mass, momentum and energy:

$$\partial_t \int_{\mathbb{R}^3} f\phi\, dv = 0, \quad \phi \in \{1, v, |v|^2\}. \tag{1.16}$$

We now introduce the spaces that will be used in the formulation of the well-posedness result for the wave kinetic equation (1.7). We shall be using the notation that for $y \in \mathbb{R}^3$,

$$\langle y \rangle^2 = 1 + |y|^2$$

to define the following function space.

Definition 1.2. *For each $T > 0$, $p, q > 1$ and $\alpha, \beta > 0$ we define the space*

$$X_{p,q,\alpha,\beta} := \left\{ f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \text{ measurable functions such that } \|f\|_{p,q,\alpha,\beta} < \infty \right\},$$

where we define the norm as

$$\|f\|_{p,q,\alpha,\beta} := \left\| \langle \alpha x \rangle^p \langle \beta v \rangle^q f(x,v) \right\|_{L^\infty}.$$

We additionally define the functional spaces in time as

$$X_{p,q,\alpha,\beta,T} = \mathcal{C}([0,T], X_{p,q,\alpha,\beta}),$$

where we are using the natural supremum norm on this space:

$$|||f(\cdot)|||_{p,q,\alpha,\beta,T} := \sup_{t\in[0,T]} \|f(t)\|_{p,q,\alpha,\beta}.$$

Before we define a concept of the solution we will be working with, we note that we will denote by T_1 the transport operator, which is defined by its action on a function $g : [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ as follows:

$$T_1^s g^{(k)}(t,x,v) := g^{(k)}(t, x - sv, v). \tag{1.17}$$

With this notation, by Duhamel's formula, the wave kinetic equation (1.7) can be formally written in mild form as:

$$f(t) = T_1^t f_0 + \int_0^t T_1^{t-s}\mathcal{C}[f](s)\,ds, \quad t \in [0,T], \tag{1.18}$$

or equivalently after applying T_1^{-t},

$$T_1^{-t} f(t) = f_0 + \int_0^t T_1^{-s}\mathcal{C}[f](s)\,ds, \quad t \in [0,T]. \tag{1.19}$$

We can now state precisely the definition of a mild solution we will be using in this paper.

Definition 1.3 (Mild solution of the wave kinetic equation)**.** *Let $T > 0$, $p, q > 1$ and $\alpha, \beta > 0$, and consider initial data $f_0 \in X_{p,q,\alpha,\beta}$. A measurable function $f : [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is called a mild solution to the wave kinetic equation* (1.7) *in $[0,T]$ corresponding to the initial data f_0 if*

$$T_1^{-(\cdot)} f(\cdot) \in X_{p,q,\alpha,\beta,T}, \tag{1.20}$$

and

$$T_1^{-t} f(t,x,v) = f_0 + \int_0^t T_1^{-s}\mathcal{C}[f](s,x,v)\,ds, \quad t \in [0,T]. \tag{1.21}$$

We now state the main result regarding the well-posedness of the wave kinetic equation (1.7).

Theorem 1.4. *Let $p > 1, q > 3$, $\alpha, \beta > 0$ and $T > 0$. Let $M > 0$ with $M < (24C_{p,q,\alpha,\beta})^{-1/2}$, where $C_{p,q,\alpha,\beta}$ is given by* (2.2)*. Consider $f_0 \in X_{p,q,\alpha,\beta}$, with $\|f_0\|_{p,q,\alpha,\beta} \le \frac{M}{2}$. Then there exists a unique mild solution to the wave kinetic equation* (1.7)*, in the class of functions satisfying:*

$$|||T_1^{-(\cdot)} f(\cdot)|||_{p,q,\alpha,\beta,T} \le M. \tag{1.22}$$

If $f_0 \ge 0$, the solution remains non-negative. Additionally, assuming that f and g are the mild solutions corresponding to initial data f_0 and g_0 respectively, we have the continuity with respect to initial data estimate:

$$|||T_1^{-(\cdot)} f(\cdot) - T_1^{-(\cdot)} g(\cdot)|||_{p,q,\alpha,\beta,T} \le 2\|f_0 - g_0\|_{p,q,\alpha,\beta}. \tag{1.23}$$

In particular

$$|||T_1^{-(\cdot)} f(\cdot)|||_{p,q,\alpha,\beta,T} \le 2\|f_0\|_{p,q,\alpha,\beta}. \tag{1.24}$$

Moreover, the solution satisfies the following conservation laws: for any $t \in [0,T]$ and a.e. $x \in \mathbb{R}^3$:

$$\text{If } p > 3,\ q > 4: \int_{\mathbb{R}^3} f(t,x,v)\,dv = \int_{\mathbb{R}^3} f_0(x,v)\,dv, \tag{1.25}$$

$$\text{If } p > 3,\ q > 5\ :\ \int_{\mathbb{R}^3} v f(t,x,v)\,dv = \int_{\mathbb{R}^3} v f_0(x,v)\,dv, \tag{1.26}$$

$$\text{If } p > 3,\ q > 6\ :\ \int_{\mathbb{R}^3} |v|^2 f(t,x,v)\,dv = \int_{\mathbb{R}^3} |v|^2 f_0(x,v)\,dv. \tag{1.27}$$

Remark 1.5. *We note that we work in dimension $d = 3$ because one of the key estimates (Lemma A.9) is established only for this dimension.*

1.4. **Wave kinetic hierarchy: notation and the main result.** As mentioned in subsection 1.2, we introduce the spatially inhomogeneous 4-wave kinetic hierarchy as follows:

$$\partial_t f^{(k)}(t, X_k, V_k) + \sum_{j=1}^{k} v_k \cdot \nabla_{x_k} f^{(k)} = \mathfrak{C}^{k+2} f^{(k+2)}, \tag{1.28}$$

with collision operator defined by

$$\begin{aligned} \mathfrak{C}^{k+2} f^{(k+2)} &= \sum_{j=1}^{k} \mathfrak{C}_{j,k+2} f^{(k+2)} && (1.29)\\ &:= \sum_{j=1}^{k} \left(\mathfrak{C}^{L_0}_{j,k+2} f^{(k+2)} + \mathfrak{C}^{L_1}_{j,k+2} f^{(k+2)} - \mathfrak{C}^{L_2}_{j,k+2} f^{(k+2)} - \mathfrak{C}^{L_3}_{j,k+2} f^{(k+2)} \right), && (1.30) \end{aligned}$$

where

$$\begin{aligned} & C^{L_0}_{j,k+2} f^{(k+2)}(t, X_k, V_k) \\ &= \int_{\mathbb{R}^9} dv_{k+1} dv_{k+2} dv_{k+3}\, \delta(\Sigma_{j,k+2})\, \delta(\Omega_{j,k+2})\, f^{(k+2)}(t, X_k, x_j, x_j,\ V_k^{j,v_{k+1}}, v_{k+2}, v_{k+3}), && (1.31)\\ & \mathfrak{C}^{L_1}_{j,k+2} f^{(k+2)}(t, X_k, V_k) \\ &= \int_{\mathbb{R}^9} dv_{k+1} dv_{k+2} dv_{k+3}\, \delta(\Sigma_{j,k+2})\, \delta(\Omega_{j,k+2})\, f^{(k+2)}(t, X_k, x_j, x_j,\ V_k, v_{k+2}, v_{k+3}), && (1.32)\\ & \mathfrak{C}^{L_2}_{j,k+2} f^{(k+2)}(t, X_k, V_k) \\ &= \int_{\mathbb{R}^9} dv_{k+1} dv_{k+2} dv_{k+3}\, \delta(\Sigma_{j,k+2})\, \delta(\Omega_{j,k+2})\, f^{(k+2)}(t, X_k, x_j, x_j,\ V_k, v_{k+1}, v_{k+3}), && (1.33)\\ & \mathfrak{C}^{L_3}_{j,k+2} f^{(k+2)}(t, X_k, V_k) \\ &= \int_{\mathbb{R}^9} dv_{k+1} dv_{k+2} dv_{k+3}\, \delta(\Sigma_{j,k+2})\, \delta(\Omega_{j,k+2})\, f^{(k+2)}(t, X_k, x_j, x_j,\ V_k, v_{k+1}, v_{k+2}), && (1.34) \end{aligned}$$

and

$$\begin{aligned} \Sigma_{j,k+2} &= v_j + v_{k+1} - v_{k+2} - v_{k+3}, \\ \Omega_{j,k+2} &= |v_j|^2 + |v_{k+1}|^2 - |v_{k+2}|^2 - |v_{k+3}|^2, \end{aligned} \tag{1.35}$$

and

$$V_k^{j,v_{k+1}} = (v_1, \ldots, v_{j-1}, \underbrace{v_{k+1}}_{j-\mathrm{th}}, v_{j+1}, \ldots, v_k). \tag{1.36}$$

One can also represent each operator $\mathfrak{C}_{j,k+2}$ as a difference:

$$\mathfrak{C}_{j,k+2} = \mathfrak{C}^{+}_{j,k+2} - \mathfrak{C}^{-}_{j,k+2}, \tag{1.37}$$

where

$$\mathfrak{C}^{+}_{j,k+2} = \mathfrak{C}^{L_0}_{j,k+2} + \mathfrak{C}^{L_1}_{j,k+2}, \tag{1.38}$$

$$\mathfrak{C}^{-}_{j,k+2} = \mathfrak{C}^{L_2}_{j,k+2} + \mathfrak{C}^{L_3}_{j,k+2}. \tag{1.39}$$

With this notation we have

$$\mathfrak{C}^{k+2} = \sum_{j=1}^{k} \mathfrak{C}^{+}_{j,k+2} - \mathfrak{C}^{-}_{j,k+2}. \tag{1.40}$$

Motivated by [6], we define Banach spaces that will be used throughout the paper. For $Y_k = (y_1, y_2, \dots, y_k) \in \mathbb{R}^{3k}$, we define

$$\langle\langle Y_k \rangle\rangle := \prod_{i=1}^{k} \langle y_i \rangle, \quad \langle y_i \rangle := \sqrt{1+|y_i|^2}, \quad i = 1, \cdots, k. \tag{1.41}$$

We are ready to define the following Banach spaces and their corresponding norms:

- Given $k \in \mathbb{N}$, $p, q > 1$, $\alpha, \beta > 0$, we define

$$X^k_{p,q,\alpha,\beta} := \left\{ g^{(k)} : \mathbb{R}^{3k} \times \mathbb{R}^{3k} \to \mathbb{R}, \text{ measurable and symmetric} : \|g^{(k)}\|_{k,p,q,\alpha,\beta} < \infty \right\}, \tag{1.42}$$

$$\|g^{(k)}\|_{k,p,q,\alpha,\beta} := \sup_{X_k, V_k} \langle\langle \alpha X_k \rangle\rangle^p \langle\langle \beta V_k \rangle\rangle^q \left| g^{(k)}(X_k, V_k) \right|. \tag{1.43}$$

Here by symmetric, we mean

$$g^{(k)} \circ \sigma_k = g^{(k)}, \quad \text{for any permutation } \sigma_k \text{ of pairs of variables } \{x_i, v_i\}_{i=1}^k. \tag{1.44}$$

- Given $T > 0$, $k \in \mathbb{N}$, $p, q > 1$, $\alpha, \beta > 0$, we define

$$X^k_{p,q,\alpha,\beta,T} := C([0,T], X^k_{p,q,\alpha,\beta}), \tag{1.45}$$

$$\begin{aligned} |||g^{(k)}(\cdot)|||_{k,p,q,\alpha,\beta,T} &:= \sup_{t \in [0,T]} \|g^{(k)}(t)\|_{k,p,q,\alpha,\beta} \\ &= \sup_{t \in [0,T]} \sup_{X_k, V_k} \langle\langle \alpha X_k \rangle\rangle^p \langle\langle \beta V_k \rangle\rangle^q \left| g^{(k)}(t, X_k, V_k) \right|. \end{aligned} \tag{1.46}$$

In order to be able to look at a solution of the infinite hierarchy (1.28) as a single object $F = (f^{(k)})_{k=1}^\infty$, we also introduce the following spaces:

- Given $p, q > 1$, $\alpha, \beta > 0$, $\mu \in \mathbb{R}$, we define

$$\mathcal{X}^\infty_{p,q,\alpha,\beta,\mu} := \left\{ G = (g^{(k)})_{k=1}^\infty \in \prod_{k=1}^\infty X^k_{p,q,\alpha,\beta} \;:\; \|G\|_{p,q,\alpha,\beta,\mu} < \infty \right\}, \tag{1.47}$$

$$\|G\|_{p,q,\alpha,\beta,\mu} = \sup_{k \in \mathbb{N}} e^{\mu k} \|g^{(k)}\|_{k,p,q,\alpha,\beta} \tag{1.48}$$

$$= \sup_{k \in \mathbb{N}} e^{\mu k} \sup_{X_k, V_k} \langle\langle \alpha X_k \rangle\rangle^p \langle\langle \beta V_k \rangle\rangle^q \left| g^{(k)}(X_k, V_k) \right|. \tag{1.49}$$

- Given $T > 0$, $p, q > 1$, $\alpha, \beta > 0$, we define

$$\mathcal{X}^\infty_{p,q,\alpha,\beta,\mu,T} := C([0,T], \mathcal{X}^\infty_{p,q,\alpha,\beta,\mu}) \tag{1.50}$$

$$\begin{aligned} |||G(\cdot)|||_{p,q,\alpha,\beta,\mu,T} &:= \sup_{t \in [0,T]} \|G(t)\|_{p,q,\alpha,\beta,\mu} \\ &= \sup_{t \in [0,T]} \sup_{k \in \mathbb{N}} e^{\mu k} \sup_{X_k, V_k} \langle\langle \alpha X_k \rangle\rangle^p \langle\langle \beta V_k \rangle\rangle^q \left| g^{(k)}(t, X_k, V_k) \right|. \end{aligned} \tag{1.51}$$

With definition of functional spaces in hand, we are ready to give a precise definition of mild solutions to the wave kinetic hierarchy (1.28), which we motivate by the following observation. For $k \in \mathbb{N}$, we denote the transport operator acting on a function $g^{(k)} : [0,\infty) \times \mathbb{R}^{3k} \times \mathbb{R}^{3k} \to \mathbb{R}$ by

$$T_k^s g^{(k)}(t, X_k, V_k) := g^{(k)}(t, X_k - sV_k, V_k). \tag{1.52}$$

With this notation, in analogy to (1.19), a mild solution of the wave kinetic hierarchy (1.28) is formally given by

$$T_k^{-t} f^{(k)}(t) = f_0^{(k)} + \int_0^t T_k^{-s} \mathfrak{C}^{k+2} f^{(k+2)}(s) ds, \quad \forall k \in \mathbb{N}, \ \forall t \in [0,T]. \tag{1.53}$$

Definition 1.6 (Mild solution to the wave kinetic hierarchy)**.** *Let $T > 0$, $p, q > 1$, $\alpha, \beta > 0$, $\mu \in \mathbb{R}$, and consider initial data $F_0 = (f_0^{(k)})_{k=1}^\infty \in \mathcal{X}^\infty_{p,q,\alpha,\beta,\mu}$. A sequence $F = (f^{(k)})_{k=1}^\infty$ of measurable functions $f^{(k)} : [0,T] \times \mathbb{R}^{3k} \times \mathbb{R}^{3k} \to \mathbb{R}$ is called a mild μ-solution to the wave kinetic hierarchy* (1.28) *in $[0,T]$, corresponding to the initial data F_0, if*

$$\mathcal{T}^{-(\cdot)} F(\cdot) := (T_k^{-(\cdot)} f^{(k)}(\cdot))_{k=1}^\infty \in \mathcal{X}^\infty_{p,q,\alpha,\beta,\mu,T}, \tag{1.54}$$

and

$$T_k^{-t} f^{(k)}(t) = f_0^{(k)} + \int_0^t T_k^{-s} \mathfrak{C}^{k+2} f^{(k+2)}(s)\, ds, \quad \forall t \in [0,T], \quad \forall k \in \mathbb{N}. \tag{1.55}$$

Our main result for the wave kinetic hierarchy (1.7) concerns its well-posedness, which means existence and stability of solutions and their uniqueness. While uniqueness part does not require special structure of the initial data, our existence part utilizes the concept of admissible initial data. In order to be able to state the entire well-posedness result, we first recall the notion of admissibility.

Definition 1.7 (Admissibility)**.** *The set of admissible functions, denoted by $\mathcal{A}$, is defined by*

$$\mathcal{A} := \Big\{ (g^{(k)})_{k=1}^\infty \in \prod_{k=1}^\infty L^1_{X_k,V_k} : \forall k \in \mathbb{N} \text{ we have } g^{(k)} \geq 0, \ g^{(k)} \text{ is symmetric (1.44)}, \\ \int_{\mathbb{R}^{6k}} g^{(k)}\, dX_k\, dV_k = 1, \ g^{(k)} = \int_{\mathbb{R}^6} g^{(k+1)} dv_{k+1} dx_{k+1} \Big\}. \tag{1.56}$$

We are now ready to state our main result on the well-posedness of the wave kinetic hierarchy.

Theorem 1.8 (Global well-posedness for the wave kinetic hierarchy)**.** *Consider the wave kinetic hierarchy* (1.28) *in dimension $d = 3$. Let $T > 0$, $p > 1$, $q > 3$, $\alpha, \beta > 0$ and let $\mu \in \mathbb{R}$ be such that $e^{2\mu} > 32 C_{p,q,\alpha,\beta}$, where $C_{p,q,\alpha,\beta}$ is given by* (2.2)*. Consider admissible initial data $F_0 = (f_0^{(k)})_{k=1}^\infty \in \mathcal{A} \cap \mathcal{X}^\infty_{p,q,\alpha,\beta,\mu'}$, where $\mu' = \mu + \ln 2$. Then, there exists a unique mild μ-solution $F = (f^{(k)})_{k=1}^\infty$ of the wave kinetic hierarchy* (1.28)*, with $f^{(k)} \geq 0$ for all k. In addition, the solution satisfies the estimate*

$$|||\mathcal{T}^{-(\cdot)} F(\cdot)|||_{p,q,\alpha,\beta,\mu,T} \leq 1. \tag{1.57}$$

Moreover, the following k-particle conservation laws hold for any $t \in [0,T]$ and a.e. $X_k \in \mathbb{R}^{3k}$:

$$\text{If } p>3,\ q>4: \quad \int_{\mathbb{R}^{3k}} f^{(k)}(t,X_k,V_k)\,dV_k = 1, \tag{1.58}$$

$$\text{If } p>3,\ q>5: \quad \int_{\mathbb{R}^{3k}} V_k f^{(k)}(t,X_k,V_k)\,dV_k = \int_{\mathbb{R}^{3k}} V_k f_0^{(k)}(X_k,V_k)\,dV_k, \tag{1.59}$$

$$\text{If } p>3, q>6: \quad \int_{\mathbb{R}^{3k}} |V_k|^2 f^{(k)}(t,X_k,V_k)\,dV_k = \int_{\mathbb{R}^{3k}} |V_k|^2 f_0^{(k)}(X_k,V_k)\,dV_k. \tag{1.60}$$

In the case that the initial data are tensorized i.e. $F_0 = (f_0^{\otimes k})_{k=1}^\infty \in \mathcal{A} \cap \mathcal{X}^\infty_{p,q,\alpha,\beta,\mu'}$*, there holds the stability estimate*

$$|||\mathcal{T}^{-(\cdot)}F(\cdot)|||_{p,q,\alpha,\beta,\mu,T} \leq \|F_0\|_{p,q,\alpha,\beta,\mu'}. \tag{1.61}$$

Organization of the paper. In Section 2 we address global well-posedness of the wave kinetic equation. Global well-posedness of the wave kinetic hierarchy is proved in Section 3. In the appendix we gather auxiliary results used throughout the paper, including properties of tensorized functions in Appendix A.1, properties of the resonant manifold in Appendix A.2, and various integral estimates in Appendix A.3.

