On the uniqueness of the mild solution of the critical quasi-geostrophic equation

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Abstract. We demonstrate that the uniqueness of the mild solution of the two-dimensional quasi-geostrophic equation with the critical dissipation holds in the scaling critical homogeneous Besov space $\dot{B}^0_{\infty,1}$. We consider a solustion of integral equation, and our result does not need regularity assumption.

1. Introduction.

In this paper, we consider the critical quasi-geostrophic equation in $\mathbb{R}^2$.

$$\begin{cases} \partial_t\theta + \Lambda\theta + u\cdot\nabla\theta = 0, & t>0, x\in\mathbb{R}^2, \\ u = \nabla^\perp\Lambda^{-1}\theta, & t>0, x\in\mathbb{R}^2, \\ \theta(0,x) = \theta_0(x), & x\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ and $\Lambda = (-\Delta)^{\frac{1}{2}}$. The real-valued function $\theta(t,x)$ denotes the potential temperature of the fluid. The quasi-geostrophic equation is an important model in geophysical dynamics, which describes large scale atmospheric and oceantic motion with small Rossby and Ekman numbers (see [12]). From a mathematical point of view, the quasi-geostrophic equation has similar structure with the there-dimensional Euler and Navier-Stokes equations (see [4]).

Let us recall several known results about the existence and the uniqueness. We consider Λ^α with $1\le\alpha\le 2$ instead of Λ in the equation of (1.1). In the cace when $1<\alpha\le 2$, Constantin and Wu [5] proved that the strong solution $\theta\in L^\infty(0,T;L^2)\cap L^2(0,T;H^{\frac{\alpha}{2}})\cap L^q(0,T;L^p)$ exists uniquelly for $\theta_0\in L^2, p\ge 1, q>1, 1/p+\alpha/2q = \alpha/2-1/2$. If $1<\alpha<2$, Ferreira [6] established the uniqueness of the mild solution $\theta\in C([0,T];L^{\frac{2}{\alpha-1}})$.

For $\alpha=2$, Iwabuchi, Ueda [8] showed the uniqueness of mild solution $\theta\in C([0,T];L^2)$. If $0<\alpha\le 1$, the existence of the local unique solution $\theta\in C([0,T];\mathring{B}^{1-\alpha}_{\infty,q})\cap\widetilde{L}^1_T B^1_{\infty,q}$ for $\theta_0\in\mathring{B}^{1-\alpha}_{\infty,q}, 1\le q<\infty$, is proved by Wang and Zhang ([13]). Here $\mathring{B}^{1-\alpha}_{\infty,q}$ is the closure of the Schwartz functions in the norm of the inhomogenous Besov space $B^{1-\alpha}_{\infty,q}$. This paper is devoted to the uniqueness of mild solutions of (1.1) in a critical space $\dot{B}^0_{\infty,1}$.

We also mention some literature on the Navier-Stokes equations. The scale critical spaces of the two-dimensional surface quasi-geostrophic equation are same as those of the two-dimensional incompressible Navier-Stokes equations. It is a classical result that the weak solustion $u\in L^\infty(0,T;L^2)\cap L^2(0,T;\dot{H}^1)$ is unique if $u_0\in L^2$ for the two-dimensional incompressible Navier-Stokes equations. The non-uniqueness in $C(0,T;L^p)$

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with $1 \le p < 2$ for the two-dimensional incompressible Navier-Stokes equations on $\mathbb{T}^2$ is provided by Cheskidov-Luo [3]. In the case when the space dimension is three, Meyer ([9]), Furioli, Lemarié-Rieusset, Terraneo ([7]) and Monniaux ([11]) proved the uniqueness of the mild solution in $C([0,T];L^3)$ for the incompressible Navier-Stokes equations. The corresponding result for our problem is the uniqueness for $\theta \in C([0,T];L^\infty)$, but there are some difficulty. Hence we consider (1.1) in $\dot{B}^0_{\infty,1}$ which is close to L^∞.

Let us recall the definition of homogeneous Besov space. We refer to the book by Bahouri, Chemin, Danchin ([1]).

Definition 1. Let $\{\phi_j\}_{j\in\mathcal{Z}} \subset \mathcal{S}(\mathbb{R}^2)$ be such that

$$\text{supp } \widehat{\phi}_j \subset \{\xi \in \mathbb{R}^2 \mid 2^{j-1} \le |\xi| \le 2^{j+1}\} \text{ for any } j \in \mathbb{Z}, \quad \sum_{j\in\mathbb{Z}} \widehat{\phi}_j(\xi) = 1 \text{ for any } \xi \in \mathbb{R}^2\backslash\{0\}.$$

Let $\mathcal{S}'_h$ be the space of tempred distribution f such that

$$\lim_{\lambda\to\infty} \left\| \mathcal{F}^{-1}\left[\psi(\lambda\xi)\widehat{f}(\xi)\right]\right\|_{L^\infty} = 0 \quad \text{for any } \psi \in C_c^\infty(\mathbb{R}^2).$$

Then

$$f = \sum_{j\in\mathbb{Z}} \phi_j * f$$

holds in $\mathcal{S}'_h$. For $s \in \mathbb{R}$ and $1 \le p,q \le \infty$, we define the homogeneous Besov spaces as follows.

$$\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^2) := \{f \in \mathcal{S}'_h \mid \|f\|_{\dot{B}^s_{p,q}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} := \left\| \left\{2^{sj}\|\phi_j * f\|_{L^p}\right\}_{j\in\mathbb{Z}}\right\|_{l^q}.$$

Remark. It is known that if $s < 2/p$ or $s = 2/p$ and $q = 1$, $\dot{B}^s_{p,q}$ is a Banach space.