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2. Global well-posedness of the wave kinetic equation

In this section, we obtain global in time existence, uniqueness and stability of mild solutions of the inhomogeneous wave kinetic equation (1.7) for initial data polynomially close to vacuum. When the initial data are non-negative, we show that the corresponding solution remains non-negative, which is physically anticipated since the equation describes a localized point energy spectrum. Additionally, we prove that the solution conserves mass, momentum and energy for sufficiently decaying initial data.

The strategy for proving the global well-posedness of (1.7) relies on the following global in time a-priori estimate.

Proposition 2.1. *Let* $p>1, q>3$, $\alpha,\beta>0$ *and* $T>0$*. For any* $j=0,1,2,3$*, and* $t\in[0,T]$*, the following bound holds*

$$\left\| \int_0^t T_1^{-s} L_j(g,h,l)(s)\,ds \right\|_{p,q,\alpha,\beta} \leq C_{p,q,\alpha,\beta} |||T_1^{-(\cdot)}g|||_{p,q,\alpha,\beta,T} |||T_1^{-(\cdot)}h|||_{p,q,\alpha,\beta,T} |||T_1^{-(\cdot)}l|||_{p,q,\alpha,\beta,T} \tag{2.1}$$

where L_j*, with* $j\in\{0,1,2,3\}$ *are defined in* (1.11)-(1.14)*. One can take*

$$C_{p,q,\alpha,\beta} = \frac{16\,p\,\pi^3}{\alpha(p-1)} \left(\frac{1}{3}+\frac{1}{q-3}\right) \max\{\beta^q, \beta^{-3q}\}. \tag{2.2}$$

Remark 2.2. *The proof of this Proposition is inspired by the work of Toscani* [48] *for the Boltzmann equation. However, the presence of higher order multilinear operators L_j, $j \in \{0,1,2,3\}$ in the wave kinetic equation allows simplifications, as mentioned in subsection 1.2.*

Proof. Without loss of generality, assume that

$$|||T_1^{-(\cdot)} g|||_{p,q,\alpha,\beta,T} = |||T_1^{-(\cdot)} h|||_{p,q,\alpha,\beta,T} = |||T_1^{-(\cdot)} l|||_{p,q,\alpha,\beta,T} = 1. \tag{2.3}$$

Given $v, v_1, v_2, v_3 \in \mathbb{R}^3$, we will use the following notation

$$u_{i,j} := v_i - v_j, \quad i,j \in \{0,1,2,3\},\ i \neq j,$$

where we abbreviate notation denoting $v_0 = v$.

• *Proof for L_0.* For any $x, v \in \mathbb{R}^3$ and $t \in [0,T]$, we have

$$\begin{aligned}
I_{L_0}(t,x,v) &:= \langle \alpha x\rangle^p \langle \beta v\rangle^q \left| \int_0^t T_1^{-s} L_0(g,h,l)(s,x,v)\,ds \right| \\
&= \langle \alpha x\rangle^p \langle \beta v\rangle^q \left| \int_0^t [L_0(g,h,l)](s,x+sv,v) ds \right| \\
&\leq \langle \alpha x\rangle^p \langle \beta v\rangle^q \int_0^t \int_{\mathbb{R}^9} dv_1\,dv_2\,dv_3 ds\ \delta(\Sigma)\delta(\Omega)\,|g(s,x+sv,v_1)h(s,x+sv,v_2)l(s,x+sv,v_3)| \\
&= \langle \alpha x\rangle^p \langle \beta v\rangle^q \int_0^t \int_{\mathbb{R}^9} dv_1\,dv_2\,dv_3\,ds\ \delta(\Sigma)\delta(\Omega) \\
&\qquad \left|T_1^{-s} g(s,x+s(v-v_1),v_1)\,T_1^{-s} h(s,x+s(v-v_2),v_2)\,T_1^{-s} l(s,x+s(v-v_3),v_3)\right| \qquad (2.4) \\
&\leq \langle \alpha x\rangle^p \int_{\mathbb{R}^9} \delta(\Sigma)\delta(\Omega) \frac{\langle \beta v\rangle^q}{\langle \beta v_1\rangle^q \langle \beta v_2\rangle^q \langle \beta v_3\rangle^q} \left(\int_0^t \langle \alpha x + \alpha s u_{0,2}\rangle^{-p} \langle \alpha x + \alpha s u_{0,3}\rangle^{-p}\,ds \right) dv_1\,dv_2\,dv_3, \qquad (2.5)
\end{aligned}$$

where we use the notation (1.9) and where in the last inequality we used (2.3) and the fact that $\langle \alpha x + \alpha s u_{0,1}\rangle^{-p} \leq 1$.

Now, by (A.6), (A.9) from Lemma A.3, on the resonant manifold determined by Σ and Ω we have

$$u_{0,2} \cdot u_{0,3} = 0, \tag{2.6}$$

as well as

$$\min\{|u_{0,2}|, |u_{0,3}|\} \geq \frac{|u_{0,1}|}{2} \sqrt{1 - (\hat{u}_{0,1} \cdot \hat{u}_{2,3})^2}. \tag{2.7}$$

Thus, applying Lemma A.5 for $x := \alpha x$, $\xi := \alpha u_{0,2}$, $\eta := \alpha u_{0,3}$, and using the above estimate, we obtain

$$\int_0^t \langle \alpha x + \alpha s u_{0,2}\rangle^{-p} \langle \alpha x + \alpha s u_{0,3}\rangle^{-p}\,ds \leq \frac{8p}{\alpha(p-1)} \frac{\langle \alpha x\rangle^{-p}}{|u_{0,1}|\sqrt{1 - (\hat{u}_{0,1} \cdot \hat{u}_{2,3})^2}}. \tag{2.8}$$

Combining (2.5) and (2.8), we obtain

$$I_{L_0}(t,x,v) \leq \frac{8p}{\alpha(p-1)} \int_{\mathbb{R}^9} \frac{dv_1\,dv_2\,dv_3\ \delta(\Sigma)\delta(\Omega)}{|u_{0,1}|\sqrt{1 - (\hat{u}_{0,1} \cdot \hat{u}_{2,3})^2}} \frac{\langle \beta v\rangle^q}{\langle \beta v_1\rangle^q \langle \beta v_2\rangle^q \langle \beta v_3\rangle^q}.$$

Finally, we can use that

$$\frac{\langle \beta v\rangle^q}{\langle \beta v_1\rangle^q\langle \beta v_2\rangle^q\langle \beta v_3\rangle^q} \le \max\{\beta^q, \beta^{-3q}\}\frac{\langle v\rangle^q}{\langle v_1\rangle^q\langle v_2\rangle^q\langle v_3\rangle^q},$$

together with (A.24) from Lemma A.9, to obtain

$$I_{L_0}(t,x,v) \le \frac{8p}{\alpha(p-1)}\ 2\pi^3\left(\frac{1}{3}+\frac{1}{q-3}\right)\ \max\{\beta^q,\beta^{-3q}\}.$$

Since x, v, t were arbitrary, estimate (2.1) follows for $j = 0$.

• *Proof for* L_1. For any $x, v \in \mathbb{R}^3$ and $t \in [0, T]$, we have

$$\begin{aligned}
I_{L_1}(t,x,v) &:= \langle \alpha x\rangle^p\langle \beta v\rangle^q\left|\int_0^t T_1^{-s}L_1(g,h,l)(s,x,v)\,ds\right|\\
&\le \langle \alpha x\rangle^p\langle \beta v\rangle^q\int_0^t\int_{\mathbb{R}^9} dv_1\,dv_2\,dv_3 ds\ \delta(\Sigma)\delta(\Omega)\,|g(s,x+sv,v)h(s,x+sv,v_2)l(s,x+sv,v_3)|\\
&= \langle \alpha x\rangle^p\langle \beta v\rangle^q\int_0^t\int_{\mathbb{R}^9} dv_1\,dv_2\,dv_3\,ds\ \delta(\Sigma)\delta(\Omega)\\
&\qquad\times\left|T_1^{-s}g(s,x,v)\,T_1^{-s}h(s,x+s(v-v_2),v_2)\,T_1^{-s}g(s,x+s(v-v_3),v_3)\right|\\
&\le \int_{\mathbb{R}^9} dv_1\,dv_2\,dv_3\delta(\Sigma)\delta(\Omega)\left(\int_0^t\langle \alpha x+\alpha s u_{0,2}\rangle^{-p}\langle \alpha x+\alpha s u_{0,3}\rangle^{-p}\,ds\right)\frac{1}{\langle \beta v_2\rangle^q\langle \beta v_3\rangle^q}.
\end{aligned}$$

Applying again estimate (2.8) on the time integral above, the fact that $\langle \beta v_2\rangle^{-q}\langle \beta v_3\rangle^{-q} \le \max\{1,\beta^{-2q}\}\langle v_2\rangle^{-q}\langle v_3\rangle^{-q}$ together with the fact that $\langle \alpha x\rangle^{-p} \le 1$, and estimate (A.25) from Lemma A.9, we obtain

$$\begin{aligned}
I_{L_1}(t,x,v) &\le \frac{8p}{\alpha(p-1)}\max\{1,\beta^{-2q}\}\int_{\mathbb{R}^9}\frac{dv_1\,dv_2\,dv_3\ \delta(\Sigma)\delta(\Omega)}{|u_{0,1}|\sqrt{1-(\hat u_{0,1}\cdot\hat u_{2,3})^2}\,\langle v_2\rangle^q\langle v_3\rangle^q}\\
&\le \frac{8p}{\alpha(p-1)}\ 2\pi^3\left(\frac{1}{3}+\frac{1}{q-3}\right)\ \max\{1,\beta^{-2q}\},
\end{aligned}$$

and estimate (2.1) follows for $j = 1$.

• *Proof for* L_2. We note that the proof for L_3 is identical. For any $x, v \in \mathbb{R}^3$ and $t \in [0, T]$, we have

$$\begin{aligned}
I_{L_2}(t,x,v) &:= \langle \alpha x\rangle^p\langle \beta v\rangle^q\left|\int_0^t T_1^{-s}L_2(g,h,l)(s,x,v)\,ds\right|\\
&\le \langle \alpha x\rangle^p\langle \beta v\rangle^q\int_0^t\int_{\mathbb{R}^9} dv_1\,dv_2\,dv_3 ds\ \delta(\Sigma)\delta(\Omega)\,|g(s,x+sv,v)h(s,x+sv,v_1)h(s,x+sv,v_3)|\\
&= \langle \alpha x\rangle^p\langle \beta v\rangle^q\int_0^t\int_{\mathbb{R}^9} dv_1\,dv_2\,dv_3\,ds\ \delta(\Sigma)\delta(\Omega)\\
&\qquad\times\left|T_1^{-s}g(s,x,v)|T_1^{-s}h(s,x+s(v-v_1),v_1)|T_1^{-s}l(s,x+s(v-v_3),v_3)\right|\\
&\le \int_{\mathbb{R}^9} dv_1\,dv_2\,dv_3\ \delta(\Sigma)\delta(\Omega)\frac{1}{\langle \beta v_1\rangle^q\langle \beta v_3\rangle^q}\left(\int_0^t\langle \alpha x+\alpha s u_{0,1}\rangle^{-p}\,ds\right),
\end{aligned}\tag{2.9}$$

since $\langle \alpha x + \alpha s u_{0,3} \rangle^{-p} \le 1$. Now, by Lemma A.4 we have

$$\int_{-\infty}^{+\infty} \langle \alpha x + \alpha s u_{0,1} \rangle^{-p} \, ds \le \frac{2p}{\alpha(p-1)} |u_{0,1}|^{-1}.$$

Therefore by using that $\langle \beta v_3 \rangle^{-q} \le 1$ and the fact that $\langle \beta v_1 \rangle^{-q} \le \max\{1, \beta^{-q}\} \langle v_1 \rangle^{-q}$, we have

$$\begin{aligned} I_{L_2}(t,x,v) &\le \frac{2p}{\alpha(p-1)} \max\{1, \beta^{-q}\} \int_{\mathbb{R}^9} \frac{\delta(\Sigma)\delta(\Omega)}{|u_{0,1}| \langle v_1 \rangle^q} \, dv_1 \, dv_2 \, dv_3 \\ &\le \frac{8p}{\alpha(p-1)} \, \pi^2 \left(\frac{1}{3} + \frac{1}{q-3} \right) \max\{1, \beta^{-q}\}, \end{aligned}$$

where in the last inequality we used Lemma A.8 for $d = 3$. Estimate (2.1) then follows for $j = 2$. $\square$

Now, using the above a-priori estimate in conjunction with the contraction mapping principle, we can prove the global well-posedness of (1.7) for arbitrary signed data which are polynomially close to vacuum. In order to guarantee non-negativity of the solution when the initial data are non-negative, one needs a more refined argument taking advantage of the monotonicity properties of the equation. This can be achieved by employing a classical tool from the kinetic theory of particles, namely the Kaniel-Shinbrot iteration [32, 31, 7, 48, 49, 1, 2]. Recently in [4], this technique was applied for the first time in the context of wave turbulence by the first author of this paper, who addressed the problem for exponentially decaying initial data.

Proof of Theorem 1.4. In order to obtain existence, uniqueness, non-negativity and stability of solutions to (1.7), we can essentially follow the strategy of the proofs in [4] (using Proposition 2.1 instead of the exponential type a priori estimate of [4] when needed), so we omit details of the proof, and show the setup only. In particular, let us define the mapping Φ as follows

$$\Phi(g(t)) = f_0 + \int_0^t T_1^{-s} \mathcal{C}[T_1^s g](s) ds. \tag{2.10}$$

Then using Proposition 2.1 it can be shown that $\Phi : B^M_{X_{p,q,\alpha,\beta,T}} \mapsto B^M_{X_{p,q,\alpha,\beta,T}}$, where

$$B^M_{X_{p,q,\alpha,\beta,T}} := \{ h \in X_{p,q,\alpha,\beta,T} \quad \text{with} \quad |||h|||_{p,q,\alpha,\beta,T} \le M \}, \tag{2.11}$$

is a contraction in $B^M_{X_{p,q,\alpha,\beta,T}}$. Hence, there exists a unique fixed point $g \in B^M_{X_{p,q,\alpha,\beta,T}}$ such that

$$g(t) = \Phi(g(t)). \tag{2.12}$$

By letting $f(\cdot) = T_1^{(\cdot)} g(\cdot)$, this is equivalent to

$$T_1^{-t} f(t) = \Phi(T_1^{-t} f(t)) = f_0 + \int_0^t T_1^{-s} \mathcal{C}[f](s) dx, \tag{2.13}$$

and we also have $T_1^{-t} f \in B^M_{X_{p,q,\alpha,\beta,T}}$ as claimed. This completes proof of well-posedness.

It remains to verify the conservation laws (1.25)-(1.27) for the constructed solution f. For this, we define the space

$$L_v^{1,\ell} = \left\{ g : \mathbb{R}^3 \to \mathbb{R} \text{ measurable such that } \int_{\mathbb{R}^3} \langle v \rangle^\ell g(v) \, dv < \infty \right\},$$

where $\ell \ge 0$.

Assume that $p > 3$ and $q > 4 + i$, where $i \in \{0, 1, 2\}$. By (1.22), for any $t \in [0, T]$ and a.e. $x, v \in \mathbb{R}^3$, we have

$$\langle \beta v \rangle^{1+i} f(t, x, v) \leq M \langle \alpha(x - vt) \rangle^{-p} \langle \beta v \rangle^{1+i-q} \leq M \langle \beta v \rangle^{1+i-q}. \tag{2.14}$$

Since $q > 4 + i$, integrating (2.14) in velocity we obtain that $f(t, x, v) \in L_v^{1,1+i}$, for any $t \in [0, T]$ and a.e. $x \in \mathbb{R}^d$. Therefore, by Lemma A.7, the weak form (1.15) is valid for any test function ϕ with $|\phi(v)| \leq C\langle v \rangle^i$. Hence, using (1.15) and the resonant conditions, we obtain

$$\int_{\mathbb{R}^3} \mathcal{C}[f]\phi(v)\, dv = 0, \quad \forall t \in [0, T], \text{ a.e. } x \in \mathbb{R}^3, \tag{2.15}$$

where $\phi = 1$ if $i = 0$, $\phi \in \{1, v\}$ if $i = 1$, and $\phi \in \{1, v, |v|^2\}$ if $i = 2$.

Additionally $f \in L^\infty([0, T], L_x^1 L_v^{1,1+i})$, which can be seen by integrating the first inequality of (2.14) in x, v and taking supremum in time:

$$\begin{aligned} \sup_{t \in [0,T]} \int_{\mathbb{R}^6} \langle \beta v \rangle^{1+i} f(t, x, v)\, dx\, dv &\leq M \sup_{t \in [0,T]} \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \langle \alpha(x - vt) \rangle^{-p}\, dx \right) \langle \beta v \rangle^{1+i-q}\, dv \\ &= M\alpha^{-3}\beta^{-3} \left(\int_{\mathbb{R}^3} \langle x' \rangle^{-p}\, dx' \right) \left(\int_{\mathbb{R}^3} \langle v' \rangle^{1+i-q}\, dv' \right) < \infty, \end{aligned} \tag{2.16}$$

since $p > 3$ and $q > 4 + i$. By Lemma A.7, we also obtain that $Q(f, f) \in L^\infty([0, T], L_x^1 L_v^{1,i})$.

Now let $\phi = 1$ if $i = 0$, $\phi \in \{1, v\}$ if $i = 1$, and $\phi \in \{1, v, |v|^2\}$ if $i = 2$. We integrate (1.21) in x, v and use Fubini's theorem and (2.15), to obtain that for any $t \in [0, T]$ we have

$$\begin{aligned} \int_{\mathbb{R}^6} f(t, x + tv, v)\phi(v)\, dx\, dv &= \int_{\mathbb{R}^6} f_0(x, v)\phi(v)\, dxdv + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{C}[f](s, x + sv, v)\phi(v) dxdvds \\ &= \int_{\mathbb{R}^6} f_0(x, v)\phi(v)\, dx\, dv + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{C}[f](s, x, v)\phi(v)\, dx\, dv\, ds \\ &= \int_{\mathbb{R}^6} f_0(x, v)\phi(v)\, dx\, dv. \end{aligned}$$

Thus for any $t \in [0, T]$, we have

$$\int_{\mathbb{R}^6} f(t, x, v)\phi(v)\, dx\, dv = \int_{\mathbb{R}^6} f(t, x + vt, v)\, \phi(v)\, dx\, dv = \int_{\mathbb{R}^6} f_0(x, v)\phi(v)\, dx\, dv,$$

and (1.25)-(1.27) follow.

□

3. Global well-posedness of wave kinetic hierarchy

In this section we construct the unique global in time mild solution to the wave kinetic hierarchy (1.28) for admissible initial data and for a range of values of the parameter μ. The construction is inspired by a similar construction of mild solutions for the Boltzmann hierarchy in [6]. The strategy consists of the following steps. First, since the initial data will be assumed to be admissible, we will be able to employ a Hewitt-Savage representation [30] tailored to our norms, to express such initial data as a convex combination of tensorized states under an appropriate probability measure π. Then, each element in the support of the measure π can be treated as initial data to the wave kinetic *equation* for which we have a mild solution. Finally, by taking the same convex combination of these tensorized solutions to the wave kinetic *equation* under the same probability measure π, we will prove that one obtains a mild solution to the wave kinetic *hierarchy*. Finally, we will utilize the uniqueness result (Theorem 3.4) to conclude that that mild solution is unique.