We define the mild solution of (1.1).

Definition 2. Let $T > 0, \theta_0 \in \dot{B}^0_{\infty,1}$. If $\theta : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ satisfies

$$\begin{cases} \theta \in C([0,T];\dot{B}^0_{\infty,1}), \\ \displaystyle\int_{\mathbb{R}^2} \theta(t,x)\phi(x)\, \mathrm{d}x = \int_{\mathbb{R}^2} e^{-t\Lambda}\theta_0(x)\,\phi(x)\,\mathrm{d}x + \int_{\mathbb{R}^2}\left(\int_0^t e^{-(t-s)\Lambda}(u\theta)\,\mathrm{d}s\right)\cdot\nabla\phi(x)\,\mathrm{d}x, \end{cases}$$

for all $t \in [0,T]$ and $\phi \in \mathcal{S}(\mathbb{R}^2)$, then we call θ a mild solution of (1.1).

We state our main result.

Theorem 1.1. Let $T > 0$ and $\theta^{(1)}, \theta^{(2)} \in C([0,T];\dot{B}^0_{\infty,1})$ are mild solutions of (1.1) such that $\theta^{(1)}(0) = \theta^{(2)}(0)$. Then $\theta^{(1)}(t) = \theta^{(2)}(t)$ in $\dot{B}^0_{\infty,1}$ for all $t \in [0,T]$.

In the proof of Theorem 1.1, we divide the equation (1.1) into high frequency part and low frequency part and consider the norms $\dot{B}^{-\eta}_{\infty,\infty}$ and $\dot{B}^0_{\infty,1}$ with $0 < \eta < 1$, respectively. After that using the maximum principle in homogeneous Besov space that is proved by [10]. To this end, we introduce some notations.

Notation. For any $j \in \mathbb{Z}$, we denote

$$f_j := \phi_j * f, \quad f_{\text{high}} := \sum_{j=1}^{\infty} \phi_j * f \quad \text{and} \quad f_{\text{low}} := f - f_{\text{high}}. \tag{1.2}$$

Also we introduce a notation for nonlinear term in the integral equation of (1.1),

$$\theta(t) = e^{-t\Lambda}\theta_0 - \int_0^t \nabla \cdot e^{-(t-s)\Lambda}(u\theta)\, \mathrm{d}s =: e^{-t\Lambda}\theta_0 - B(u\theta)(t). \tag{1.3}$$

2. Preliminarys.

In this section, we introduce several lemmas which are elemental properties of Besov spaces and heat kernel.

Lemma 2.1. ([1, 14]) Let $s \in \mathbb{R}$ and $1 \le p \le \infty$. Then there exist constants $C, c > 0$ such that

$$\|\nabla(\phi_j * f)\|_{L^p} \le C2^j \|\phi_j * f\|_{L^p},\ \|\Lambda^{-1}(\phi_j * f)\|_{L^p} \le C2^{-j}\|\phi_j * f\|_{L^p} \tag{2.4}$$

and

$$\|\phi_j * e^{-t\Lambda} f\|_{L^p} \le Ce^{-ct2^j}\|\phi_j * f\|_{L^p} \tag{2.5}$$

for all $f \in L^p, t > 0, j \in \mathbb{Z}$.

Lemma 2.2. ([1]) Let $s \in \mathbb{R},\ s' < 0$, $1 \le p, q \le \infty$ and $1/q = 1/q_1 + 1/q_2 \le 1$. Then

$$\Big\|\sum_{k \le l-2} f_k g_l\Big\|_{\dot{B}^s_{p,q}} \le C\|f\|_{L^\infty}\|g\|_{\dot{B}^s_{p,q}} \text{ for all } f \in L^\infty,\ g \in \dot{B}^s_{p,q},$$

and

$$\Big\|\sum_{k \le l-2} f_k g_l\Big\|_{\dot{B}^{s+s'}_{p,q}} \le C\|f\|_{\dot{B}^{s'}_{\infty,q_1}}\|g\|_{\dot{B}^s_{p,q_2}} \text{ for all } f \in \dot{B}^{s'}_{\infty,q_1},\ g \in \dot{B}^s_{p,q}.$$