3.1. **Proof of existence for the wave kinetic hierarchy.** In this section we prove the existence part of Theorem 1.8. In particular we claim the following.

Theorem 3.1 (Existence of solutions for the wave kinetic hierarchy)**.** *Consider the wave kinetic hierarchy* (1.28) *in dimension $d = 3$. Let $T > 0$, $p > 1$, $q > 3$, $\alpha, \beta > 0$ and let $\mu \in \mathbb{R}$ be such that $e^{2\mu} > 32C_{p,q,\alpha,\beta}$, where $C_{p,q,\alpha,\beta}$ is given by* (2.2)*. Consider admissible initial data $F_0 = (f_0^{(k)})_{k=1}^\infty \in \mathcal{A} \cap \mathcal{X}^\infty_{p,q,\alpha,\beta,\mu'}$, where $\mu' = \mu + \ln 2$. Then, there exists a non-negative mild μ-solution $F = (f^{(k)})_{k=1}^\infty$ of the wave kinetic hierarchy* (1.28)*. In addition, this solution satisfies the estimate*

$$|||\mathcal{T}^{-(\cdot)}F(\cdot)|||_{p,q,\alpha,\beta,\mu,T} \leq 1. \tag{3.1}$$

We start by stating the Hewitt-Savage theorem that is tailored to our norms. It is exactly the same as Proposition 4.4. in [6], but we include it here so that the paper is self-contained. Similar versions of this theorem can be found in, for example, [24, Proposition 6.1.3], [23, Theorem 2.6], [43], [17].

Proposition 3.2 (Hewitt-Savage)**.** *Suppose $G = (g^{(k)})_{k=1}^\infty$ is admissible in the sense of Definition 1.7. Then, there exists a unique Borel probability measure π on the set of probability measures $\mathcal{P}$, where*

$$\mathcal{P} = \left\{ h \in L^1(\mathbb{R}^{2d}) : h \geq 0, \quad \int_{\mathbb{R}^{2d}} h(x,v)\, dx\, dv = 1 \right\}, \tag{3.2}$$

such that

$$g^{(k)} = \int_{\mathcal{P}} h^{\otimes k} d\pi(h), \qquad \forall\, k \in \mathbb{N}. \tag{3.3}$$

If additionally $G \in \mathcal{X}^\infty_{p,q,\alpha,\beta,\mu'}$, for some $p, q > 1$, $\alpha, \beta > 0$ and $\mu' \in \mathbb{R}$, then

$$\operatorname{supp}(\pi) \subseteq \left\{ h \in \mathcal{P} : \|h\|_{p,q,\alpha,\beta} \leq e^{-\mu'} \right\}. \tag{3.4}$$

Remark 3.3. *We note that representation* (3.3)*, and the support condition* (3.4) *imply that $\mathcal{A} \cap \mathcal{X}^\infty_{p,q,\alpha,\beta,\mu'} = \mathcal{A} \cap B_{\mathcal{X}^\infty_{p,q,\alpha,\beta,\mu'}}$, where $B_{\mathcal{X}^\infty_{p,q,\alpha,\beta,\mu'}}$ denotes the unit ball of $\mathcal{X}^\infty_{p,q,\alpha,\beta,\mu'}$.*

We are now ready to prove Theorem 3.1 in a similar manner to the proof of the analogous result for the Boltzmann hierarchy (see proof of [6, Theorem 2.1]).

Proof of Theorem 3.1. Let μ be such that $e^{2\mu} > 32C_{p,q,\alpha,\beta}$, let $\mu' = \mu + \ln 2$, and consider an initial datum $F_0 = (f_0^{(k)})_{k=1}^\infty \in \mathcal{A} \cap \mathcal{X}^\infty_{p,q,\alpha,\beta,\mu'}$. By the Hewitt-Savage theorem (Proposition 3.2), there exists a Borel probability measure π on $\mathcal{P}$ such that

$$f_0^{(k)} = \int_{\mathcal{P}} h_0^{\otimes k} d\pi(h_0), \tag{3.5}$$

and

$$\operatorname{supp}(\pi) \subseteq \left\{ h_0 \in \mathcal{P} : \|h_0\|_{p,q,\alpha,\beta} \leq e^{-\mu'} \right\}. \tag{3.6}$$

Therefore, for π-almost any $h_0 \in \mathcal{P}$, we have that $\|h_0\|_{p,q,\alpha,\beta} \leq e^{-\mu'} = \frac{e^{-\mu}}{2}$. Let $M = e^{-\mu}$ and note that then $\|h_0\|_{p,q,\alpha,\beta} \leq \frac{M}{2}$, and that due to the assumption $e^{2\mu} > 32C_{p,q,\alpha,\beta}$, we have that $M = e^{-\mu} < (32C_{p,q,\alpha,\beta})^{-1/2} < (24C_{p,q,\alpha,\beta})^{-1/2}$. Therefore, we can apply Theorem 1.4 with $M = e^{-\mu}$ and initial data h_0 to conclude that there is a mild solution $h(t)$ to the wave kinetic equation corresponding to the initial datum h_0. By (1.24), we have that this solution satisfies

$$||T_1^{-t}h(t)\|_{p,q,\alpha,\beta} \leq 2\|h_0\|_{p,q,\alpha,\beta} \leq e^{-\mu}, \quad \forall\, t \in [0,T]. \tag{3.7}$$

Thanks to the continuity with respect to initial data (1.23), given $t \in [0,T]$, the map $h_0 \mapsto h(t)$ is continuous and thus Borel measurable.

Now we construct an infinite sequence of functions, which we will show will be a mild solution of the wave kinetic hierarchy (1.28). Namely, we define $F = (f^{(k)})_{k=1}^{\infty}$, by

$$f^{(k)}(t) := \int_{\mathcal{P}} h(t)^{\otimes k} d\pi(h_0), \quad t \in [0,T],\ k \in \mathbb{N}. \tag{3.8}$$

For any $k \in \mathbb{N}$ and $t \in [0,T]$, we have

$$\begin{aligned} e^{\mu k}\|T_k^{-t}f^{(k)}(t)\|_{k,p,q,\alpha,\beta} &\leq e^{\mu k}\int_{\mathcal{P}} \|T_k^{-t}h^{\otimes k}(t)\|_{k,p,q,\alpha,\beta} d\pi(h_0) \\ &= e^{\mu k}\int_{\mathcal{P}} \|T_1^{-t}h(t)\|_{p,q,\alpha,\beta}^k d\pi(h_0) \\ &\leq 1, \end{aligned}$$

where the equality follows from (A.3) and the last inequality follows from (3.7). Since this estimate is true for any $k \in \mathbb{N}$ and $t \in [0,T]$, the estimate (3.1) follows, which also implies that $\mathcal{T}^{-(\cdot)}F(\cdot) \in \mathcal{X}^{\infty}_{p,q,\alpha,\beta,\mu,T}$. Now that we know that F is in the right space, a standard computation shows that F, defined via (3.8), is a mild μ-solution to the wave kinetic hierarchy (1.28), corresponding to initial data $F_0 \in \mathcal{X}^{\infty}_{p,q,\alpha,\beta,\mu'} \subset \mathcal{X}^{\infty}_{p,q,\alpha,\beta,\mu}$. This solution is non-negative since $h(t)$ in (3.8) are solutions to the wave kinetic equation with non-negative initial data h_0.

□

3.2. **Proof of uniqueness of mild solutions to the wave kinetic hierarchy.** The goal of this section is to prove uniqueness of solutions to the wave kinetic hierarchy stated as follows:

Theorem 3.4 (Uniqueness of solutions to the wave kinetic hierarchy)**.** *Consider the wave kinetic hierarchy* (1.28) *in dimension $d = 3$. Let $T > 0$ and assume that $p > 1, q > 3$ and $\alpha, \beta > 0$. Let $\mu \in \mathbb{R}$ be such that $e^{2\mu} > 32C_{p,q,\alpha,\beta}$ for $C_{p,q,\alpha,\beta}$ as in* (2.2)*. Let $F_0 = (f_0^{(k)})_{k=1}^{\infty} \in X^{\infty}_{p,q,\alpha,\beta}$ and assume $F = (f^{(k)})_{k=1}^{\infty} \in X^{\infty}_{p,q,\alpha,\beta,T}$ is a mild μ-solution to wave kinetic hierarchy* (1.28) *corresponding to the initial data F_0. Then F is unique.*

Since the wave kinetic hierarchy in linear, it suffices to show that if $F_0 = 0$ and F is a mild solution, then $F = 0$. In order to motivate the proof, let us start by recalling that a mild solution $F = (f^{(k)})_{k=1}^{\infty}$ to the wave kinetic hierarchy (1.28) is given by

$$T_k^{-t}f^{(k)}(t) = f_0^{(k)} + \int_0^t T_k^{-s}\mathfrak{C}^{k+2}f^{(k+2)}(s)ds, \quad \forall k \in \mathbb{N},\ \forall\, t \in [0,T].$$

When initial data is zero ($F_0 = 0$), one can apply this formula iteratively $n \in \mathbb{N}$ times to obtain

$$T_k^{-t} f^{(k)}(t) = \int_0^t \int_0^{t_{k+2}} \cdots \int_0^{t_{k+2n-2}} dt_{k+2n} \cdots dt_{k+4} dt_{k+2}$$
$$T_k^{-t_{k+2}} \mathfrak{C}^{k+2} T_{k+2}^{t_{k+2}-t_{k+4}} \mathfrak{C}^{k+4} \cdots T_{k+2n-2}^{t_{k+2n-2}-t_{k+2n}} \mathfrak{C}^{k+2n} f^{(k+2n)}(t_{k+2n}). \tag{3.9}$$

In order to prove that $F = 0$, it will suffice to show that $\left\|T_k^{-t} f^{(k)}(t)\right\|_{p,q,\alpha,\beta} \lesssim C^n$, where $C < 1$. Then by letting $n \to \infty$ one can conclude that for each $k \in \mathbb{N}$, $f^{(k)} = 0$, thus completing the proof. Recall that each $\mathfrak{C}^{k+2j}$ is a sum of $k+2j-2$ operators, and thus the iterated Duhamel formula (3.9) contains a factorial number of terms. For each of them, we will use an (iterated) a priori estimate (see Proposition 3.6). This fact alone is not enough to obtain the needed geometric growth estimate since the number of terms is factorial. This is where a board game argument inspired by [35, 10] will come into play and will allow us to rearrange the factorial number of terms in (3.9) in an exponential number of equivalence classes, and the sum over each equivalence class will be bounded by a constant, thus allowing us to obtain the geometric growth estimate $\left\|T_k^{-t} f^{(k)}(t)\right\|_{p,q,\alpha,\beta} \lesssim C^n$.

We have organized this section into three subsections. The first one establishes the (iterated) a priori estimate. The second subsection discusses the combinatorial board game argument. Finally, in the third subsection, we combine the a priori estimate and the board game argument to prove the uniqueness of solutions to the wave kinetic hierarchy as stated in Theorem 3.4.

Throughout this section, we will use the following notation

$$u_{i,j} = v_i - v_j. \tag{3.10}$$

Also for any vector $v \in \mathbb{R}^3$, we denote by $\hat{v}$ its unit vector

$$\hat{v} = \frac{v}{|v|}. \tag{3.11}$$

3.2.1. *A priori estimate.* The following estimate is the hierarchy level analogue of the nonlinear a priori estimate presented in Proposition 2.1.

Proposition 3.5. *Let $T > 0$, $p > 1$ and $q > 3$. Let $\mathfrak{C}^{\lambda}_{j,k+2}$, where $\lambda \in \{L_0, L_1, L_2, L_3\}$, be defined in* (1.31)-(1.34)*. Then for any $t \in [0,T]$, $k \in \mathbb{N}$, $j \in \{1, \cdots, k\}$, and $\lambda \in \{L_0, L_1, L_2, L_3\}$ we have*

$$\left\| \int_0^t T_k^{-s} \mathfrak{C}^{\lambda}_{j,k+2} g^{(k+2)}(s)\, ds \right\|_{k,p,q,\alpha,\beta} \leq C_{p,q,\alpha,\beta} |||T_{k+2}^{-(\cdot)} g^{(k+2)}(\cdot)|||_{k+2,p,q,\alpha,\beta,T}, \tag{3.12}$$

where $C_{p,q,\alpha,\beta}$ is the constant appearing in (2.2)*.*

Proof. The proof of the estimate (3.12) is analogous to the proof of the nonlinear estimate (2.1) at the level of the equation. As a result, we will only present the proof for $\lambda = L_0$ to demonstrate this. The other cases are proved in a similar spirit to the proof of Proposition 2.1.

Fix $k \in \mathbb{N}$, $j \in \{1, \cdots, k\}$ and $\lambda = L_0$. Recall notation introduced in (3.10). For any $X_k, V_k \in \mathbb{R}^{3k}$ we have

$$
\begin{aligned}
I_{L_0}(X_k, V_k) &:= \langle\langle \alpha X_k \rangle\rangle^p \langle\langle \beta V_k \rangle\rangle^q \left| \int_0^t T_k^{-s} \mathfrak{C}_{j,k+2}^{L_0} g^{(k+2)}(s, X_k, V_k)\, ds \right| \\
&= \langle\langle \alpha X_k \rangle\rangle^p \langle\langle \beta V_k \rangle\rangle^q \left| \int_0^t [\mathfrak{C}_{j,k+2}^{L_0} g^{(k+2)}](s, X_k + sV_k, V_k) ds \right| \\
&\leq \langle\langle \alpha X_k \rangle\rangle^p \langle\langle \beta V_k \rangle\rangle^q \int_0^t \int_{\mathbb{R}^9} dv_{k+1}\, dv_{k+2}\, dv_{k+3} ds\ \delta(\Sigma_{j,k+2}) \delta(\Omega_{j,k+2}) \\
&\qquad \times \left| g^{(k+2)}(s, X_k + sV_k, x_j + sv_j, x_j + sv_j,\ V_k^{j,v_{k+1}}, v_{k+2}, v_{k+3}) \right| \\
&= \langle\langle \alpha X_k \rangle\rangle^p \langle\langle \beta V_k \rangle\rangle^q \int_0^t \int_{\mathbb{R}^9} dv_{k+1}\, dv_{k+2}\, dv_{k+3}\, ds\ \delta(\Sigma_{j,k+2}) \delta(\Omega_{j,k+2}) \\
&\qquad \left| T_{k+2}^{-s} g^{(k+2)}(s, X_k + s(V_k - V_k^{j,v_{k+1}}),\ x_j + su_{j,k+2},\ x_k + su_{j,k+3},\ V_k^{j,v_{k+1}}, v_{k+2}, v_{k+3}) \right| \\
&\leq \langle \alpha x_j \rangle^p |||T_{k+2}^{-(\cdot)} g^{(k+2)}(\cdot)|||_{k+2,p,q,\alpha,\beta,T} \int_{\mathbb{R}^9} \delta(\Sigma_{j,k+2}) \delta(\Omega_{j,k+2}) \frac{\langle \beta v_j \rangle^q}{\langle \beta v_{k+1} \rangle^q \langle \beta v_{k+2} \rangle^q \langle \beta v_{k+3} \rangle^q} \\
&\qquad \times \left(\int_0^t \langle \alpha x_j + \alpha s u_{j,k+2} \rangle^{-p} \langle \alpha x_j + \alpha s u_{j,k+3} \rangle^{-p}\, ds \right) dv_{k+1}\, dv_{k+2}\, dv_{k+3},
\end{aligned} \tag{3.13}
$$

where in the last inequality we used that $\langle \alpha x_j + \alpha s u_{j,k+1} \rangle^{-p} \leq 1$.

Now, by (A.6), (A.9) from Lemma A.3, on the resonant manifold determined by $\Sigma_{j,k+2}$ and $\Omega_{j,k+2}$ we have

$$u_{j,k+2} \cdot u_{j,k+3} = 0,$$

as well as

$$\min\{|u_{j,k+2}|, |u_{j,k+3}|\} \geq \frac{|u_{j,k+1}|}{2} \sqrt{1 - (\hat{u}_{j,k+1} \cdot \hat{u}_{k+2,k+3})^2}. \tag{3.14}$$

Applying Lemma A.5 for $x = \alpha x_j$, $\xi = \alpha u_{j,k+2}$, $\eta = \alpha u_{j,k+3}$, and using the above estimate (3.14), yields

$$\int_0^t \langle \alpha x_j + \alpha s u_{j,k+2} \rangle^{-p} \langle \alpha x_j + \alpha s u_{j,k+3} \rangle^{-p}\, ds \leq \frac{8p}{\alpha(p-1)} \frac{\langle \alpha x_j \rangle^{-p}}{|u_{j,k+1}| \sqrt{1 - (\hat{u}_{j,k+1} \cdot \hat{u}_{k+2,k+3})^2}}. \tag{3.15}$$

Combining (3.13) and (3.15), we obtain

$$
\begin{aligned}
I_{L_0}(X_k, V_K) \leq \frac{8p}{\alpha(p-1)} |||T_{k+2}^{-(\cdot)} g^{(k+2)}(\cdot)|||_{k+2,p,q,\alpha,\beta,T} \int_{\mathbb{R}^9} dv_{k+1}\, dv_{k+2}\, dv_{k+3}\ \delta(\Sigma_{j,k+2}) \delta(\Omega_{j,k+2}) \\
\times \frac{1}{|u_{j,k+1}| \sqrt{1 - (\hat{u}_{j,k+1} \cdot \hat{u}_{k+2,k+3})^2}} \frac{\langle \beta v_j \rangle^q}{\langle \beta v_{k+1} \rangle^q \langle \beta v_{k+2} \rangle^q \langle \beta v_{k+3} \rangle^q}.
\end{aligned}
$$

Finally, we can use that

$$\frac{\langle \beta v_j \rangle^q}{\langle \beta v_{k+1} \rangle^q \langle \beta v_{k+2} \rangle^q \langle \beta v_{k+3} \rangle^q} \leq \max\{\beta^q, \beta^{-3q}\} \frac{\langle v_j \rangle^q}{\langle v_{k+1} \rangle^q \langle v_{k+2} \rangle^q \langle v_{k+3} \rangle^q},$$

together with Lemma A.9, to obtain

$$I_{L_0}(X_k, V_K) \leq \frac{8p}{\alpha(p-1)}\ 2\pi^3 \left(\frac{1}{3} + \frac{1}{q-3}\right)\ \max\{\beta^q, \beta^{-3q}\}\ |||T_{k+2}^{-(\cdot)} g^{(k+2)}(\cdot)|||_{k+1,p,q,\alpha,\beta,T}.$$

Since t, X_k, V_k were arbitrary, estimate (3.12) follows. □

We next derive an iterated estimate by recursively applying Proposition 3.5. We start by recalling the notation introduced in (1.38)-(1.39).