Lemma 2.3. ([1]) Let $s_1, s_2 \in \mathbb{R},\ s_1 + s_2 > 0, 1 \le p, q \le \infty, 1/p = 1/p_1 + 1/_2 \le 1$ and $1/q = 1/q_1 + 1/q_2 \le 1$. Then

$$\Big\|\sum_{|k-l| \le 1} f_k g_l\Big\|_{\dot{B}^{s_1+s_2}_{p,q}} \le C\|f\|_{\dot{B}^{s_1}_{p_1,q_1}}\|g\|_{\dot{B}^{s_2}_{p_2,q_2}} \text{ for all } f \in \dot{B}^{s_1}_{p_1,q_1},\ g \in \dot{B}^{s_2}_{p_2,q_2}.$$

Lemma 2.4. ([1]) Let $s > 0,\ 1 \le p, q \le \infty$. Then

$$\|fg\|_{\dot{B}^s_{p,q}} \le C\|f\|_{L^\infty \cap \dot{B}^s_{p,q}}\|g\|_{L^\infty \cap \dot{B}^s_{p,q}} \text{ for all } f, g \in L^\infty \cap \dot{B}^s_{p,q}.$$

Next lemma is concerned with the boundness of biliner Fourier multiplier in L^∞. We notice the frequency of the function is restricted.

Lemma 2.5. Let $j, j' \in \mathbb{Z}$ and $|j - j'| \le 1$. Then

$$\Big\|\int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{(\xi, -\eta)}{|\xi| + |\eta|} \widehat{f}_j(\xi)(\eta)\widehat{g_{j'}}(\eta) e^{i(\cdot,\cdot)\cdot(\xi,\eta)}\, \mathrm{d}(\xi, \eta)\Big\|_{L^\infty} \le C\|f_j\|_{L^\infty}\|g_{j'}\|_{L^\infty} \tag{2.6}$$

for all $f, g \in L^\infty$, where $f_j, g_{j'}$ are defined by (1.2)

Proof. We can write left side of (2.6) using Fourier multiplier in $\mathbb{R}^4$, which we write by $\mathcal{F}^{-1}_{\mathbb{R}^4}$. We denote $\widetilde{\phi}_j = \phi_{j-1}+\phi_j+\phi_{j+1}$. Then $\widetilde{\phi}_j * \phi_j * f = \phi_j * f$. Using Young's inequality, we have

$$\left\|\mathcal{F}^{-1}_{\mathbb{R}^4}\left[\frac{(\xi,-\eta)}{|\xi|+|\eta|}\widehat{\widetilde{\phi}}_j(\xi)\widehat{f_j}(\xi)\widehat{\widetilde{\phi}}_{j'}(\eta)\widehat{g_{j'}}(\eta)\right]\right\|_{L^\infty} \le C\left\|\mathcal{F}^{-1}_{\mathbb{R}^4}\left[\frac{(\xi,-\eta)}{|\xi|+|\eta|}\widehat{\widetilde{\phi}}_j(\xi)\widehat{\widetilde{\phi}_{j'}}(\eta)\right]\right\|_{L^1}\|f_j\|_{L^\infty}\|g_{j'}\|_{L^\infty}.$$

and since $|j-j'|\le 1$, we have

$$\begin{aligned}\left\|\mathcal{F}^{-1}_{\mathbb{R}^4}\left[\frac{(\xi,-\eta)}{|\xi|+|\eta|}\widehat{\widetilde{\phi}}_j(\xi)\widehat{\widetilde{\phi}}_{j'}(\eta)\right]\right\|_{L^1} &= \left\|\mathcal{F}^{-1}_{\mathbb{R}^4}\left[\frac{(\xi,-\eta)}{|\xi|+|\eta|}\widehat{\widetilde{\phi}}_{j-j'}(\xi)\widehat{\widetilde{\phi}}_0(\eta)\right]\right\|_{L^1}\\ &\le Constant,\end{aligned}$$

where the constant is independent of j. ∎

Lemma 2.6. Let $T>0$ and $\theta \in C([0,T];\dot{B}^0_{\infty,1})$ be a mild solution of (1.1). Then for any $j\in\mathbb{Z}$, $\partial_t\theta_j \in C([0,T];\dot{B}^0_{\infty,1})$ and for all $t\in(0,T)$, θ_j satisfies

$$\partial_t\theta_j + \Lambda\theta_j + \nabla\phi_j * (u\theta) = 0 \text{ in } \dot{B}^0_{\infty,1}.$$

Proof. Fix $t\in(0,T)$ and $x\in\mathbb{R}^2$. By definition of the mild solution, for $\phi_j(x-\cdot)\in\mathcal{S}(\mathbb{R}^2)$, we obtain

$$\phi_j * \theta(t) = \phi_j * e^{-t\Lambda}\theta_0 - \nabla\phi_j * \left(\int_0^t e^{-(t-s)\Lambda}(u\theta)\,\mathrm{d}s\right).$$

Since $\nabla\phi_j *$ is a bounded operator in L^∞, we can write

$$\nabla\phi_j * \left(\int_0^t e^{-(t-s)\Lambda}(u\theta)\,\mathrm{d}s\right) = \int_0^t e^{-(t-s)\Lambda}(\nabla\phi_j*(u\theta))\,\mathrm{d}s, \quad t>0, \quad x\in\mathbb{R}^2.$$

To prove Lemma 2.6, we show that

$$\partial_t\left(\int_0^t e^{-(t-s)\Lambda}(\nabla\phi_j*(u\theta))\,\mathrm{d}s\right) = -\Lambda\left(\int_0^t e^{-(t-s)\Lambda}(\nabla\phi_j*(u\theta))\,\mathrm{d}s\right) + \nabla\phi_j*(u\theta).$$

For $h\in(0,T-t)$,

$$\begin{aligned}&\frac{1}{h}\left(\int_0^{t+h} e^{-(t+h-s)\Lambda}(\nabla\phi_j*(u\theta))\,\mathrm{d}s - \int_0^t e^{-(t-s)\Lambda}(\nabla\phi_j*(u\theta))\,\mathrm{d}s\right)\\ &= \frac{1}{h}\int_t^{t+h} e^{-(t+h-s)\Lambda}(\nabla\phi_j*(u\theta))\,\mathrm{d}s + \frac{1}{h}\int_0^t (e^{-(t+h-s)\Lambda}-e^{-(t-s)\Lambda})(\nabla\phi_j*(u\theta))\,\mathrm{d}s\\ &=: I + II.\end{aligned}$$

The continuity of $e^{-t\Lambda}$ and $\nabla\phi_j*(u\theta)$ with respect to time, we have

$$\lim_{h\to 0} I = \nabla\phi_j*(u(t)\theta(t)) \text{ in } \dot{B}^0_{\infty,1}.$$

By the boundness of $e^{-h\Lambda}$ in $\dot{B}^0_{\infty,1}$ and $\Lambda\nabla\phi_j*(u\theta)\in C([0,T];\dot{B}^0_{\infty,1})$, we can justify the limit for II and we obtain

$$\lim_{h\to0} II = \frac{e^{-h\Lambda}-Id}{h}\int_0^t e^{-(t-s)\Lambda}(\nabla\phi_j*(u\theta))\,\mathrm{d}s = -\Lambda\left(\int_0^t e^{-(t-s)\Lambda}(\nabla\phi_j*(u\theta))\,\mathrm{d}s\right) \text{ in } \dot{B}^0_{\infty,1}.$$

Therefore θ_j is differentiable in time and

$$\partial_t\theta_j = -\Lambda\theta_j - \nabla\phi_j*(u\theta) \text{ in } \dot{B}^0_{\infty,1}.$$

∎

We consider the transport-diffusion equation

$$\begin{cases}\partial_t\theta+\Lambda\theta+u\cdot\nabla\theta=f+g, & t>0, x\in\mathbb{R}^2,\\ \theta(0,x)=\theta_0(x), & x\in\mathbb{R}^2,\end{cases}\tag{2.7}$$

where u is a given vector field such that div $u=0$ and $f,\ g$ are given external forces.

The following proposition is a corollary of Theorem 1.2 in [10]. We give a brief proof.

Proposition 2.7. Let $-1<s<1$. Assume that θ is a smooth solution of (2.7). Then θ satisfies

$$\|\theta\|_{L^\infty(0,T;\dot B^s_{\infty,\infty})}\le C\exp\big(C\|\nabla u\|_{L^1(0,T;\dot B^0_{\infty,\infty})}\big)\big(\|\theta_0\|_{\dot B^s_{\infty,1}}+\|f\|_{L^1(0,T;\dot B^s_{\infty,\infty})}+\|g\|_{L^\infty(0,T;\dot B^{s-1}_{\infty,\infty})}\big)\tag{2.8}$$

Proof. Let ψ_j be the flow of $\displaystyle\sum_{k\le j-2}u_k$. We denote $\overline{f}_j:=f_j\circ\psi_j$. Applying ϕ_j* to (2.7) and change of variable $x\mapsto\psi_j$, we have

$$\partial_t\overline{\theta}_j+\Lambda\overline{\theta}_j=\overline{f}_j+\overline{g}_j+\overline{R}_j+G_j,\tag{2.9}$$

where $R_j:=\Big(\displaystyle\sum_{k\le j-2}u_k-u\Big)\cdot\nabla\theta_j-[\phi_j*,u\cdot\nabla]\theta$ and $G_j:=\Lambda\overline{\theta}_j-(\Lambda\theta_j)\circ\psi_j$. We write $\theta_{j,j'}:=\phi_{j'}*\phi_j*\theta$. Applying $\phi_{j'}*$ and using Lemma 2.1 (2.5), we get

$$\begin{aligned}\|\overline{\theta}_{j,j'}\|_{L^\infty}&\le Ce^{-ct2^{j'}}\|\theta_{0,j,j'}\|_{L^\infty}\\&\quad+C\int_0^t e^{-c(t-\tau)2^{j'}}\big(\|\overline{f}_{j,j'}(\tau)\|_{L^\infty}+\|\overline{g}_{j,j'}(\tau)\|_{L^\infty}+\|\overline{R}_{j,j'}(\tau)\|_{L^\infty}+\|G_{j,j'}(\tau)\|_{L^\infty}\big)\,\mathrm{d}\tau.\end{aligned}$$