Proposition 3.6. *Let $T > 0$, $p > 1, q > 3$ and $k, n \in \mathbb{N}$ with $n \geq 2$. For each $\ell \in \{1, 2, \ldots, n\}$, let $j_{k+2\ell}$ be a number in the set $\{1, 2, \ldots, k + 2\ell - 2\}$ and let $\pi_{k+2\ell} \in \{+, -\}$. Then we have*

$$\begin{aligned} &\Bigg\| \int_{[0,T]^n} T_k^{-t_{k+2}} \mathfrak{C}^{\pi_{k+2}}_{j_{k+2},k+2} T_{k+2}^{t_{k+2}-t_{k+4}} \mathfrak{C}^{\pi_{k+4}}_{j_{k+4},k+4} \cdots \\ &\qquad \cdots T_{k+2n-2}^{t_{k+2n-2}-t_{k+2n}} \mathfrak{C}^{\pi_{k+2n}}_{j_{k+2n},k+2n} f^{(k+2n)}(t_{k+2n})\ dt_{k+2n} \cdots dt_{k+4} dt_{k+2} \Bigg\|_{k,p,q,\alpha,\beta} \\ &\leq (2C_{p,q,\alpha,\beta})^n \left|\left|\left| T_{k+2n}^{-(\cdot)} f^{(k+2n)} \right|\right|\right|_{k+2n,p,q,\alpha,\beta,T}, \end{aligned} \tag{3.16}$$

where $C_{p,q,\alpha,\beta}$ is given in (2.2).

Proof. Let

$$\begin{aligned} I :=& \Bigg\| \int_{[0,T]^n} T_k^{-t_{k+2}} \mathfrak{C}^{\pi_{k+2}}_{j_{k+2},k+2} T_{k+2}^{t_{k+2}-t_{k+4}} \mathfrak{C}^{\pi_{k+4}}_{j_{k+4},k+4} \cdots \\ &\qquad \cdots T_{k+2n-2}^{t_{k+2n-2}-t_{k+2n}} \mathfrak{C}^{\pi_{k+2n}}_{j_{k+2n},k+2n} f^{(k+2n)}(t_{k+2n})\ dt_{k+2n} \cdots dt_{k+4} dt_{k+2} \Bigg\|_{k,p,q,\alpha,\beta} \\ =& \left\| \int_0^T T_k^{-t_{k+2}} \mathfrak{C}^{\pi_{k+2}}_{j_{k+2},k+2}\ G^{(k+2)}(t_{k+2})\ dt_{k+2} \right\|_{k,p,q,\alpha,\beta} \end{aligned}$$

where

$$\begin{aligned} G^{(k+2)}(t_{k+2}) = \int_{[0,T]^{n-1}} & dt_{k+2n} \cdots dt_{k+4} \\ & T_{k+2}^{t_{k+2}-t_{k+4}} \mathfrak{C}^{\pi_{k+4}}_{j_{k+4},k+4} \cdots T_{k+2n-2}^{t_{k+2n-2}-t_{k+2n}} \mathfrak{C}^{\pi_{k+2n}}_{j_{k+2n},k+2n} f^{(k+2n)}(t_{k+2n}). \end{aligned}$$

By Proposition 3.5, we have

$$\begin{aligned}
I &\leq 2C_{p,q,\alpha,\beta}\left|\left|\left|T_{k+2}^{-(\cdot)}G^{(k+2)}\right|\right|\right|_{k+2,p,q,\alpha,\beta,T} \\
&= 2C_{p,q,\alpha,\beta}\sup_{t_{k+2}\in[0,T]}\Bigg\| T_{k+2}^{-t_{k+2}}\int_{[0,T]^{n-1}} dt_{k+2n}\cdots dt_{k+4} \\
&\qquad T_{k+2}^{t_{k+2}-t_{k+4}}\mathfrak{C}^{\pi_{k+4}}_{j_{k+4},k+4}\cdots T_{k+2n-2}^{t_{k+2n-2}-t_{k+2n}}\mathfrak{C}^{\pi_{k+2n}}_{j_{k+2n},k+2n}f^{(k+2n)}(t_{k+2n})\Bigg\|_{k+2,p,q,\alpha,\beta} \\
&= 2C_{p,q,\alpha,\beta}\Bigg\|\int_{[0,T]^{n-1}} dt_{k+2n}\cdots dt_{k+4} \\
&\qquad T_{k+2}^{-t_{k+4}}\mathfrak{C}^{\pi_{k+4}}_{j_{k+4},k+4}\cdots T_{k+2n-2}^{t_{k+2n-2}-t_{k+2n}}\mathfrak{C}^{\pi_{k+2n}}_{j_{k+2n},k+2n}f^{(k+2n)}(t_{k+2n})\Bigg\|_{k+2,p,q,\alpha,\beta}.
\end{aligned}$$

Repeating these calculations $(n-1)$ more times, we obtain

$$I \leq (2C_{p,q,\alpha,\beta})^n\left|\left|\left|T_{k+2n}^{-(\cdot)}f^{(k+2n)}\right|\right|\right|_{k+2n,p,q,\alpha,\beta,T}.$$

□

3.2.2. *Combinatorial board game argument.* In this section, inspired by [35] and [10], we devise a combinatorial board game argument which, together with the a priori estimates from the previous section, will allow us to prove uniqueness of solutions to the wave kinetic hierarchy (1.28). As indicated earlier in this section, the wave kinetic hierarchy is linear, and thus uniqueness boils down to proving that the solution corresponding to zero initial data is zero. Recall that such a solution has the expansion (3.9), and that the integrand in the iterated time integral has a factorial number of terms since each operator $\mathfrak{C}^{k+2\ell}$ is a sum of $k+2\ell-2$ operators. The goal of this section is to find a way to reorganize the integral (3.9) in such a way that allows us to estimate it with an exponential number of terms instead. In order to achieve this, we start by introducing some notation, starting with the notation for the integrand in (3.9).

Definition 3.7. *Let $k,n\in\mathbb{N}$ with $n\geq 2$, and let $\underline{t}_{k+2n}=(t_{k+2},t_{k+4},\dots,t_{k+2n})\in\mathbb{R}^n_+$. We define the operator $J_{n,k}$ by*

$$J_{n,k}(\underline{t}_{k+2n})f^{(k+2n)} = T_k^{-t_{k+2}}\mathfrak{C}^{k+2}T_{k+2}^{t_{k+2}-t_{k+4}}\mathfrak{C}^{k+4}\cdots T_{k+2n-2}^{t_{k+2n-2}-t_{k+2n}}\mathfrak{C}^{k+2n}f^{(k+2n)}(t_{k+2n}). \tag{3.17}$$

By (1.29), each $\mathfrak{C}^{k+2\ell}$ is a sum of $k+2\ell-2$ terms, so we can write the $J_{n,k}$ operator as a sum

$$J_{n,k}(\underline{t}_{k+2n}) = \sum_{\mu\in M_{n,k}} J_{n,k}(\underline{t}_{k+2n},\mu), \tag{3.18}$$

where

$$M_{n,k} = \Big\{\mu:\{k+2,k+4,\dots,k+2n\}\to\{1,2,\dots,k+2n-2\},\ \text{with } \forall j\ \mu(j)<j-1\Big\} \tag{3.19}$$

and

$$\begin{aligned}
J_{n,k}(\underline{t}_{k+2n},\mu)f^{(k+2n)} := T_k^{-t_{k+2}}&\mathfrak{C}_{\mu(k+2),k+2}T_{k+2}^{t_{k+2}-t_{k+4}}\mathfrak{C}_{\mu(k+4),k+4}\cdots \\
&\cdots T_{k+2n-2}^{t_{k+2n-2}-t_{k+2n}}\mathfrak{C}_{\mu(k+2n),k+2n}f^{(k+2n)}(t_{k+2n}).
\end{aligned} \tag{3.20}$$

We next define operator $\mathcal{I}_{n,k}$ as a time integral of the operator $J_{n,k}$. In what follows, we use the following notation for the group of all permutations of elements $\{k+2,\dots,k+2n\}$:

$$S_{n,k} = S(\{k+2,\dots,k+2n\}). \tag{3.21}$$

Definition 3.8. *Let $k,n \in \mathbb{N}$ with $n \geq 2$. For each $(\mu,\sigma) \in M_{n,k} \times S_{n,k}$ define the operator $\mathcal{I}_{n,k}(\mu,\sigma)$ by the expression*

$$\mathcal{I}_{n,k}(\mu,\sigma) := \int_{t \geq t_{\sigma(k+2)} \geq t_{\sigma(k+4)} \geq \cdots \geq t_{\sigma(k+2n)} \geq 0} J_{n,k}(\underline{t}_{k+2n};\mu) dt_{k+2n} dt_{k+2n-2} \dots dt_{k+2}. \tag{3.22}$$

Note that it is equivalent to write

$$\mathcal{I}_{n,k}(\mu,\sigma) = \int_{t \geq t_{k+2} \geq t_{k+4} \geq \cdots \geq t_{k+2n} \geq 0} J_{n,k}(\sigma^{-1}(\underline{t}_{k+2n});\mu) dt_{k+2n} dt_{k+2n-2} \dots dt_{k+2}, \tag{3.23}$$

where

$$\sigma^{-1}(\underline{t}_{k+2n}) := (t_{\sigma^{-1}(k+2)},\dots,t_{\sigma^{-1}(k+2n)}). \tag{3.24}$$

The operator $\mathcal{I}_{n,k}$ can be represented on a $(k+2n-2)\times n$ board with carved in names $\mathfrak{C}_{i,j}$, with $1 \leq i \leq j-2$, for $j \in \{k+2, k+4, ..., k+2n\}$, arranged as in (3.25). The board also has an extra top row that keeps track of the order of times in the operator $\mathcal{I}_{n,k}$. For each $(\mu,\sigma) \in M_{n,k} \times S_{n,k}$ we associate a state of the game, where μ determines which $\mathfrak{C}$ operators on the board are circled, and σ determines the order of the times in the top row.

$$\begin{bmatrix}
t_{\sigma^{-1}(k+2)} & t_{\sigma^{-1}(k+4)} & \dots\dots & t_{\sigma^{-1}(k+2n)} & \\
 & & & & \\
\mathfrak{C}_{1,k+2} & \mathfrak{C}_{1,k+4} & \dots\dots & \mathfrak{C}_{1,k+2n} & \text{row } 1 \\
\mathfrak{C}_{2,k+2} & \mathfrak{C}_{2,k+4} & \dots\dots & \mathfrak{C}_{2,k+2n} & \text{row } 2 \\
\dots & \dots & \dots\dots & \dots & \dots \\
\dots & \mathfrak{C}_{\mu(k+4),k+4} & \dots\dots & \dots & \dots \\
\dots & \dots & \dots\dots & \dots & \dots \\
\mathfrak{C}_{\mu(k+2),k+2} & \dots & \dots\dots & \dots & \dots \\
\dots & \dots & \dots\dots & \dots & \dots \\
\mathfrak{C}_{k,k+2} & \mathfrak{C}_{k,k+4} & \dots\dots & \dots & \dots \\
0 & \mathfrak{C}_{k+1,k+4} & \dots\dots & \mathfrak{C}_{\mu(k+2n),k+2n} & \dots \\
0 & \mathfrak{C}_{k+2,k+4} & \dots\dots & \dots & \dots \\
\dots & 0 & \dots\dots & \dots & \dots \\
\dots & \dots & \dots\dots & \dots & \dots \\
0 & 0 & \dots\dots & \mathfrak{C}_{k+2n-2,k+2n} & \text{row } k+2n-3 \\
0 & 0 & \dots\dots & \mathfrak{C}_{k+2n-2,k+2n} & \text{row } k+2n-2 \\
 & & & & \\
\text{col } k+2 & \text{col } k+4 & \dots\dots & \text{col } k+2n &
\end{bmatrix}. \tag{3.25}$$

The strategy is to define an "acceptable move" on the board which will allow us to move the circles in such a way that the value of the iterated Duhamel integral (3.22) is invariant. Ultimately, this will enable us to define an equivalence relation between integrals of the type (3.22), and the sum over all integrals in the same equivalence class will be estimated by a single time integral (see Proposition 3.17), while the number of equivalence classes will be exponential (see Proposition 3.41), thus resolving the issue of the factorial number of terms.

Inspired by [10] we define an acceptable move as follows.

Definition 3.9 (Acceptable move). *Let $k, n \in \mathbb{N}$ with $n \geq 2$, and let $M_{k,n}$ and $S_{n,k}$ be defined as in* (3.19) *and* (3.21)*. Suppose $(\mu, \sigma) \in M_{n,k} \times S_{n,k}$ is a state of the game for which one has $\mu(j+2) < \mu(j)$ for some $j \in \{k+2, k+4, \dots, k+2n-2\}$. An acceptable move transforms (μ, σ) to (μ', σ'), where*

$$\begin{aligned} \mu' &= (j-1, j+1) \circ (j, j+2) \circ \mu \circ (j, j+2), \\ \sigma' &= (j, j+2) \circ \sigma. \end{aligned} \tag{3.26}$$

Note that under an acceptable move, due to $\mu(j+2) < \mu(j) < j-1$ and $\mu(j-1) < j-2$, we have

$$\mu'(\ell) = \begin{cases} (j-1, j+1) \circ (j, j+2) \circ \mu(\ell) & \text{for } \ \ell \in \{k+2, k+4, \dots k+2n\} \setminus \{j, j+2\}, \\ \mu(j+2) & \text{for } \ \ell = j, \\ \mu(j) & \text{for } \ \ell = j+2, \end{cases} \tag{3.27}$$

and

$$\sigma'^{-1}(\ell) = \begin{cases} \sigma^{-1}(\ell) & \text{for } \ \ell \in \{k+2, k+4, \dots k+2n\} \setminus \{j, j+2\}, \\ \sigma^{-1}(j+2) & \text{for } \ \ell = j, \\ \sigma^{-1}(j) & \text{for } \ \ell = j+2. \end{cases} \tag{3.28}$$

Therefore, the effect of an acceptable move to the board is:

- it exchanges positions of times in column j and column $j+2$ (due to σ'), and
- it exchanges positions of circles in column j and column $j+2$ (due to μ'), and
- it exchanges positions of circles in row j and row $j+2$ if such rows exist (due to μ') (if one of those rows does not exist, no changes are made at the level of rows), and
- it exchanges positions of circles in row $j-1$ and row $j+1$ if such rows exist (due to μ') (if one of those rows does not exist, no changes are made at the level of rows).

Before we show that an acceptable move doesn't change the value of the integral $\mathcal{I}_{n,k}(\mu, \sigma)$, let us introduce the following operator.

Definition 3.10 (Operator $S_{j,j+2}$). *We define $S_{j,j+2}$ to be an operator that exchanges x variables in positions $j-1, j$ with x variables in positions $j+1, j+2$, and exchanges v variables in positions $j-1, j$ with v variables in positions $j+1, j+2$. In other words, for $\ell > j+1$, we define*

$$\begin{aligned} \Big[S_{j,j+2} f^{(\ell)}\Big](X_\ell, V_\ell) := f^{(\ell)}(&x_1, \dots, x_{j-2}, x_{j+1}, x_{j+2}, x_{j-1}, x_j, x_{j+3}, \dots, x_\ell; \\ &v_1, \dots, v_{j-2}, v_{j+1}, v_{j+2}, v_{j-1}, v_j, v_{j+3}, \dots, v_\ell), \end{aligned} \tag{3.29}$$

Now we are ready to state and prove the invariance of integrals $\mathcal{I}_{n,k}(\mu, \sigma)$ under acceptable moves.

Proposition 3.11 (Acceptable move invariance). *Suppose (μ, σ) and (μ', σ') are as in Definition 3.9. Then $(\mu', \sigma') \in M_{n,k} \times S_{n,k}$ and*

$$\mathcal{I}_{n,k}(\mu, \sigma) = \mathcal{I}_{n,k}(\mu', \sigma').$$

Proof. Suppose j is as in Definition 3.9, that is, suppose $j \in \{k+2, k+4, \ldots, k+2n-2\}$ is such that $\mu(j+2) < \mu(j)$. We first write what $\mathcal{I}_{n,k}(\mu', \sigma') f^{(k+2n)}$ is:

$$\begin{aligned}
\mathcal{I}_{n,k}(\mu', \sigma') f^{(k+2n)} &= \int_{t_k \geq t_{k+2} \geq t_{k+4} \geq \cdots \geq t_{k+2n} \geq 0} J_{n,k}(\sigma'^{-1}(\underline{t}_{k+2n}); \mu')\, dt_{k+2n} \ldots dt_{k+4} dt_{k+2} \\
&= \int_{t_k \geq t_{k+2} \geq t_{k+4} \geq \cdots \geq t_{k+2n} \geq 0} T_k^{-t_{\sigma'^{-1}(k+2)}} \mathfrak{C}_{\mu'(k+2),k+2}\, T_{k+2}^{t_{\sigma'^{-1}(k+2)} - t_{\sigma'^{-1}(k+4)}} \mathfrak{C}_{\mu'(k+4),k+4} \\
&\qquad \ldots T_{j-2}^{t_{\sigma'^{-1}(j-2)} - t_{\sigma'^{-1}(j)}} \mathfrak{C}_{\mu'(j),j}\, T_j^{t_{\sigma'^{-1}(j)} - t_{\sigma'^{-1}(j+2)}} \mathfrak{C}_{\mu'(j+2),j+2}\, T_{j+2}^{t_{\sigma'^{-1}(j+2)} - t_{\sigma'^{-1}(j+4)}} \ldots \\
&\qquad \cdots T_{k+2n-2}^{t_{\sigma'^{-1}(k+2n-2)} - t_{\sigma'^{-1}(k+2n)}} \mathfrak{C}_{\mu'(k+2n),k+2n}\, f^{(k+2n)}(t_{\sigma'^{-1}(k+2n)}) dt_{k+2n} \ldots dt_{k+2}.
\end{aligned} \tag{3.30}$$

According to properties (3.27) and (3.28), for the operators appearing before T_{j-2} in (3.30), each μ' and σ'^{-1} can be replaced by μ and σ^{-1}, respectively. For operators appearing after T_{j+2} in (3.30) one can drop primes from σ'^{-1} and turn μ' to $(j-1, j+1) \circ (j, j+2) \circ \mu$. Finally, for the operators appearing between and including T_{j-2} and T_{j+2} in (3.30), the evaluation of σ'^{-1} and μ' at j (resp. $j+2$) turns into an evaluation of σ^{-1} and μ at $j+2$ (resp. j). For all other σ'^{-1}, one can drop the prime. Therefore, we can rewrite $\mathcal{I}_{n,k}(\mu', \sigma')$ in terms of μ and σ as follows

$$\begin{aligned}
\mathcal{I}_{n,k}(\mu', \sigma') f^{(k+2n)} &= \int_{t_k \geq t_{k+2} \geq t_{k+4} \geq \cdots \geq t_{k+2n} \geq 0} T_k^{-t_{\sigma^{-1}(k+2)}} \mathfrak{C}_{\mu(k+2),k+2}\, T_{k+2}^{t_{\sigma^{-1}(k+2)} - t_{\sigma^{-1}(k+4)}} \mathfrak{C}_{\mu(k+4),k+4} \\
&\qquad \ldots T_{j-2}^{t_{\sigma^{-1}(j-2)} - t_{\sigma^{-1}(j+2)}} \mathfrak{C}_{\mu(j+2),j}\, T_j^{t_{\sigma^{-1}(j+2)} - t_{\sigma^{-1}(j)}} \mathfrak{C}_{\mu(j),j+2}\, T_{j+2}^{t_{\sigma^{-1}(j)} - t_{\sigma^{-1}(j+4)}} \ldots \\
&\qquad \cdots T_{k+2n-2}^{t_{\sigma^{-1}(k+2n-2)} - t_{\sigma^{-1}(k+2n)}} \mathfrak{C}_{(j-1,j+1)\circ(j,j+2)\circ\mu(k+2n),k+2n}\, f^{(k+2n)}(t_{\sigma^{-1}(k+2n)}) dt_{k+2n} \ldots dt_{k+2}.
\end{aligned}$$