According to the property of flow and commutator estimate, we obtain

$$\begin{aligned}\|\overline{\theta}_{j,j'}\|_{L^\infty}&\le Ce^{-ct2^{j'}}\|\theta_{0,j,j'}\|_{L^\infty}\\&\quad+C2^{j-j'}\int_0^t e^{-c(t-\tau)2^{j'}}\exp\big(C\|\nabla u\|_{L^1(0,\tau;L^\infty)}\big)\\&\qquad\times\big(\|f_j(\tau)\|_{L^\infty}+\|g_j(\tau)\|_{L^\infty}+2^{-js}\|\nabla u(\tau)\|_{\dot B^0_{\infty,1}}\|\theta(\tau)\|_{\dot B^s_{\infty,\infty}}\\&\qquad\quad+2^{j'}\|\nabla u\|^{\frac12}_{L^1(0,\tau;L^\infty)}\|\theta_j(\tau)\|_{L^\infty}\big)\,\mathrm{d}\tau.\end{aligned}$$

Taking the L^∞ norm over $[0,t]$ and multiplying both sides by 2^{js}, we have

$$\begin{aligned}&2^{js}\|\overline{\theta}_{j,j'}\|_{L^\infty}\\&\le C2^{js}\|\theta_{0,j,j'}\|_{L^\infty}\\&\quad+C\max\{1,2^{j-j'}\}2^{j-j'}\exp\big(C\|\nabla u\|_{L^1(0,t;L^\infty)}\big)\big(2^{js}\|f_j\|_{L^1(0,t;L^\infty)}+2^{j(s-1)}\|g_j\|_{L^\infty(0,t;L^\infty)}\big)\\&\quad+C2^{j-j'}\int_0^t\exp\big(C\|\nabla u\|_{L^1(0,\tau;L^\infty)}\big)\|\nabla u(\tau)\|_{\dot B^0_{\infty,1}}\|\theta(\tau)\|_{\dot B^s_{\infty,\infty}}\,\mathrm{d}\tau\\&\quad+C2^{j-j'}\exp\big(C\|\nabla u\|_{L^1(0,t;L^\infty)}\big)\|\nabla u\|^{\frac12}_{L^1(0,t;L^\infty)}2^{js}\|\theta_j\|_{L^\infty(0,t;L^\infty)}.\end{aligned}$$

The rest of the proof is the same as [10]. □

Thanks to Lemma 2.1 (2.5), we obtain convergence of linear solustion.

Lemma 2.8. Suppose that $\theta_0 \in \dot{B}^0_{\infty,1}$. Then

$$\lim_{T\to 0} \|e^{-t\Lambda}\theta_0\|_{L^1(0,T;\dot{B}^1_{\infty,1})} = 0.$$

Also next lemma implies convergence of the nonlinear term in the mild solustion of (1.1).

Lemma 2.9. If θ is a mild solustion of (1.1), then

$$\lim_{t\to 0} \|\theta(t) - e^{-t\Lambda}\theta_0\|_{\dot{B}^0_{\infty,1}} = 0.$$

Following lemma are valid for functions with high or low frequency restriction.

Lemma 2.10. Let $s \in \mathbb{R},\ s' > 0$ and $1 \le p, q \le \infty$. Then

$$\|f_{\mathrm{high}}\|_{\dot{B}^{s-s'}_{p,q}} \le C\|f_{\mathrm{high}}\|_{\dot{B}^s_{p,q}} \quad \text{and} \quad \|f_{\mathrm{low}}\|_{\dot{B}^{s+s'}_{p,q}} \le C\|f_{\mathrm{low}}\|_{\dot{B}^s_{p,q}}.$$

3. Proof of Theorem 1.1.

Let $w = \theta^{(1)} - \theta^{(2)},\ u^{(i)} = \nabla^\perp\Lambda^{-1}\theta^{(i)}$ $(i = 1, 2)$. We show that $\|w\|_{\dot{B}^0_{\infty,1}} = 0$ in $[0, T_0]$ for some small $T_0 > 0$. To that end, we consider the following:

$$\|w_{\mathrm{high}}\|_{\dot{B}^{-\eta}_{\infty,\infty}} + \|w_{\mathrm{low}}\|_{\dot{B}^0_{\infty 1}} \text{ with } 0 < \eta < 1,$$

where w_{high} and w_{low} are defined by (1.2). By Lemma 2.6, for any $j \in \mathbb{Z},\ w_j = \phi_j * w$ satisfies

$$\begin{cases} \partial_t w_j + \Lambda w_j + \nabla\phi_j * (u^{(1)}\theta^{(2)} - u^{(1)}\theta^{(2)}) = 0, \\ w_j(0, x) = 0 \end{cases} \tag{3.10}$$

and

$$u^{(1)}\theta^{(1)} - u^{(2)}\theta^{(2)} = \frac{1}{2}\sum_{i=1}^{2}((\nabla^\perp\Lambda^{-1}w)\theta^{(i)} + (\nabla^\perp\Lambda^{-1}\theta^{(i)})w).$$

Moreover, since $\theta^{(1)}, \theta^{(2)}$ are mild solustions of (1.1) and are written by the sum of the linear and the nonlinear parts,

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{2}((\nabla^\perp\Lambda^{-1}w)\theta^{(i)} + (\nabla^\perp\Lambda^{-1}\theta^{(i)})w) \\ &= (\nabla^\perp\Lambda^{-1}w)e^{-t\Lambda}\theta_0 + (\nabla^\perp\Lambda^{-1}(e^{-t\Lambda}\theta_0))w \\ &\quad - \frac{1}{2}\sum_{i=1}^{2}((\nabla^\perp\Lambda^{-1}w)B(u^{(i)}\theta^{(i)}) + (\nabla^\perp\Lambda^{-1}B(u^{(i)}\theta^{(i)}))w) \end{aligned}$$

where B is defined by (1.3).

First, we consider w_{high} i.e., high frequency part of w. Applying $\sum_{j'=1}^{\infty} \phi_{j'} *$ to (3.10), we denote $w_{\mathrm{high},j} := \phi_j * \sum_{j'=1}^{\infty} \phi_{j'} * w$ and obtain

$$\begin{aligned}
&\partial_t w_{\mathrm{high},j} + \Lambda w_{\mathrm{high},j} + \nabla \phi_j * \Big((\nabla^\perp \Lambda^{-1}(e^{-t\Lambda}\theta_0)) w \Big)_{\mathrm{high}} \\
&= -\nabla \phi_j * \Big((\nabla^\perp \Lambda^{-1} w) e^{-t\Lambda}\theta_0 - \frac{1}{2}\sum_{i=1}^{2} \big((\nabla^\perp \Lambda^{-1} w) B(u^{(i)}\theta^{(i)}) + (\nabla^\perp \Lambda^{-1} B(u^{(i)}\theta^{(i)})) w \big) \Big)_{\mathrm{high}}.
\end{aligned} \tag{3.11}$$

Now, we can write

$$\begin{aligned}
\Big((\nabla^\perp \Lambda^{-1}(e^{-t\Lambda}\theta_0)) w \Big)_{\mathrm{high}} &= \Big((\nabla^\perp \Lambda^{-1}(e^{-t\Lambda}\theta_0)) w_{\mathrm{high}} \Big)_{\mathrm{high}} + \Big((\nabla^\perp \Lambda^{-1}(e^{-t\Lambda}\theta_0)) w_{\mathrm{low}} \Big)_{\mathrm{high}} \\
&= (\nabla^\perp \Lambda^{-1}(e^{-t\Lambda}\theta_0)) w_{\mathrm{high}} \\
&\quad - \Big((\nabla^\perp \Lambda^{-1}(e^{-t\Lambda}\theta_0)) w_{\mathrm{high}} \Big)_{\mathrm{low}} + \Big((\nabla^\perp \Lambda^{-1}(e^{-t\Lambda}\theta_0)) w_{\mathrm{low}} \Big)_{\mathrm{high}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\partial_t w_{\mathrm{high},j} + \Lambda w_{\mathrm{high},j} + \nabla \phi_j * ((\nabla^\perp \Lambda^{-1}(e^{-t\Lambda}\theta_0)) w_{\mathrm{high}}) \\
&= \nabla \phi_j * \Bigg(\Big((\nabla^\perp \Lambda^{-1}(e^{-t\Lambda}\theta_0)) w_{\mathrm{high}} \Big)_{\mathrm{low}} - \Big((\nabla^\perp \Lambda^{-1}(e^{-t\Lambda}\theta_0)) w_{\mathrm{low}} \Big)_{\mathrm{high}} \\
&\quad - \Big((\nabla^\perp \Lambda^{-1} w) e^{-t\Lambda}\theta_0 - \frac{1}{2}\sum_{i=1}^{2} \big((\nabla^\perp \Lambda^{-1} w) B(u^{(i)}\theta^{(i)}) + (\nabla^\perp \Lambda^{-1} B(u^{(i)}\theta^{(i)})) w \big) \Big)_{\mathrm{high}} \Bigg).
\end{aligned} \tag{3.12}$$