In order to complete the proof of the proposition, it suffices to show two identities:

$$T_{j-2}^{a-b} \mathfrak{C}_{\alpha,j} T_j^{b-c} \mathfrak{C}_{\beta,j+2} T_{j+2}^{c-d} = T_{j-2}^{a-c} \mathfrak{C}_{\beta,j} T_j^{c-b} \mathfrak{C}_{\alpha,j+2} T_{j+2}^{b-d} S_{j,j+2}, \tag{3.31}$$

$$S_{j,j+2} \mathfrak{C}_{(j-1,j+1)\circ(j,j+2)\circ\mu(\ell),\ell} = \mathfrak{C}_{\mu(\ell),\ell}\, S_{j,j+2}. \tag{3.32}$$

Namely, if these two identities are true, then an application of the identity (3.31) with $a = t_{\sigma^{-1}(j-2)}$, $b = t_{\sigma^{-1}(j+2)}$, $c = t_{\sigma^{-1}(j)}$, $\alpha = \mu(j+2)$, $\beta = \mu(j)$ one would get

$$\begin{aligned}
\mathcal{I}_{n,k}(\mu', \sigma') f^{(k+2n)} &= \int_{t_k \geq t_{k+2} \geq t_{k+4} \geq \cdots \geq t_{k+2n} \geq 0} T_k^{-t_{\sigma^{-1}(k+2)}} \mathfrak{C}_{\mu(k+2),k+2}\, T_{k+2}^{t_{\sigma^{-1}(k+2)} - t_{\sigma^{-1}(k+4)}} \mathfrak{C}_{\mu(k+4),k+4} \\
&\qquad \ldots T_{j-2}^{t_{\sigma^{-1}(j-2)} - t_{\sigma^{-1}(j)}} \mathfrak{C}_{\mu(j),j}\, T_j^{t_{\sigma^{-1}(j)} - t_{\sigma^{-1}(j+2)}} \mathfrak{C}_{\mu(j+2),j+2}\, T_{j+2}^{t_{\sigma^{-1}(j+2)} - t_{\sigma^{-1}(j+4)}} S_{j,j+2} \ldots \\
&\qquad \cdots T_{k+2n-2}^{t_{\sigma^{-1}(k+2n-2)} - t_{\sigma^{-1}(k+2n)}} \mathfrak{C}_{(j-1,j+1)\circ(j,j+2)\circ\mu(k+2n),k+2n}\, f^{(k+2n)}(t_{\sigma^{-1}(k+2n)}) dt_{k+2n} \ldots dt_{k+2}.
\end{aligned}$$

Then by using the identity (3.32) iteratively, the fact that $S_{j,j+2}$ commutes with translation operators and that f^{k+2n} is a symmetric function, one would be able to conclude that $\mathcal{I}_{n,k}(\mu', \sigma') = \mathcal{I}_{n,k}(\mu, \sigma)$. Thus it remains to prove identities (3.31) and (3.32), which will be done the next two lemmata. ☐

We first prove identity (3.32).

Lemma 3.12. *For $\ell > j+2$ and for any $\lambda \in \{L_0, L_1, L_2, L_3\}$, we have*

$$S_{j,j+2} \mathfrak{C}^{\lambda}_{(j-1,j+1)\circ(j,j+2)\circ\mu(\ell),\ell} = \mathfrak{C}^{\lambda}_{\mu(\ell),\ell}\, S_{j,j+2}. \tag{3.33}$$

Proof of Lemma 3.12. We will present the proof for $\lambda = L_0$ and $\lambda = L_1$. The cases $\lambda = L_2$ and $\lambda = L_3$ are done analogously to $\lambda = L_1$. Throughout the proof, we will use notation (1.35):

$$\Sigma_{p,q} = v_p + v_{q-1} - v_q - v_{q+1}, \tag{3.34}$$

$$\Omega_{p,q} = |v_p|^2 + |v_{q-1}|^2 - |v_q|^2 - |v_{q+2}|^2. \tag{3.35}$$

• We first prove (3.33) with $\lambda = L_0$ by considering three cases.

Case 1: assume that $\ell > j+2$ and $\mu(\ell) \notin \{j-1, j, j+1, j+2\}$. Then $(j-1, j+1) \circ (j, j+2) \circ \mu(\ell) = \mu(\ell)$, and so for any non-negative measurable function $f^{(\ell)}$ we have

$$\begin{aligned}
&[S_{j,j+2}\mathfrak{C}^{L_0}_{(j-1,j+1)\circ(j,j+2)\circ\mu(\ell),\ell} f^{(\ell)}](t, X_{\ell-2}, V_{\ell-2}) = [S_{j,j+2}\mathfrak{C}^{L_0}_{\mu(\ell),\ell} f^{(\ell)}](t, X_{\ell-2}, V_{\ell-2}) \\
&= [\mathfrak{C}^{L_0}_{\mu(\ell),\ell} f^{(\ell)}](x_1, \dots, x_{j+1}, x_{j+2}, x_{j-1}, x_j, \dots, x_{\ell-2};\ v_1, \dots, v_{j+1}, v_{j+2}, v_{j-1}, v_j, \dots, v_{\ell-2}) \\
&= \int_{\mathbb{R}^{3d}} dv_{\ell-1} dv_\ell dv_{\ell+1}\, \delta(\Sigma_{\mu(\ell),\ell})\, \delta(\Omega_{\mu(\ell),\ell}) \\
&\qquad \Big(f^{(\ell)}(x_1, \dots, x_{j+1}, x_{j+2}, x_{j-1}, x_j, \dots\dots\dots, x_{\ell-2}, x_{\mu(\ell)}, x_{\mu(\ell)}; \\
&\qquad\qquad v_1, \dots, v_{j+1}, v_{j+2}, v_{j-1}, v_j, \dots, \underbrace{v_{\ell-1}}_{\mu(\ell)-\mathrm{th}}, \dots, v_{\ell-2}, v_\ell, v_{\ell+1}) \\
&= \int_{\mathbb{R}^{3d}} dv_{\ell-1} dv_\ell dv_{\ell+1}\, \delta(\Sigma_{\mu(\ell),\ell})\, \delta(\Omega_{\mu(\ell),\ell}) [S_{j,j+2} f^{(\ell)}](X_{\ell-2}, x_{\mu(\ell)}, x_{\mu(\ell)};\ V^{\mu(\ell), v_{\ell-1}}_{\ell-2}, v_\ell, v_{\ell+1}) \\
&= [\mathfrak{C}^{L_0}_{\mu(\ell),\ell} S_{j,j+2} f^{(\ell)}](t, X_{\ell-2}, V_{\ell-2}).
\end{aligned}$$

Case 2: assume that $\ell > j+2$ and $\mu(\ell) \in \{j, j+2\}$. Without loss of generality, $\mu(\ell) = j$. Then $(j-1, j+1) \circ (j, j+2) \circ \mu(\ell) = j+2$, and so for any non-negative measurable $f^{(\ell)}$ we have

$$\begin{aligned}
&[S_{j,j+2}\mathfrak{C}^{L_0}_{(j-1,j+1)\circ(j,j+2)\circ\mu(\ell),\ell} f^{(\ell)}](t, X_{\ell-2}, V_{\ell-2}) = [S_{j,j+2}\mathfrak{C}^{L_0}_{j+2,\ell} f^{(\ell)}](t, X_{\ell-2}, V_{\ell-2}) \\
&= [\mathfrak{C}^{L_0}_{j+2,\ell} f^{(\ell)}](x_1, \dots, x_{j+1}, x_{j+2}, x_{j-1}, \underbrace{x_j}_{(j+2)\mathrm{nd}}, \dots, x_{\ell-2};\ v_1, \dots, v_{j+1}, v_{j+2}, v_{j-1}, \underbrace{v_j}_{(j+2)\mathrm{nd}}, \dots, v_{\ell-2}) \\
&= \int_{\mathbb{R}^{3d}} dv_{\ell-1} dv_\ell dv_{\ell+1}\, \delta(\Sigma_{j,\ell})\, \delta(\Omega_{j,\ell}) \\
&\qquad \Big(f^{(\ell)}(x_1, \dots, x_{j+1}, x_{j+2}, x_{j-1}, x_j, \dots\dots, x_{\ell-2}, x_j, x_j; \\
&\qquad\qquad v_1, \dots, v_{j+1}, v_{j+2}, v_{j-1}, \underbrace{v_{\ell-1}}_{(j+2)\mathrm{nd}}, \dots, v_{\ell-2}, v_\ell, v_{\ell+1}) \\
&= \int_{\mathbb{R}^{3d}} dv_{\ell-1} dv_\ell dv_{\ell+1}\, \delta(\Sigma_{j,\ell})\, \delta(\Omega_{j,\ell}) [S_{j,j+2} f^{(\ell)}](X_{\ell-2}, x_j, x_j;\ V^{j, v_{\ell-1}}_{\ell-2}, v_\ell, v_{\ell+1}) \\
&= [\mathfrak{C}^{L_0}_{j,\ell} S_{j,j+2} f^{(\ell)}](t, X_{\ell-2}, V_{\ell-2}) = [\mathfrak{C}^{L_0}_{\mu(\ell),\ell} S_{j,j+2} f^{(\ell)}](t, X_{\ell-2}, V_{\ell-2}).
\end{aligned}$$

Case 3: assume that $\ell > j+2$ and $\mu(\ell) \in \{j-1, j+1\}$. Without loss of generality, $\mu(\ell) = j-1$. Then $(j-1, j+1) \circ (j, j+2) \circ \mu(\ell) = j+1$, and so for any non-negative measurable $f^{(\ell)}$ we have

$$
\begin{aligned}
&[S_{j,j+2}\mathfrak{C}^{L_0}_{(j-1,j+1)\circ(j,j+2)\circ\mu(\ell),\ell}f^{(\ell)}](t,X_{\ell-2},V_{\ell-2}) = [S_{j,j+2}\mathfrak{C}^{L_0}_{j+1,\ell}f^{(\ell)}](t,X_{\ell-2},V_{\ell-2})\\
&= [\mathfrak{C}^{L_0}_{j+1,\ell}f^{(\ell)}](x_1,\dots,x_{j+1},x_{j+2},\underbrace{x_{j-1}}_{(j+1)\text{st}},x_j,\dots,x_{\ell-2};\ v_1,\dots,v_{j+1},v_{j+2},\underbrace{v_{j-1}}_{(j+1)\text{st}},v_j,\dots,v_{\ell-2})\\
&= \int_{\mathbb{R}^{3d}} dv_{\ell-1}dv_\ell dv_{\ell+1}\,\delta(\Sigma_{j-1,\ell})\,\delta(\Omega_{j-1,\ell})\\
&\qquad \Big(f^{(\ell)}(x_1,\dots,x_{j+1},x_{j+2},x_{j-1},x_j,\dots,x_{\ell-2},x_{j-1},x_{j-1};\\
&\qquad\qquad v_1,\dots,v_{j+1},v_{j+2},\underbrace{v_{\ell-1}}_{(j+1)\text{st}},v_j,\dots,v_{\ell-2},v_\ell,v_{\ell+1}\Big)\\
&= \int_{\mathbb{R}^{3d}} dv_{\ell-1}dv_\ell dv_{\ell+1}\,\delta(\Sigma_{j-1,\ell})\,\delta(\Omega_{j-1,\ell})[S_{j,j+2}f^{(\ell)}](X_{\ell-2},x_{j-1},x_{j-1};\ V_{\ell-2}^{j-1,v_{\ell-1}},v_\ell,v_{\ell+1})\\
&= [\mathfrak{C}^{L_0}_{j-1,\ell}S_{j,j+2}f^{(\ell)}](t,X_{\ell-2},V_{\ell-2}) = [\mathfrak{C}^{L_0}_{\mu(\ell),\ell}S_{j,j+2}f^{(\ell)}](t,X_{\ell-2},V_{\ell-2}).
\end{aligned}
$$

- We next prove (3.32) with $\lambda = L_1$ by again considering three cases.

Case 1: assume that $\ell > j+2$ and $\mu(\ell)\notin\{j-1,j,j+1,j+2\}$. Then, $(j-1,j+1)\circ(j,j+2)\circ\mu(\ell) = \mu(\ell)$, and so for any non-negative measurable function $f^{(\ell)}$ we have

$$
\begin{aligned}
&[S_{j,j+2}\mathfrak{C}^{L_1}_{(j-1,j+1)\circ(j,j+2)\circ\mu(\ell),\ell}f^{(\ell)}](t,X_{\ell-2},V_{\ell-2}) = [S_{j,j+2}\mathfrak{C}^{L_1}_{\mu(\ell),\ell}f^{(\ell)}](t,X_{\ell-2},V_{\ell-2})\\
&= [\mathfrak{C}^{L_1}_{\mu(\ell),\ell}f^{(\ell)}](x_1,\dots,x_{j+1},x_{j+2},x_{j-1},x_j,\dots,x_{\ell-2};\ v_1,\dots,v_{j+1},v_{j+2},v_{j-1},v_j,\dots,v_{\ell-2})\\
&= \int_{\mathbb{R}^{3d}} dv_{\ell-1}dv_\ell dv_{\ell+1}\,\delta(\Sigma_{\mu(\ell),\ell})\,\delta(\Omega_{\mu(\ell),\ell})f^{(\ell)}(x_1,\dots,x_{j+1},x_{j+2},x_{j-1},x_j,\dots,x_{\ell-2},x_{\mu(\ell)},x_{\mu(\ell)};\\
&\qquad\qquad v_1,\dots,v_{j+1},v_{j+2},v_{j-1},v_j,\dots,v_{\ell-2},v_\ell,v_{\ell+1})\\
&= \int_{\mathbb{R}^{3d}} dv_{\ell-1}dv_\ell dv_{\ell+1}\,\delta(\Sigma_{\mu(\ell),\ell})\,\delta(\Omega_{\mu(\ell),\ell})[S_{j,j+2}f^{(\ell)}](X_{\ell-2},x_{\mu(\ell)},x_{\mu(\ell)};\ V_{\ell-2},v_\ell,v_{\ell+1})\\
&= [\mathfrak{C}^{L_1}_{\mu(\ell),\ell}S_{j,j+2}f^{(\ell)}](t,X_{\ell-2},V_{\ell-2}).
\end{aligned}
$$

Case 2: assume that $\ell > j+2$ and $\mu(\ell)\in\{j,j+2\}$. Without loss of generality, assume that $\mu(\ell)=j$. Then, $(j-1,j+1)\circ(j,j+2)\circ\mu(\ell) = j+2$, and so for any non-negative measurable function $f^{(\ell)}$ we have

$$
\begin{aligned}
&[S_{j,j+2}\mathfrak{C}^{L_1}_{(j-1,j+1)\circ(j,j+2)\circ\mu(\ell),\ell}f^{(\ell)}](t,X_{\ell-2},V_{\ell-2}) = [S_{j,j+2}\mathfrak{C}^{L_1}_{j+2,\ell}f^{(\ell)}](t,X_{\ell-2},V_{\ell-2})\\
&= [\mathfrak{C}^{L_1}_{j+2,\ell}f^{(\ell)}](x_1,\dots,x_{j+1},x_{j+2},x_{j-1},\underbrace{x_j}_{(j+2)\text{nd}},\dots,x_{\ell-2};\ v_1,\dots,v_{j+1},v_{j+2},v_{j-1},\underbrace{v_j}_{(j+2)\text{nd}},\dots,v_{\ell-2})\\
&= \int_{\mathbb{R}^{3d}} dv_{\ell-1}dv_\ell dv_{\ell+1}\,\delta(\Sigma_{j,\ell})\,\delta(\Omega_{j,\ell})f^{(\ell)}(x_1,\dots,x_{j+1},x_{j+2},x_{j-1},x_j,\dots,x_{\ell-2},x_j,x_j;\\
&\qquad\qquad v_1,\dots,v_{j+1},v_{j+2},v_{j-1},v_j,\dots,v_{\ell-2},v_\ell,v_{\ell+1})\\
&= \int_{\mathbb{R}^{3d}} dv_{\ell-1}dv_\ell dv_{\ell+1}\,\delta(\Sigma_{j,\ell})\,\delta(\Omega_{j,\ell})[S_{j,j+2}f^{(\ell)}](X_{\ell-2},x_j,x_j;\ V_{\ell-2},v_\ell,v_{\ell+1})\\
&= [\mathfrak{C}^{L_1}_{j,\ell}S_{j,j+2}f^{(\ell)}](t,X_{\ell-2},V_{\ell-2}) = [\mathfrak{C}^{L_1}_{\mu(\ell),\ell}S_{j,j+2}f^{(\ell)}](t,X_{\ell-2},V_{\ell-2}).
\end{aligned}
$$

Case 3: assume that $\ell > j+2$ and $\mu(\ell) \in \{j-1, j+1\}$. Without loss of generality, assume that $\mu(\ell) = j-1$. Then, $(j-1, j+1) \circ (j, j+2) \circ \mu(\ell) = j+1$, and so for any non-negative measurable function $f^{(\ell)}$ we have

$$\begin{aligned}
&[S_{j,j+2}\mathfrak{C}^{L_1}_{(j-1,j+1)\circ(j,j+2)\circ\mu(\ell),\ell} f^{(\ell)}](t, X_{\ell-2}, V_{\ell-2}) = [S_{j,j+2}\mathfrak{C}^{L_1}_{j+1,\ell} f^{(\ell)}](t, X_{\ell-2}, V_{\ell-2}) \\
&= [\mathfrak{C}^{L_1}_{j+1,\ell} f^{(\ell)}](x_1, \ldots, x_{j+1}, x_{j+2}, \underbrace{x_{j-1}}_{(j+1)\text{st}}, x_j, \ldots, x_{\ell-2};\ v_1, \ldots, v_{j+1}, v_{j+2}, \underbrace{v_{j-1}}_{(j+1)\text{st}}, v_j, \ldots, v_{\ell-2}) \\
&= \int_{\mathbb{R}^{3d}} dv_{\ell-1} dv_\ell dv_{\ell+1}\, \delta(\Sigma_{j-1,\ell})\, \delta(\Omega_{j-1,\ell}) f^{(\ell)}(x_1, \ldots, x_{j+1}, x_{j+2}, x_{j-1}, x_j, \ldots, x_{\ell-2}, x_{j-1}, x_{j-1}; \\
&\qquad\qquad v_1, \ldots, v_{j+1}, v_{j+2}, v_{j-1}, v_j, \ldots, v_{\ell-2}, v_\ell, v_{\ell+1}) \\
&= \int_{\mathbb{R}^{3d}} dv_{\ell-1} dv_\ell dv_{\ell+1}\, \delta(\Sigma_{j-1,\ell})\, \delta(\Omega_{j-1,\ell}) [S_{j,j+2} f^{(\ell)}](X_{\ell-2}, x_{j-1}, x_{j-1};\ V_{\ell-2}, v_\ell, v_{\ell+1}) \\
&= [\mathfrak{C}^{L_1}_{j-1,\ell} S_{j,j+2} f^{(\ell)}](t, X_{\ell-2}, V_{\ell-2}) = [\mathfrak{C}^{L_1}_{\mu(\ell),\ell} S_{j,j+2} f^{(\ell)}](t, X_{\ell-2}, V_{\ell-2}).
\end{aligned}$$

□

Next we prove identity (3.31).