Let $0 < \eta < 1$. We divide w into w_{high} and w_{low} and consider $\|w_{\mathrm{high}}\|_{\dot{B}^{-\eta}_{\infty,\infty}}$ instead of the $\dot{B}^0_{\infty,1}$ norm. Using maximum principle (Proposition 2.7) to (3.12), w_{high} satisfies

$$
\begin{aligned}
&\|w_{\mathrm{high}}\|_{L^\infty(0,T;\dot{B}^{-\eta}_{\infty,\infty})} \\
&\le C\exp\Big(C\|\nabla e^{-t\Lambda}\theta_0\|_{L^1(0,T;\dot{B}^0_{\infty,1})}\Big)\bigg(\Big\|\nabla\cdot\Big((\nabla^\perp\Lambda^{-1}(e^{-t\Lambda}\theta_0))w_{\mathrm{high}}\Big)_{\mathrm{low}}\Big\|_{L^1(0,T;\dot{B}^{-\eta}_{\infty,\infty})} \\
&\quad+\Big\|\nabla\cdot\Big((\nabla^\perp\Lambda^{-1}(e^{-t\Lambda}\theta_0))w_{\mathrm{low}}\Big)_{\mathrm{high}}\Big\|_{L^1(0,T;\dot{B}^{-\eta}_{\infty,\infty})} \\
&\quad+\Big\|\nabla\cdot\Big((\nabla^\perp\Lambda^{-1}w_{\mathrm{high}})e^{-t\Lambda}\theta_0\Big)_{\mathrm{high}}\Big\|_{L^1(0,T;\dot{B}^{-\eta}_{\infty,\infty})} \\
&\quad+\Big\|\nabla\cdot\Big((\nabla^\perp\Lambda^{-1}w_{\mathrm{low}})e^{-t\Lambda}\theta_0\Big)_{\mathrm{high}}\Big\|_{L^1(0,T;\dot{B}^{-\eta}_{\infty,\infty})} \\
&\quad+\sum_{i=1}^{2}\Big\|\nabla\cdot\Big((\nabla^\perp\Lambda^{-1}w_{\mathrm{high}})B(u^{(i)}\theta^{(i)})+(\nabla^\perp\Lambda^{-1}B(u^{(i)}\theta^{(i)}))w_{\mathrm{high}}\Big)_{\mathrm{high}}\Big\|_{L^\infty(0,T;\dot{B}^{-\eta-1}_{\infty,\infty})} \\
&\quad+\sum_{i=1}^{2}\Big\|\nabla\cdot\Big((\nabla^\perp\Lambda^{-1}w_{\mathrm{low}})B(u^{(i)}\theta^{(i)})+(\nabla^\perp\Lambda^{-1}B(u^{(i)}\theta^{(i)}))w_{\mathrm{low}}\Big)_{\mathrm{high}}\Big\|_{L^\infty(0,T;\dot{B}^{-\eta-1}_{\infty,\infty})}\bigg) \\
&=: C\exp\Big(C\|\nabla e^{-t\Lambda}\theta_0\|_{L^1(0,T;\dot{B}^0_{\infty,1})}\Big)((I\text{-}1)+(I\text{-}2)+(II\text{-}1)+(II\text{-}2)+(III\text{-}1)+(III\text{-}2)).
\end{aligned}
\tag{3.13}
$$

By Bony's decomposition ([2]) , $\nabla\cdot\nabla^\perp\Lambda^{-1}w_{\mathrm{high},k}=0$, Lemma 2.1 (2.4) and Lemma 2.10, we write

$$
\begin{aligned}
(I\text{-}1) \le C\Bigg(&\Big\|\Big(\sum_{k\le l-2}(\nabla^\perp\Lambda^{-1}(e^{-t\Lambda}\theta_0)_k)w_{\mathrm{high},l}\Big)_{\mathrm{low}}\Big\|_{L^1(0,T;\dot{B}^{-\eta}_{\infty,\infty})} \\
&+\Big\|\Big(\sum_{|k-l|\le 1}(\nabla^\perp\Lambda^{-1}(e^{-t\Lambda}\theta_0)_k)w_{\mathrm{high},l}\Big)_{\mathrm{low}}\Big\|_{L^1(0,T;\dot{B}^{-\eta+1}_{\infty,\infty})} \\
&+\Big\|\Big(\sum_{l\le k-2}(\nabla^\perp\Lambda^{-1}(e^{-t\Lambda}\theta_0)_k)w_{\mathrm{high},l}\Big)_{\mathrm{low}}\Big\|_{L^1(0,T;\dot{B}^{-\eta+1}_{\infty,\infty})}\Bigg),
\end{aligned}
$$

$$(II)\text{-}1 \le C\left(\left\|\left(\sum_{k\le l-2}(\nabla^\perp\Lambda^{-1}w_{\mathrm{high},k})(e^{-t\Lambda}\theta_0)_l\right)_{\mathrm{high}}\right\|_{L^1(0,T;\dot B^{-\eta+1}_{\infty,\infty})}\right.$$
$$+\left\|\left(\sum_{|k-l|\le 1}(\nabla^\perp\Lambda^{-1}w_{\mathrm{high},k})(e^{-t\Lambda}\theta_0)_l\right)_{\mathrm{high}}\right\|_{L^1(0,T;\dot B^{-\eta+1}_{\infty,\infty})}$$
$$\left.+\left\|\left(\sum_{l\le k-2}(\nabla^\perp\Lambda^{-1}w_{\mathrm{high},k})\nabla(e^{-t\Lambda}\theta_0)_l\right)_{\mathrm{high}}\right\|_{L^1(0,T;\dot B^{-\eta}_{\infty,\infty})}\right)$$