Lemma 3.13. *For any $a, b, c, d \geq 0$, any $\beta < \alpha < j$ and any $k, \ell \in \{0, 1, 2, 3\}$, we have*

$$T^{a-b}_{j-2}\mathfrak{C}^{L_k}_{\alpha,j} T^{b-c}_j \mathfrak{C}^{L_\ell}_{\beta,j+2} T^{c-d}_{j+2} = T^{a-c}_{j-2}\mathfrak{C}^{L_\ell}_{\beta,j} T^{c-b}_j \mathfrak{C}^{L_k}_{\alpha,j+2} T^{b-d}_{j+2} S_{j,j+2} \tag{3.36}$$

Proof of Lemma 3.13. By applying the operator T^{c-a}_{j-2} from the left and the operator T^{d-b}_{j+2} from the right, the identity can be written in its equivalent form

$$T^{c-b}_{j-2}\mathfrak{C}^{L_k}_{\alpha,j} T^{b-c}_j \mathfrak{C}^{L_\ell}_{\beta,j+2} T^{c-b}_{j+2} = \mathfrak{C}^{L_\ell}_{\beta,j} T^{c-b}_j \mathfrak{C}^{L_k}_{\alpha,j+2} S_{j,j+2}. \tag{3.37}$$

If we introduce notation

$$\tau = b - c,$$

it suffices to show that for any $\beta < \alpha < j$ and any $k, \ell \in \{0, 1, 2, 3\}$, we have

$$T^{-\tau}_{j-2}\mathfrak{C}^{L_k}_{\alpha,j} T^{\tau}_j \mathfrak{C}^{L_\ell}_{\beta,j+2} T^{-\tau}_{j+2} = \mathfrak{C}^{L_\ell}_{\beta,j} T^{-\tau}_j \mathfrak{C}^{L_k}_{\alpha,j+2} S_{j,j+2}. \tag{3.38}$$

The strategy of the proof is to expand the left-hand side (LHS) and the right-hand side (RHS) of these identities separately and then compare their formulas.

Note that the operator $\mathfrak{C}_{\alpha,j}$ comes with three integrating variables v_{j-1}, v_j and v_{j+1}, while the corresponding variables for $\mathfrak{C}_{\beta,j+2}$ are v_{j+1}, v_{j+2} and v_{j+3}. In order to avoid confusion due to the repeated letter v_{j+1}, we will add stars to the variables corresponding to $\mathfrak{C}_{\alpha,j}$, primes for $\mathfrak{C}_{\beta,j+2}$, sharps for $\mathfrak{C}_{\beta,j}$ and tildes for $\mathfrak{C}_{\alpha,j+2}$. This notation will be used only within this lemma.

• We first prove identity (3.38) with $k=\ell=0$. We start by computing its left-hand side.

$$
\begin{aligned}
\mathrm{LHS}_{00} &= [T^{-\tau}_{j-2}\mathfrak{C}^{L_0}_{\alpha,j}T^{\tau}_{j}\mathfrak{C}^{L_0}_{\beta,j+2}T^{-\tau}_{j+2}f^{(j+2)}](t,X_{j-2},V_{j-2})\\
&= [\mathfrak{C}^{L_0}_{\alpha,j}T^{\tau}_{j}\mathfrak{C}^{L_0}_{\beta,j+2}T^{-\tau}_{j+2}f^{(j+2)}](t,X_{j-2}+\tau V_{j-2},V_{j-2})\\
&= \int_{\mathbb{R}^{3d}} dv^*_{j-1}dv^*_j dv^*_{j+1}\,\delta(\Sigma^*_{\alpha,j})\,\delta(\Omega^*_{\alpha,j})\\
&\qquad [T^{\tau}_{j}\mathfrak{C}^{L_0}_{\beta,j+2}T^{-\tau}_{j+2}f^{(j+2)}](t,X_{j-2}+\tau V_{j-2},x_\alpha+\tau v_\alpha,x_\alpha+\tau v_\alpha;\,V^{\alpha,v^*_{j-1}}_{j-2},v^*_j,v^*_{j+1})\\
&= \int_{\mathbb{R}^{3d}} dv^*_{j-1}dv^*_j dv^*_{j+1}\,\delta(\Sigma^*_{\alpha,j})\,\delta(\Omega^*_{\alpha,j})\\
&\qquad [\mathfrak{C}^{L_0}_{\beta,j+2}T^{-\tau}_{j+2}f^{(j+2)}](t,X_{j-2}+\tau(V_{j-2}-V^{\alpha,v^*_{j-1}}_{j-2}),x_\alpha+\tau(v_\alpha-v^*_j),x_\alpha+\tau(v_\alpha-v^*_{j+1});\;V^{\alpha,v^*_{j-1}}_{j-2},v^*_j,v^*_{j+1})\\
&= \int_{\mathbb{R}^{6d}} dv^*_{j-1}dv^*_j dv^*_{j+1}dv'_{j+1}dv'_{j+2}dv'_{j+3}\,\delta(\Sigma^*_{\alpha,j})\,\delta(\Omega^*_{\alpha,j})\,\delta(\Sigma'_{\beta,j+2})\,\delta(\Omega'_{\beta,j+2})\\
&\qquad [T^{-\tau}_{j+2}f^{(j+2)}](t,X_{j-2}+\tau(V_{j-2}-V^{\alpha,v^*_{j-1}}_{j-2}),x_\alpha+\tau(v_\alpha-v^*_j),x_\alpha+\tau(v_\alpha-v^*_{j+1}),x_\beta,x_\beta;\\
&\qquad\qquad V^{\beta,v'_{j+1};\alpha,v^*_{j-1}}_{j-2},v^*_j,v^*_{j+1},v'_{j+2},v'_{j+3})\\
&= \int_{\mathbb{R}^{6d}} dv^*_{j-1}dv^*_j dv^*_{j+1}dv'_{j+1}dv'_{j+2}dv'_{j+3}\,\delta(\Sigma^*_{\alpha,j})\,\delta(\Omega^*_{\alpha,j})\,\delta(\Sigma'_{\beta,j+2})\,\delta(\Omega'_{\beta,j+2})\\
&\qquad f^{(j+2)}(t,X_{j-2}+\tau V^{\beta,v'_{j+1}}_{j-2},x_\alpha+\tau v_\alpha,x_\alpha+\tau v_\alpha,x_\beta+\tau v'_{j+2},x_\beta+\tau v'_{j+3};\\
&\qquad\qquad V^{\beta,v'_{j+1};\alpha,v^*_{j-1}}_{j-2},v^*_j,v^*_{j+1},v'_{j+2},v'_{j+3}),
\end{aligned}
$$

where in the last equality we used that $V_{j-2}-V^{\alpha,v^*_{j-1}}_{j-2}+V^{\beta,v'_{j+1};\alpha,v^*_{j-1}}_{j-2}=V^{\beta,v'_{j+1}}_{j-2}$.

We next calculate the right-hand side of (3.38) with $k=\ell=0$.

$$
\begin{aligned}
\mathrm{RHS}_{00} &= [\mathfrak{C}^{L_0}_{\beta,j}T^{-\tau}_{j}\mathfrak{C}^{L_0}_{\alpha,j+2}S_{j,j+2}f^{(j+2)}](t,X_{j-2},V_{j-2})\\
&= \int_{\mathbb{R}^{3d}} dv^{\#}_{j-1}dv^{\#}_j dv^{\#}_{j+1}\,\delta(\Sigma^{\#}_{\beta,j})\,\delta(\Omega^{\#}_{\beta,j})[T^{-\tau}_{j}\mathfrak{C}^{L_0}_{\alpha,j+2}S_{j,j+2}f^{(j+2)}](t,X_{j-2},x_\beta,x_\beta;\,V^{\beta,v^{\#}_{j-1}}_{j-2},v^{\#}_j,v^{\#}_{j+1})\\
&= \int_{\mathbb{R}^{3d}} dv^{\#}_{j-1}dv^{\#}_j dv^{\#}_{j+1}\,\delta(\Sigma^{\#}_{\beta,j})\,\delta(\Omega^{\#}_{\beta,j})\\
&\qquad [\mathfrak{C}^{L_0}_{\alpha,j+2}S_{j,j+2}f^{(j+2)}](t,X_{j-2}+\tau V^{\beta,v^{\#}_{j-1}}_{j-2},x_\beta+\tau v^{\#}_j,x_\beta+\tau v^{\#}_{j+1};\,V^{\beta,v^{\#}_{j-1}}_{j-2},v^{\#}_j,v^{\#}_{j+1})\\
&= \int_{\mathbb{R}^{6d}} dv^{\#}_{j-1}dv^{\#}_j dv^{\#}_{j+1}d\widetilde{v}_{j+1}d\widetilde{v}_{j+2}d\widetilde{v}_{j+3}\,\delta(\Sigma^{\#}_{\beta,j})\,\delta(\Omega^{\#}_{\beta,j})\delta(\widetilde{\Sigma}_{\alpha,j+2})\,\delta(\widetilde{\Omega}_{\alpha,j+2})\\
&\qquad [S_{j,j+2}f^{(j+2)}](t,X_{j-2}+\tau V^{\beta,v^{\#}_{j-1}}_{j-2},x_\beta+\tau v^{\#}_j,x_\beta+\tau v^{\#}_{j+1},x_\alpha+\tau v_\alpha,x_\alpha+\tau v_\alpha;\\
&\qquad\qquad V^{\beta,v^{\#}_{j-1};\,\alpha,\widetilde{v}_{j+1}}_{j-2},v^{\#}_j,v^{\#}_{j+1},\widetilde{v}_{j+2},\widetilde{v}_{j+3})\\
&= \int_{\mathbb{R}^{6d}} dv^{\#}_{j-1}dv^{\#}_j dv^{\#}_{j+1}d\widetilde{v}_{j+1}d\widetilde{v}_{j+2}d\widetilde{v}_{j+3}\,\delta(\Sigma^{\#}_{\beta,j})\,\delta(\Omega^{\#}_{\beta,j})\delta(\widetilde{\Sigma}_{\alpha,j+2})\,\delta(\widetilde{\Omega}_{\alpha,j+2})\\
&\qquad f^{(j+2)}(t,X_{j-2}+\tau V^{\beta,v^{\#}_{j-1}}_{j-2},x_\alpha+\tau v_\alpha,x_\alpha+\tau v_\alpha,x_\beta+\tau v^{\#}_j,x_\beta+\tau v^{\#}_{j+1};\\
&\qquad\qquad V^{\beta,v^{\#}_{j-1};\,\alpha,\widetilde{v}_{j+1}}_{j-2},\widetilde{v}_{j+2},\widetilde{v}_{j+3},v^{\#}_j,v^{\#}_{j+1}).
\end{aligned}
$$

By applying the change of variables:

$$\begin{cases} v^{\#}_{j-1} & \mapsto v'_{j+1} \\ v^{\#}_{j} & \mapsto v'_{j+2} \\ v^{\#}_{j+1} & \mapsto v'_{j+3} \end{cases} \quad \text{and} \quad \begin{cases} \widetilde{v}_{j+1} & \mapsto v^*_{j-1} \\ \widetilde{v}_{j+2} & \mapsto v^*_{j} \\ \widetilde{v}_{j+3} & \mapsto v^*_{j+1} \end{cases} \tag{3.39}$$

one can see that $\mathrm{RHS}_{00} = \mathrm{LHS}_{00}$, which completes the proof of (3.38) with $k = \ell = 0$.

• Next we prove identity (3.38) for $k = 0$ and $\ell = 1$. When $\ell = 2$ or $\ell = 3$, the proof can be done analogously. We start by expanding the left-hand side:

$$\begin{aligned}
&\mathrm{LHS}_{01} = [T^{-\tau}_{j-2}\mathfrak{C}^{L_0}_{\alpha,j}T^{\tau}_{j}\mathfrak{C}^{L_1}_{\beta,j+2}T^{-\tau}_{j+2}f^{(j+2)}](t, X_{j-2}, V_{j-2}) \\
&= [\mathfrak{C}^{L_0}_{\alpha,j}T^{\tau}_{j}\mathfrak{C}^{L_1}_{\beta,j+2}T^{-\tau}_{j+2}f^{(j+2)}](t, X_{j-2} + \tau V_{j-2}, V_{j-2}) \\
&= \int_{\mathbb{R}^{3d}} dv^*_{j-1}dv^*_{j}dv^*_{j+1}\,\delta(\Sigma^*_{\alpha,j})\,\delta(\Omega^*_{\alpha,j}) \\
&\qquad [T^{\tau}_{j}\mathfrak{C}^{L_1}_{\beta,j+2}T^{-\tau}_{j+2}f^{(j+2)}](t, X_{j-2} + \tau V_{j-2}, x_\alpha + \tau v_\alpha, x_\alpha + \tau v_\alpha;\, V^{\alpha,v^*_{j-1}}_{j-2}, v^*_{j}, v^*_{j+1}) \\
&= \int_{\mathbb{R}^{3d}} dv^*_{j-1}dv^*_{j}dv^*_{j+1}\,\delta(\Sigma^*_{\alpha,j})\,\delta(\Omega^*_{\alpha,j}) \\
&\quad [\mathfrak{C}^{L_1}_{\beta,j+2}T^{-\tau}_{j+2}f^{(j+2)}](t, X_{j-2} + \tau(V_{j-2} - V^{\alpha,v^*_{j-1}}_{j-2}), x_\alpha + \tau(v_\alpha - v^*_{j}), x_\alpha + \tau(v_\alpha - v^*_{j+1});\; V^{\alpha,v^*_{j-1}}_{j-2}, v^*_{j}, v^*_{j+1}) \\
&= \int_{\mathbb{R}^{6d}} dv^*_{j-1}dv^*_{j}dv^*_{j+1}dv'_{j+1}dv'_{j+2}dv'_{j+3}\,\delta(\Sigma^*_{\alpha,j})\,\delta(\Omega^*_{\alpha,j})\,\delta(\Sigma'_{\beta,j+2})\,\delta(\Omega'_{\beta,j+2}) \\
&\qquad [T^{-\tau}_{j+2}f^{(j+2)}](t, X_{j-2} + \tau(V_{j-2} - V^{\alpha,v^*_{j-1}}_{j-2}), x_\alpha + \tau(v_\alpha - v^*_{j}), x_\alpha + \tau(v_\alpha - v^*_{j+1}), x_\beta, x_\beta; \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad V^{\alpha,v^*_{j-1}}_{j-2}, v^*_{j}, v^*_{j+1}, v'_{j+2}, v'_{j+3}) \\
&= \int_{\mathbb{R}^{6d}} dv^*_{j-1}dv^*_{j}dv^*_{j+1}dv'_{j+1}dv'_{j+2}dv'_{j+3}\,\delta(\Sigma^*_{\alpha,j})\,\delta(\Omega^*_{\alpha,j})\,\delta(\Sigma'_{\beta,j+2})\,\delta(\Omega'_{\beta,j+2}) \\
&\quad f^{(j+2)}(t, X_{j-2} + \tau V_{j-2}, x_\alpha + \tau v_\alpha, x_\alpha + \tau v_\alpha, x_\beta + \tau v'_{j+2}, x_\beta + \tau v'_{j+3}; V^{\alpha,v^*_{j-1}}_{j-2}, v^*_{j}, v^*_{j+1}, v'_{j+2}, v'_{j+3}).
\end{aligned}$$

On the other hand, the right-hand side of (3.38) for $k = 0$ and $\ell = 1$ can be expanded as follows

$$\begin{aligned}
&\mathrm{RHS}_{01} = [\mathfrak{C}^{L_1}_{\beta,j}T^{-\tau}_{j}\mathfrak{C}^{L_0}_{\alpha,j+2}S_{j,j+2}f^{(j+2)}](t, X_{j-2}, V_{j-2}) \\
&= \int_{\mathbb{R}^{3d}} dv^{\#}_{j-1}dv^{\#}_{j}dv^{\#}_{j+1}\,\delta(\Sigma^{\#}_{\beta,j})\,\delta(\Omega^{\#}_{\beta,j})[T^{-\tau}_{j}\mathfrak{C}^{L_0}_{\alpha,j+2}S_{j,j+2}f^{(j+2)}](t, X_{j-2}, x_\beta, x_\beta;\, V_{j-2}, v^{\#}_{j}, v^{\#}_{j+1}) \\
&= \int_{\mathbb{R}^{3d}} dv^{\#}_{j-1}dv^{\#}_{j}dv^{\#}_{j+1}\,\delta(\Sigma^{\#}_{\beta,j})\,\delta(\Omega^{\#}_{\beta,j})[\mathfrak{C}^{L_0}_{\alpha,j+2}S_{j,j+2}f^{(j+2)}](t, X_{j-2} + \tau V_{j-2}, x_\beta + \tau v^{\#}_{j}, x_\beta + \tau v^{\#}_{j+1};\, V_{j-2}, v^{\#}_{j}, v^{\#}_{j+1}) \\
&= \int_{\mathbb{R}^{6d}} dv^{\#}_{j-1}dv^{\#}_{j}dv^{\#}_{j+1}d\widetilde{v}_{j+1}d\widetilde{v}_{j+2}d\widetilde{v}_{j+3}\,\delta(\Sigma^{\#}_{\beta,j})\,\delta(\Omega^{\#}_{\beta,j})\delta(\widetilde{\Sigma}_{\alpha,j+2})\,\delta(\widetilde{\Omega}_{\alpha,j+2}) \\
&\qquad [S_{j,j+2}f^{(j+2)}](t, X_{j-2} + \tau V_{j-2}, x_\beta + \tau v^{\#}_{j}, x_\beta + \tau v^{\#}_{j+1}, x_\alpha + \tau v_\alpha, x_\alpha + \tau v_\alpha; V^{\alpha,\widetilde{v}_{j+1}}_{j-2}, v^{\#}_{j}, v^{\#}_{j+1}, \widetilde{v}_{j+2}, \widetilde{v}_{j+3}) \\
&= \int_{\mathbb{R}^{6d}} dv^{\#}_{j-1}dv^{\#}_{j}dv^{\#}_{j+1}d\widetilde{v}_{j+1}d\widetilde{v}_{j+2}d\widetilde{v}_{j+3}\,\delta(\Sigma^{\#}_{\beta,j})\,\delta(\Omega^{\#}_{\beta,j})\delta(\widetilde{\Sigma}_{\alpha,j+2})\,\delta(\widetilde{\Omega}_{\alpha,j+2}) \\
&\qquad f^{(j+2)}(t, X_{j-2} + \tau V_{j-2}, x_\alpha + \tau v_\alpha, x_\alpha + \tau v_\alpha, x_\beta + \tau v^{\#}_{j}, x_\beta + \tau v^{\#}_{j+1}; V^{\alpha,\widetilde{v}_{j+1}}_{j-2}, \widetilde{v}_{j+2}, \widetilde{v}_{j+3}, v^{\#}_{j}, v^{\#}_{j+1}).
\end{aligned}$$

Under the same change of variables as in (3.39), we see that $\mathrm{RHS}_{01} = \mathrm{LHS}_{01}$, which completes the proof of the identity (3.38) for $k = 0$ and $\ell = 1$.

• Finally we prove identity (3.38) for $k = 1$ and $\ell = 2$. Any other combination of $k, \ell \in \{1, 2, 3\}$ can be proved analogously. We start by expanding the left-hand side:

$$
\begin{aligned}
&\mathrm{LHS}_{12} = [T_{j-2}^{-\tau}\mathfrak{C}_{\alpha,j}^{L_1}T_j^{\tau}\mathfrak{C}_{\beta,j+2}^{L_2}T_{j+2}^{-\tau}f^{(j+2)}](t, X_{j-2}, V_{j-2})\\
&= [\mathfrak{C}_{\alpha,j}^{L_1}T_j^{\tau}\mathfrak{C}_{\beta,j+2}^{L_2}T_{j+2}^{-\tau}f^{(j+2)}](t, X_{j-2} + \tau V_{j-2}, V_{j-2})\\
&= \int_{\mathbb{R}^{3d}} dv_{j-1}^* dv_j^* dv_{j+1}^*\, \delta(\Sigma_{\alpha,j}^*)\, \delta(\Omega_{\alpha,j}^*)\\
&\qquad [T_j^{\tau}\mathfrak{C}_{\beta,j+2}^{L_2}T_{j+2}^{-\tau}f^{(j+2)}](t, X_{j-2} + \tau V_{j-2}, x_\alpha + \tau v_\alpha, x_\alpha + \tau v_\alpha;\, V_{j-2}, v_j^*, v_{j+1}^*)\\
&= \int_{\mathbb{R}^{3d}} dv_{j-1}^* dv_j^* dv_{j+1}^*\, \delta(\Sigma_{\alpha,j}^*)\, \delta(\Omega_{\alpha,j}^*)\\
&\quad [\mathfrak{C}_{\beta,j+2}^{L_2}T_{j+2}^{-\tau}f^{(j+2)}](t, X_{j-2}, x_\alpha + \tau(v_\alpha - v_j^*), x_\alpha + \tau(v_\alpha - v_{j+1}^*);\ V_{j-2}, v_j^*, v_{j+1}^*)\\
&= \int_{\mathbb{R}^{6d}} dv_{j-1}^* dv_j^* dv_{j+1}^* dv_{j+1}' dv_{j+2}' dv_{j+3}'\, \delta(\Sigma_{\alpha,j}^*)\, \delta(\Omega_{\alpha,j}^*)\, \delta(\Sigma_{\beta,j+2}')\, \delta(\Omega_{\beta,j+2}')\\
&\qquad [T_{j+2}^{-\tau}f^{(j+2)}](t, X_{j-2}, x_\alpha + \tau(v_\alpha - v_j^*), x_\alpha + \tau(v_\alpha - v_{j+1}^*), x_\beta, x_\beta; V_{j-2}, v_j^*, v_{j+1}^*, v_{j+1}', v_{j+3}')\\
&= \int_{\mathbb{R}^{6d}} dv_{j-1}^* dv_j^* dv_{j+1}^* dv_{j+1}' dv_{j+2}' dv_{j+3}'\, \delta(\Sigma_{\alpha,j}^*)\, \delta(\Omega_{\alpha,j}^*)\, \delta(\Sigma_{\beta,j+2}')\, \delta(\Omega_{\beta,j+2}')\\
&\quad f^{(j+2)}(t, X_{j-2} + \tau V_{j-2}, x_\alpha + \tau v_\alpha, x_\alpha + \tau v_\alpha, x_\beta + \tau v_{j+1}', x_\beta + \tau v_{j+3}'; V_{j-2}, v_j^*, v_{j+1}^*, v_{j+1}', v_{j+3}').
\end{aligned}
$$

On the other hand, the right-hand side of (3.38) for $k = 1$ and $\ell = 2$ can be expanded as follows

$$
\begin{aligned}
&\mathrm{RHS}_{12} = [\mathfrak{C}_{\beta,j}^{L_2}T_j^{-\tau}\mathfrak{C}_{\alpha,j+2}^{L_1}S_{j,j+2}f^{(j+2)}](t, X_{j-2}, V_{j-2})\\
&= \int_{\mathbb{R}^{3d}} dv_{j-1}^{\#} dv_j^{\#} dv_{j+1}^{\#}\, \delta(\Sigma_{\beta,j}^{\#})\, \delta(\Omega_{\beta,j}^{\#})[T_j^{-\tau}\mathfrak{C}_{\alpha,j+2}^{L_1}S_{j,j+2}f^{(j+2)}](t, X_{j-2}, x_\beta, x_\beta;\, V_{j-2}, v_{j-1}^{\#}, v_{j+1}^{\#})\\
&= \int_{\mathbb{R}^{3d}} dv_{j-1}^{\#} dv_j^{\#} dv_{j+1}^{\#}\, \delta(\Sigma_{\beta,j}^{\#})\, \delta(\Omega_{\beta,j}^{\#})[\mathfrak{C}_{\alpha,j+2}^{L_1}S_{j,j+2}f^{(j+2)}]\\
&\qquad\qquad\qquad\qquad (t, X_{j-2} + \tau V_{j-2}, x_\beta + \tau v_{j-1}^{\#}, x_\beta + \tau v_{j+1}^{\#};\, V_{j-2}, v_{j-1}^{\#}, v_{j+1}^{\#})\\
&= \int_{\mathbb{R}^{6d}} dv_{j-1}^{\#} dv_j^{\#} dv_{j+1}^{\#} d\widetilde{v}_{j+1} d\widetilde{v}_{j+2} d\widetilde{v}_{j+3}\, \delta(\Sigma_{\beta,j}^{\#})\, \delta(\Omega_{\beta,j}^{\#})\delta(\widetilde{\Sigma}_{\alpha,j+2})\, \delta(\widetilde{\Omega}_{\alpha,j+2})[S_{j,j+2}f^{(j+2)}]\\
&\qquad (t, X_{j-2} + \tau V_{j-2}, x_\beta + \tau v_{j-1}^{\#}, x_\beta + \tau v_{j+1}^{\#}, x_\alpha + \tau v_\alpha, x_\alpha + \tau v_\alpha; V_{j-2}, v_{j-1}^{\#}, v_{j+1}^{\#}, \widetilde{v}_{j+2}, \widetilde{v}_{j+3})\\
&= \int_{\mathbb{R}^{6d}} dv_{j-1}^{\#} dv_j^{\#} dv_{j+1}^{\#} d\widetilde{v}_{j+1} d\widetilde{v}_{j+2} d\widetilde{v}_{j+3}\, \delta(\Sigma_{\beta,j}^{\#})\, \delta(\Omega_{\beta,j}^{\#})\delta(\widetilde{\Sigma}_{\alpha,j+2})\, \delta(\widetilde{\Omega}_{\alpha,j+2})\\
&\quad f^{(j+2)}(t, X_{j-2} + \tau V_{j-2}, x_\alpha + \tau v_\alpha, x_\alpha + \tau v_\alpha, x_\beta + \tau v_{j-1}^{\#}, x_\beta + \tau v_{j+1}^{\#}; V_{j-2}, \widetilde{v}_{j+2}, \widetilde{v}_{j+3}, v_{j-1}^{\#}, v_{j+1}^{\#}).
\end{aligned}
$$

Under the same change of variables as in (3.39), we see that $\mathrm{RHS}_{12} = \mathrm{LHS}_{12}$, which completes the proof of the identity (3.38) for $k = 1$ and $\ell = 2$, and thus also the proof of this lemma. □

Inspired by [35] and [10], we define a special upper echelon form as the state which does not admit any further acceptable moves.

Definition 3.14 (Special Upper Echelon Form)**.** *Let $k, n \in \mathbb{N}$ with $n \geq 2$, and let $M_{n,k}$ be defined as in* (3.19)*. We say that $\mu \in M_{n,k}$ is in special upper echelon form if for every $j \in \{k+2, k+4, \dots, k+2n\}$ we have $\mu(j) \leq \mu(j+2)$. We will denote by*

$$\mathcal{M}_{n,k} \text{ the set of all special upper echelon forms in } M_{n,k}. \tag{3.40}$$

In the same way as in [35, Lemma 3.2], one can show that every state on the board can be transformed by finitely many acceptable moves into a special upper echelon form.

Proposition 3.15. *Let $k, n \in \mathbb{N}$ with $n \geq 2$, and let $M_{n,k}$ be defined as in* (3.19)*. Any $\mu \in M_{n,k}$ can be changed to a special upper echelon form via a finite sequence of acceptable moves.*

Next we provide an upper bound on the number of special upper echelon forms. The proof of this proposition can be done exactly in the same way as in [10], since their board is of the same size as ours $(k+2n-2) \times n$.

Proposition 3.16 (Number of special upper echelon forms; [10, Lemma 7.3])**.** *Let $k, n \in \mathbb{N}$ with $n \geq 2$, and let $\mathcal{M}_{n,k}$ be the set of all special upper echelon forms defined in* (3.40)*. Then the following upper bound holds:*

$$\#\mathcal{M}_{n,k} \leq 2^{k+3n-2}. \tag{3.41}$$

Finally, one can show that the sum over all states that can be turned into the same special upper echelon form can be reorganized as follows. The proof of this proposition is identical to that of Theorem 7.4 in [10].

Proposition 3.17 (Sum over one equivalence class; [10, Theorem 7.4])**.** *Let $\mu_u \in \mathcal{M}_{n,k}$ be a special upper echelon form, and write $\mu \sim \mu_u$ if μ can be reduced to μ_u in finitely many acceptable moves. Then there exists a set $D \subset [0,t]^n$, that depends on μ_u, such that*

$$\sum_{\mu \sim \mu_u} \int_{t \geq t_{k+2} \geq \cdots \geq t_{k+2n} \geq 0} J(\underline{t}_{n,k}; \mu)\, dt_{k+2n} \dots dt_{k+2} = \int_D J(\underline{t}_{n,k}; \mu_u)\, dt_{k+2n} \dots dt_{k+2}, \tag{3.42}$$

where the sum goes over all $\mu \in M_{n,k}$ such that μ can be changed to μ_u via a finite sequence of acceptable moves.

3.2.3. *Combining a priori estimates and the board game argument.* In this section we combine the iterated a priori estimate from Proposition 3.6 and the board game argument described in the previous subsection to prove uniqueness of solutions to the wave kinetic hierarchy as stated in Theorem 3.4. Since wave kinetic hierarchy in linear, it suffices to show that if $F_0 = 0$ and F is a mild solution, then $F = 0$. Here, for $F = (f^{(k)})_{k=1}^{\infty}$, we say $F = 0$ if for each $k \in \mathbb{N}$ we have $f^{(k)} = 0$.

Proof of Theorem 3.4. Recall from (3.9) that for zero initial data, a mild solution to the wave kinetic hierarchy can be expressed as

$$\begin{aligned} T_k^{-t} f^{(k)}(t) &= \int_0^t \int_0^{t_{k+2}} \cdots \int_0^{t_{k+2n-2}} dt_{k+2n} \cdots dt_{k+4} dt_{k+2} \\ &\qquad T_k^{-t_{k+2}} \mathfrak{C}^{k+2} T_{k+2}^{t_{k+2}-t_{k+4}} \mathfrak{C}^{k+4} \cdots T_{k+2n-2}^{t_{k+2n-2}-t_{k+2n}} \mathfrak{C}^{k+2n} f^{(k+2n)}(t_{k+2n}) && (3.43) \\ &= \sum_{\mu \in M_{n,k}} \int_0^t \int_0^{t_{k+1}} \cdots \int_0^{t_{k+n-1}} J_{n,k}(\underline{t}_n; \mu)\ dt_{k+n} \cdots dt_{k+1}, && (3.44) \end{aligned}$$

where the sum is taken over all the mappings in $M_{n,k}$ given by (3.19), and where $J_{n,k}(\underline{t}_n; \mu)$ is defined as in (3.20). By Proposition 3.15 and Proposition 3.17, we can instead sum over all upper echelon forms:

$$T_k^{-t} f^{(k)}(t) = \sum_{\mu_u \in \mathcal{M}_{n,k}} \int_{D(\mu_u)} J_{n,k}(\underline{t}_n; \mu_u)\ dt_{k+n} \cdots dt_{k+1}. \tag{3.45}$$

Since each operator $\mathfrak{C}_{\mu_u(k+2j),k+2j}$ appearing in $J_{n,k}(\underline{t}_n;\mu_u)$ is a difference of two operators defined in (1.37): $\mathfrak{C}^+_{\mu_u(k+2j),k+2j} - \mathfrak{C}^-_{\mu_u(k+2j),k+2j}$, if we use notation $\boldsymbol{\pi} = (\pi_{k+2}, \pi_{k+4}, \dots, \pi_{k+2n}) \in \{+,-\}^n$ and $\operatorname{sgn}(\boldsymbol{\pi}) = \pi_{k+2} \cdot \pi_{k+4} \cdot \ldots \cdot \pi_{k+2n}$, we can write

$$\begin{aligned}T_k^{-t} f^{(k)}(t) = \sum_{\mu_u \in \mathcal{M}_{n,k}} \sum_{\boldsymbol{\pi} \in \{+,-\}^n} \operatorname{sgn}(\boldsymbol{\pi}) \int_{D(\mu_u)} T_k^{-t_{k+2}} \mathfrak{C}^{\pi_{k+2}}_{\mu_u(k+2),k+2} T_{k+2}^{t_{k+2}-t_{k+4}} \mathfrak{C}^{\pi_{k+4}}_{\mu_u(k+4),k+4} \cdots \\ \cdots T_{k+2n-2}^{t_{k+2n-2}-t_{k+2n}} \mathfrak{C}^{\pi_{k+2n}}_{\mu_u(k+2n),k+2n} f^{(k+2n)}(t_{k+2n})\ dt_{k+2n} \cdots dt_{k+4} dt_{k+2}.\end{aligned}$$

Since, $|\mathfrak{C}^{\pm}_{j,k} g^{(k)}| \le \mathfrak{C}^{\pm}_{j,k} |g^{(k)}|$ and $|T_k^{\tau} g^{(k)}| = T_k^{\tau} |g^{(k)}|$, we have

$$\begin{aligned}\left|T_k^{-t} f^{(k)}(t)\right| &\le \sum_{\mu_u \in \mathcal{M}_{n,k}} \sum_{\boldsymbol{\pi} \in \{+,-\}^n} \int_{D(\mu_u)} T_k^{-t_{k+2}} \mathfrak{C}^{\pi_{k+2}}_{\mu_u(k+2),k+2} T_{k+2}^{t_{k+2}-t_{k+4}} \mathfrak{C}^{\pi_{k+4}}_{\mu_u(k+4),k+4} \cdots \\ &\qquad \cdots T_{k+2n-2}^{t_{k+2n-2}-t_{k+2n}} \mathfrak{C}^{\pi_{k+2n}}_{\mu_u(k+2n),k+2n} \left|f^{(k+2n)}(t_{k+2n})\right|\ dt_{k+2n} \cdots dt_{k+4} dt_{k+2} \\ &\le \sum_{\mu_u \in \mathcal{M}_{n,k}} \sum_{\boldsymbol{\pi} \in \{+,-\}^n} \int_{[0,T]^n} T_k^{-t_{k+2}} \mathfrak{C}^{\pi_{k+2}}_{\mu_u(k+2),k+2} T_{k+2}^{t_{k+2}-t_{k+4}} \mathfrak{C}^{\pi_{k+4}}_{\mu_u(k+4),k+4} \cdots \\ &\qquad \cdots T_{k+2n-2}^{t_{k+2n-2}-t_{k+2n}} \mathfrak{C}^{\pi_{k+2n}}_{\mu_u(k+2n),k+2n} \left|f^{(k+2n)}(t_{k+2n})\right|\ dt_{k+2n} \cdots dt_{k+4} dt_{k+2},\end{aligned}$$

where in the last inequality we enlarged the domain of the time integration thanks to the fact that the integrand is non-negative. Multiplying both sides of the above inequality with the polynomial weights $\langle\langle \alpha X_k \rangle\rangle^p \langle\langle \beta V_k \rangle\rangle^q$, taking the supremum in X_k, V_k, and using the triangle inequality on the norm $\|\cdot\|_{k,p,q,\alpha,\beta}$, we obtain

$$\begin{aligned}\left\|T_k^{-t} f^{(k)}(t)\right\|_{k,p,q,\alpha,\beta} \le \sum_{\mu_u \in \mathcal{M}_{n,k}} \sum_{\boldsymbol{\pi} \in \{+,-\}^n} \Bigg\| \int_{[0,T]^n} T_k^{-t_{k+2}} \mathfrak{C}^{\pi_{k+2}}_{\mu_u(k+2),k+2} T_{k+2}^{t_{k+2}-t_{k+4}} \mathfrak{C}^{\pi_{k+4}}_{\mu_u(k+4),k+4} \cdots \\ \cdots T_{k+2n-2}^{t_{k+2n-2}-t_{k+2n}} \mathfrak{C}^{\pi_{k+2n}}_{\mu_u(k+2n),k+2n} \left|f^{(k+2n)}(t_{k+2n})\right|\ dt_{k+2n} \cdots dt_{k+4} dt_{k+2} \Bigg\|_{k,p,q,\alpha,\beta}.\end{aligned}$$

By Proposition 3.6, combined with the fact that $\#\mathcal{M}_{n,k} \le 2^{k+3n-2}$ (Proposition (3.16)) and $\#\{+,-\}^n = 2^n$, we have

$$\begin{aligned}\left\|T_k^{-t} f^{(k)}(t)\right\|_{k,p,q,\alpha,\beta} &\le 2^{k+5n-2}\, C^n_{p,q,\alpha,\beta} \left|\left|\left| T_{k+2n}^{-(\cdot)} \left|f^{(k+2n)}(\cdot)\right| \right|\right|\right|_{k+2n,p,q,\alpha,\beta,T} \\ &= 2^{k+5n-2}\, C^n_{p,q,\alpha,\beta} \left|\left|\left| T_{k+2n}^{-(\cdot)} f^{(k+2n)}(\cdot) \right|\right|\right|_{k+2n,p,q,\alpha,\beta,T}.\end{aligned}$$

From the definition of the norm (1.51), we can further deduce that

$$\begin{aligned}\left\|T_k^{-t} f^{(k)}(t)\right\|_{k,p,q,\alpha,\beta} &\le 2^{k+5n-2}\, C^n_{p,q,\alpha,\beta}\, e^{-\mu(k+2n)} \left|\left|\left| \mathcal{T}^{-(\cdot)} F(\cdot) \right|\right|\right|_{p,q,\alpha,\beta,T} \\ &= \frac{(2e^{-\mu})^k}{4} \left(32 e^{-2\mu} C_{p,q,\alpha,\beta}\right)^n \left|\left|\left| \mathcal{T}^{-(\cdot)} F(\cdot) \right|\right|\right|_{p,q,\alpha,\beta,T}.\end{aligned}$$

Since μ was chosen so that $e^{2\mu} > 32C_{p,q,\alpha,\beta}$, and since $\left|\left|\left|\mathcal{T}^{-(\cdot)}F(\cdot)\right|\right|\right|_{p,q,\alpha,\beta,T} < \infty$, when we let $n \to \infty$, we get that $\left\|T_k^{-t} f^{(k)}(t)\right\|_{k,p,q,\alpha,\beta} = 0$. Since $t \in [0,T]$ was arbitrary, we obtain $T_k^{-(\cdot)} f^{(k)}(\cdot) = 0$. Hence $f^{(k)} = 0$, and thus $F = 0$. □

3.3. **Proof of Theorem 1.8.** Thanks to the assumption $e^{2\mu} > 32C_{p,q,\alpha,\beta}$ in Theorem 1.8, the solution constructed in Theorem 3.1 is unique due to Theorem 3.4.

Using representation (3.8), Fubini's theorem, and the conservation laws (1.25)-(1.27) at the level of the wave kinetic equation, one can obtain the conservation laws (1.58)-(1.60) for the wave kinetic hierarchy.

Also, we prove the stability estimate (1.61) under the assumption that the initial datum $F_0 \in \mathcal{A} \cap \mathcal{X}^\infty_{p,q,\alpha,\beta,\mu'}$ is tensorised, i.e. $F_0 = (f_0^{\otimes k})_{k=1}^\infty$. For such data, by Remark 3.3, we have $\|f_0\|_{p,q,\alpha,\beta} \leq e^{-\mu'}$ and $F = (f^{\otimes k})_{k=1}^\infty$ is the solution to the wave kinetic hierarchy (1.28), where f is the mild solution of the wave kinetic equation with initial data f_0, obtained by Theorem 1.4. In particular, by (1.24), we have $\|T_1^{-t} f\|_{p,q,\alpha,\beta} \leq 2\|f_0\|_{p,q,\alpha,\beta}$. Therefore, using (A.3) and (A.1), we obtain

$$
\begin{aligned}
e^{\mu k}\|T_k^{-t} f^{\otimes k}(t)\|_{k,p,q,\alpha,\beta} = e^{\mu k}\|T_1^{-t} f(t)\|^k_{p,q,\alpha,\beta} &\leq 2^k e^{\mu k}\|f_0\|^k_{p,q,\alpha,\beta} \\
&\leq e^{\mu' k}\|f_0^{\otimes k}\|_{k,p,q,\alpha,\beta} \leq \|F_0\|_{p,q,\alpha,\beta,\mu',T}.
\end{aligned}
$$

Taking supremum over time, bound (1.57) follows.

Appendix A.

A.1. **Properties of tensorized functions.** Here we state results from [6] regarding the relationship between the norms used in this paper and tensorized products of a given function $h : \mathbb{R}^{2d} \to \mathbb{R}$ defined by

$$
h^{\otimes k}(X_k, V_k) = \prod_{i=1}^k h(x_i, v_i), \quad k \in \mathbb{N}.
$$

Remark A.1. *We note that given $k \in \mathbb{N}$, $h^{\otimes k} \in X^k_{p,q,\alpha,\beta}$ if and only if $h \in X_{p,q,\alpha,\beta}$. In particular, there holds*

$$
\|h^{\otimes k}\|_{k,p,q,\alpha,\beta} = \|h\|^k_{p,q,\alpha,\beta}, \quad \forall k \in \mathbb{N}. \tag{A.1}
$$

Remark A.2. *We note that the transport operator tensorizes as well. Namely for given $h : \mathbb{R}^{2d} \to \mathbb{R}$, we have*

$$
T_k^s h^{\otimes k} = (T_1^s h)^{\otimes k}, \quad \forall\, s \in \mathbb{R},\ \forall\, k \in \mathbb{N}. \tag{A.2}
$$

In particular, by (A.1)*, we have*

$$
\|T_k^s h^{\otimes k}\|_{k,p,q,\alpha,\beta} = \|T_1^s h\|^k_{p,q,\alpha,\beta}, \quad \forall\, s \in \mathbb{R},\ \forall\, k \in \mathbb{N}. \tag{A.3}
$$

A.2. **Resonant manifold properties.** Here we gather several properties of resonant manifolds.

Lemma A.3. *Suppose $w_0, w_1, w_2, w_3 \in \mathbb{R}^d$ lie on the resonant manifold determined by $\Sigma = \{w_0 + w_1 = w_2 + w_3\}$ and $\Omega = \left\{|w_0|^2 + |w_1|^2 = |w_2|^2 + |w_3|^2\right\}$. For any $i, j \in \{0,1,2,3\}$, define*

$$
W_{i,j} = w_i - w_j, \tag{A.4}
$$

and for any $v \in \mathbb{R}^d$, let $\hat{v}$ denote the unit vector in the direction of v, that is

$$\hat{v} = \frac{v}{|v|}. \tag{A.5}$$

Then the following identities hold

$$W_{0,2} \cdot W_{0,3} = 0, \tag{A.6}$$

$$|W_{0,1}| = |W_{2,3}|, \tag{A.7}$$

$$|W_{0,2}|^2 + |W_{0,3}|^2 = |W_{0,1}|^2 \tag{A.8}$$

Additionally, the following estimate holds

$$\min\{|W_{0,2}|, |W_{0,3}|\} \geq \frac{|W_{0,1}|}{2}\sqrt{1 - (\widehat{W}_{0,1} \cdot \widehat{W}_{2,3})^2}. \tag{A.9}$$

Proof. Suppose $w_0, w_1, w_2, w_3 \in \mathbb{R}^d$ belong to Σ and Ω, that is, they satisfy

$$w_0 + w_1 = w_2 + w_3, \tag{A.10}$$

$$|w_0|^2 + |w_1|^2 = |w_2|^2 + |w_3|^2. \tag{A.11}$$

By squaring (A.10) and subtracting from it (A.11), one has that

$$w_0 \cdot w_1 = w_2 \cdot w_3. \tag{A.12}$$

Next, by multiplying the identity (A.10) by w_0 and combining that with (A.12), one has

$$|w_0|^2 + w_2 + w_3 = w_0 \cdot (w_2 + w_3),$$

which is equivalent to (A.6).

Identity (A.7) easily follows from (A.11) and (A.12).

By combining the orthogonality property (A.6) and (A.7), we have

$$|W_{0,2}|^2 + |W_{0,3}|^2 = |W_{2,3}|^2 = |W_{0,1}|^2. \tag{A.13}$$

Finally, we prove (A.9). Momentum equation (A.10) and (A.7) imply that

$$w_2 = \frac{w_0 + w_1}{2} + \frac{|W_{0,1}|}{2}\widehat{W}_{2,3}, \tag{A.14}$$

$$w_3 = \frac{w_0 + w_1}{2} - \frac{|W_{0,1}|}{2}\widehat{W}_{2,3}. \tag{A.15}$$

Therefore,

$$|W_{0,2}| = \frac{|W_{0,1}|}{2}|\widehat{W}_{0,1} - \widehat{W}_{2,3}|, \qquad |W_{0,3}| = \frac{|W_{0,1}|}{2}|\widehat{W}_{0,1} + \widehat{W}_{2,3}|.$$

and thus

$$\begin{aligned}
|W_{0,2}|\,|W_{0,3}| &= \frac{|W_{0,1}|^2}{4}\sqrt{|\widehat{W}_{0,1} - \widehat{W}_{2,3}|^2 \;\; |\widehat{W}_{0,1} + \widehat{W}_{2,3}|^2} \\
&= \frac{|W_{0,1}|^2}{4}\sqrt{\left(2 - 2\,\widehat{W}_{0,1} \cdot \widehat{W}_{2,3}\right)\left(2 + 2\,\widehat{W}_{0,1} \cdot \widehat{W}_{2,3}\right)} \\
&= \frac{|W_{0,1}|^2}{2}\sqrt{1 - (\widehat{W}_{0,1} \cdot \widehat{W}_{2,3})^2}.
\end{aligned} \tag{A.16}$$

Then, the estimate (A.9) follows from the elementary inequality $\min\{|x|,|y|\} \geq \frac{|xy|}{\sqrt{x^2+y^2}}$ and identities (A.13) and (A.16).

□

A.3. **Various integral estimates.** In this section we gather several estimates that will be used throughout the paper. We begin by two lemmas that will be used to estimate time integrals appearing in proofs a priori estimates in Section 3.2.1.

Lemma A.4 ([6, Lemma A.1])**.** *For $p > 1$ and $x, \eta \in \mathbb{R}^d$ with $\eta \neq 0$, we have*

$$\int_{-\infty}^{\infty} \langle x + s\eta\rangle^{-p}\, ds \leq \frac{2p}{p-1}\frac{1}{|\eta|}. \tag{A.17}$$

Lemma A.5 ([6, Lemma 3.3])**.** *Let $p > 1$ and $x \in \mathbb{R}^d$. Consider $\xi, \eta \in \mathbb{R}^d$ with $\xi, \eta \neq 0$ and $\xi \cdot \eta = 0$. Then for any $t \geq 0$ there holds the bound*

$$\int_0^t \langle x + s\xi\rangle^{-p}\langle x + s\eta\rangle^{-p}\, ds \leq \frac{4p}{p-1}\,\frac{\langle x\rangle^{-p}}{\min\{|\xi|,|\eta|\}}. \tag{A.18}$$

Our next goal is to provide estimates (see Lemma A.8 and Lemma A.9) that are used to control velocity integrals in a priori estimates in Section 3.2.1. They, in turn, rely on two results - a convolution lemma from [6] (Lemma A.6) and a paramertization lemma for the integration over resonant manifolds (Lemma A.7). We start by recalling the convolution lemma from [6].

Lemma A.6 ([6, Lemma A.2.])**.** *Suppose $\delta \in (-d, 0]$ and let $q > d + \delta$. Then there exists a positive constant $L_{q,\delta}$ such that*

$$\int_{\mathbb{R}^d} |y - v|^{\delta}\langle y\rangle^{-q}\, dy \leq L_{q,\delta}, \quad \forall v \in \mathbb{R}^d. \tag{A.19}$$

One can take

$$L_{q,\delta} = \omega_{d-1}\left(\frac{1}{d} + \frac{1}{d+\delta} + \frac{2}{q-d-\delta}\right). \tag{A.20}$$

Next we prove a parametrization lemma for the integration over resonant manifolds determined by $\Sigma = v + v_1 - v_2 - v_3$ and $\Omega = |v|^2 + |v_1|^2 - |v_2|^2 - |v_3|^2$.

Lemma A.7. *For $f : \mathbb{R}^{4d} \to \mathbb{R}$ for which the integrals below make sense, we have*

$$\int_{\mathbb{R}^{3d}} \delta(\Sigma)\delta(\Omega) f(v, v_1, v_2, v_3)\, dv_1\, dv_2\, dv_3 = 2^{-d}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |v - v_1|^{d-2} f(v, v_1, v_2(\sigma), v_3(\sigma))\, d\sigma\, dv_1,$$

where $\Sigma = v + v_1 - v_2 - v_3$, $\Omega = |v|^2 + |v_1|^2 - |v_2|^2 - |v_3|^2$, and

$$v_2(\sigma) = \frac{v + v_1}{2} + \frac{|v - v_1|}{2}\sigma, \quad v_3(\sigma) = \frac{v + v_1}{2} - \frac{|v - v_1|}{2}\sigma. \tag{A.21}$$

Proof. Let us write

$$\begin{aligned} I(v) &= \int_{\mathbb{R}^{3d}} \delta(\Sigma)\delta(\Omega) f(v, v_1, v_2, v_3)\, dv_1\, dv_2\, dv_3 \\ &= \int_{\mathbb{R}^{2d}} \delta(|v_2|^2 + |v + v_1 - v_2|^2 - |v|^2 - |v_1|^2) f(v, v_1, v_2, v + v_1 - v_2)\, dv_1\, dv_2. \end{aligned}$$

It is easy to verify that

$$|v_2|^2 + |v+v_1-v_2|^2 - |v|^2 - |v_1|^2 = 2\left(\left|v_2 - \frac{v+v_1}{2}\right|^2 - \left|\frac{v-v_1}{2}\right|^2\right),$$

and so

$$I(v) = \int_{\mathbb{R}^{2d}} \delta\left(2\left|v_2 - \frac{v+v_1}{2}\right|^2 - 2\left|\frac{v-v_1}{2}\right|^2\right) f(v, v_1, v_2, v+v_1-v_2)\, dv_1\, dv_2.$$

By letting $y = \sqrt{2}\left(v_2 - \frac{v+v_1}{2}\right)$, and using polar coordinates, we obtain

$$\begin{aligned} I(v) &= 2^{-d/2}\int_{\mathbb{R}^{2d}} \delta\left(|y|^2 - \frac{|v-v_1|^2}{2}\right) f\left(v, v_1, \frac{v+v_1}{2} + \frac{y}{\sqrt{2}}, \frac{v+v_1}{2} - \frac{y}{\sqrt{2}}\right) dv_1 dy \\ &= 2^{-d/2}\int_{\mathbb{R}^d}\int_0^\infty\int_{\mathbb{S}^{d-1}} \delta\left(r^2 - \frac{|v-v_1|^2}{2}\right) f\left(v, v_1, \frac{v+v_1}{2} + \frac{r\sigma}{\sqrt{2}}, \frac{v+v_1}{2} - \frac{r\sigma}{\sqrt{2}}\right) r^{d-1} d\sigma dr dv_1. \end{aligned}$$

Next we apply another change of variables $z = r^2$ (and so $dr = \frac{dz}{2\sqrt{z}}$) to further obtain

$$\begin{aligned} I(v) &= 2^{-\frac{d}{2}-1}\int_{\mathbb{R}^d}\int_0^\infty\int_{\mathbb{S}^{d-1}} \delta\left(z - \frac{|v-v_1|^2}{2}\right) f\left(v, v_1, \frac{v+v_1}{2} + \frac{\sqrt{z}\sigma}{\sqrt{2}}, \frac{v+v_1}{2} - \frac{\sqrt{z}\sigma}{\sqrt{2}}\right) z^{\frac{d-2}{2}} d\sigma dz dv_1 \\ &= 2^{-\frac{d}{2}-1}\, 2^{-\frac{d-2}{2}} \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} |v-v_1|^{d-2}\ f\left(v, v_1, \frac{v+v_1}{2} + \frac{|v-v_1|}{2}\sigma, \frac{v+v_1}{2} - \frac{|v-v_1|}{2}\sigma\right) d\sigma dv_1, \end{aligned}$$

which completes the proof of the lemma. □

The following two lemmata, which are a consequence of Lemma A.6 and A.7, will be essential in providing estimates on velocity integrals in Section 3.2.1.

Lemma A.8 (Analogue of Lemma A.6 in delta notation). *Let $d \in \{2, 3\}$ and $q > 2d-3$. Then there exists a positive constant $\widetilde{L}_q$ such that for $\Sigma = v + v_1 - v_2 - v_3$, $\Omega = |v|^2 + |v_1|^2 - |v_2|^2 - |v_3|^2$ we have*

$$\int_{\mathbb{R}^{3d}} \delta(\Sigma)\delta(\Omega) \frac{1}{|v-v_1|\langle v_1\rangle^q}\, dv_1\, dv_2\, dv_3 \le \widetilde{L}_q, \quad \forall v \in \mathbb{R}^d. \tag{A.22}$$

One can take

$$\widetilde{L}_q = 2^{-d}\omega_{d-1}^2\left(\frac{1}{d} + \frac{1}{2d-3} + \frac{2}{q-2d+3}\right), \tag{A.23}$$

where ω_{d-1} denotes the area of the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. In particular, for $d = 3$ we have $\widetilde{L}_q = 4\pi^2\left(\frac{1}{3} + \frac{1}{q-3}\right)$.

Proof. By Lemma A.7, we have

$$I := \int_{\mathbb{R}^{3d}} \delta(\Sigma)\delta(\Omega)\frac{1}{|v-v_1|\langle v_1\rangle^q}\, dv_1\, dv_2\, dv_3 = 2^{-d}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |v-v_1|^{d-3}\langle v_1\rangle^{-q}\, d\sigma\, dv_1.$$

Since $d \in \{2, 3\}$ and $q > 2d-3$, we can apply Lemma A.6 with $\delta = d-3$ to obtain

$$I \le 2^{-d}\omega_{d-1}L_{q,d-3},$$

where $L_{q,d-3}$ is given by (A.20) with $\delta = d-3$. This completes the proof of the lemma. □

Lemma A.9. *For $d = 3$ and $q > 3$ there exists a positive constant U_q such that*

$$\sup_{v\in\mathbb{R}^d}\int_{\mathbb{R}^{3d}} \frac{\delta(\Sigma)\delta(\Omega)}{|v-v_1|\sqrt{1-(\frac{v-v_1}{|v-v_1|}\cdot\frac{v_2-v_3}{|v_2-v_3|})^2}}\frac{\langle v\rangle^q}{\langle v_1\rangle^q\langle v_2\rangle^q\langle v_3\rangle^q}\,dv_1\,dv_2\,dv_3 \le U_q, \tag{A.24}$$

$$\sup_{v\in\mathbb{R}^d}\int_{\mathbb{R}^{3d}} \frac{\delta(\Sigma)\delta(\Omega)}{|v-v_1|\sqrt{1-(\frac{v-v_1}{|v-v_1|}\cdot\frac{v_2-v_3}{|v_2-v_3|})^2}}\frac{1}{\langle v_2\rangle^q\langle v_3\rangle^q}\,dv_1\,dv_2\,dv_3 \le U_q. \tag{A.25}$$

One can take

$$U_q = 2\pi^3\left(\frac{1}{3}+\frac{1}{q-3}\right). \tag{A.26}$$

Proof. Let us denote the integral appearing in (A.24) by $I_1(v)$ and the integral in (A.25) by $I_2(v)$. By the conservation of energy, we have

$$\langle v_2\rangle^2\langle v_3\rangle^2 = (1+|v_2|^2)(1+|v_3|^2) = 1+|v|^2+|v_1|^2+|v_2|^2|v_3|^2 \ge \max\{\langle v\rangle^2,\langle v_1\rangle^2\},$$

and thus

$$\frac{\langle v\rangle^q}{\langle v_2\rangle^q\langle v_3\rangle^q}\le 1, \quad\text{and}\quad \frac{1}{\langle v_2\rangle^2\langle v_3\rangle^2}\le\frac{1}{\langle v_1\rangle^2},$$

which implies that both $I_1(v)$ and $I_2(v)$ can be estimated by the same upper bound:

$$I_1(v),\, I_2(v) \le \int_{\mathbb{R}^{3d}} \frac{\delta(\Sigma)\delta(\Omega)}{|v-v_1|\sqrt{1-(\frac{v-v_1}{|v-v_1|}\cdot\frac{v_2-v_3}{|v_2-v_3|})^2}}\frac{1}{\langle v_1\rangle^q}\,dv_1\,dv_2\,dv_3.$$

Then by Lemma A.7, we have

$$I_1(v),\, I_2(v) \le 2^{-d}\int_{\mathbb{R}^d}\frac{|v-v_1|^{d-3}}{\langle v_1\rangle^q}\int_{\mathbb{S}^{d-1}}\frac{1}{\sqrt{1-(\frac{v-v_1}{|v-v_1|}\cdot\sigma)^2}}d\sigma dv_1.$$

For $d\ge 3$, integration in spherical coordinates yields

$$\int_{\mathbb{S}^{d-1}}\frac{1}{\sqrt{1-(\hat{n}\cdot\sigma)^2}}\,d\sigma = \omega_{d-2}\int_0^\pi \sin^{d-3}(\theta)\,d\theta \le \pi\omega_{d-2}, \tag{A.27}$$

where ω_{d-2} denotes the area of the unit sphere $\mathbb{S}^{d-2}$. We next apply Lemma A.6 with $\delta = d-3$, which requires that $d\in\{2,3\}$ and $q > 2d-3$. Since the estimate (A.27) required $d\ge 3$, we need the dimension to be $d = 3$, and together with $q > 2d-3 = 3$, we have

$$I_1(v),\, I_2(v) \le 2^{-d}\pi\omega_{d-2}\int_{\mathbb{R}^d}|v-v_1|^{d-3}\langle v_1\rangle^{-q}dv_1 \le 2^{-d}\pi\omega_{d-2}L_{q,d-3},$$

where $L_{q,d-3}$ is the constant from Lemma A.6. Since $d = 3$, we in fact have

$$I_1(v),\, I_2(v) \le 2^{-3}\pi(2\pi)(4\pi)\left(\frac{1}{3}+\frac{1}{3}+\frac{2}{q-3}\right) = 2\pi^3\left(\frac{1}{3}+\frac{1}{q-3}\right).$$

This completes the proof of the lemma. □

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