and

$$\begin{aligned}
&(III\text{-}1)\\
&\le C\sum_{i=1}^{2}\left(\left\|\left(\sum_{k\le l-2}\big((\nabla^\perp\Lambda^{-1}w_{\mathrm{high},k})(B(u^{(i)}\theta^{(i)}))_l+(\nabla^\perp\Lambda^{-1}(B(u^{(i)}\theta^{(i)}))_l)w_{\mathrm{high},k}\big)\right)_{\mathrm{high}}\right\|_{L^\infty(0,T;\dot B^{-\eta}_{\infty,\infty})}\right.\\
&\quad+\left\|\nabla\cdot\left(\sum_{|k-l|\le 1}\big((\nabla^\perp\Lambda^{-1}w_{\mathrm{high},k})(B(u^{(i)}\theta^{(i)}))_l+(\nabla^\perp\Lambda^{-1}(B(u^{(i)}\theta^{(i)}))_l)w_{\mathrm{high},k}\big)\right)_{\mathrm{high}}\right\|_{L^\infty(0,T;\dot B^{-\eta-1}_{\infty,\infty})}\\
&\quad\left.+\left\|\left(\sum_{l\le k-2}\big((\nabla^\perp\Lambda^{-1}w_{\mathrm{high},k})(B(u^{(i)}\theta^{(i)}))_l+(\nabla^\perp\Lambda^{-1}(B(u^{(i)}\theta^{(i)}))_l)w_{\mathrm{high},k}\big)\right)_{\mathrm{high}}\right\|_{L^\infty(0,T;\dot B^{-\eta}_{\infty,\infty})}\right)\\
&=: (III\text{-}1a)+(III\text{-}1b)+(III\text{-}1c).
\end{aligned}$$

Using the product estimate of Besov norm (Lemma 2.2 and Lemma 2.3), we have

$$(I\text{-}1)\le C\|e^{-t\Lambda}\theta_0\|_{L^1(0,T;\dot B^0_{\infty,1}\cap\dot B^1_{\infty,1})}\|w_{\mathrm{high}}\|_{L^\infty(0,T;\dot B^{-\eta}_{\infty,\infty})}, \tag{3.14}$$

$$(II\text{-}1)\le C\|e^{-t\Lambda}\theta_0\|_{L^1(0,T;\dot B^1_{\infty,1})}\|w_{\mathrm{high}}\|_{L^\infty(0,T;\dot B^{-\eta}_{\infty,\infty})} \tag{3.15}$$

and

$$(III\text{-}1a),(III\text{-}1c)\le C\sum_{i=1}^{2}\|B(u^{(i)}\theta^{(i)})\|_{L^\infty(0,T;\dot B^0_{\infty,1})}\|w_{\mathrm{high}}\|_{L^\infty(0,T;\dot B^{-\eta}_{\infty,\infty})}.$$

In order to estimate $(III\text{-}1b)$, we write

$$(\nabla^\perp\Lambda^{-1}f)g+(\nabla^\perp\Lambda^{-1}g)f=\nabla^\perp((\Lambda^{-1}f)g)-(\Lambda^{-1}f)(\nabla^\perp g)+(\nabla^\perp\Lambda^{-1}g)f.$$

The Fourier transform in $\mathbb{R}^4$ of the above is written with the separate variables, andwe have

$$\begin{aligned}(\nabla^\perp\Lambda^{-1}g)f-(\Lambda^{-1}f)(\nabla^\perp g)&=\mathcal{F}^{-1}_{\mathbb{R}^4}\left[\eta^\perp\left(\frac{1}{|\eta|}-\frac{1}{|\xi|}\right)\widehat{f}(\xi)\widehat{g}(\eta)\right]\\&=\mathcal{F}^{-1}_{\mathbb{R}^4}\left[\frac{(\xi,\eta)\cdot(\xi,-\eta)}{|\xi|+|\eta|}\cdot\frac{1}{|\xi|}\widehat{f}(\xi)\frac{\eta^\perp}{|\eta|}\widehat{g}(\eta)\right]\\&=\nabla_{\mathbb{R}^4}\cdot m(D_1,D_2)((\Lambda^{-1}f)(\nabla^\perp\Lambda^{-1}g)).\end{aligned}\tag{3.16}$$

where $m(D_1,D_2)(fg):=\mathcal{F}^{-1}_{\mathbb{R}^4}\left[\frac{(\xi,-\eta)}{|\xi|+|\eta|}\widehat{f}(\xi)\widehat{g}(\eta)\right]$ and $\nabla_{\mathbb{R}^4}$ is the gradient in $\mathbb{R}^4$. We replace f and g in (3.16) with $w_{\mathrm{high},k}$ and $(B(u^{(i)}\theta^{(i)}))_l$ respectively. By $\nabla\cdot\nabla^\perp((\Lambda^{-1}f)g)=0$ and Lemma 2.5, we obtain

$$\begin{aligned}&(III\text{-}1b)\\&\le C\sum_{i=1}^2\left\|\left(\sum_{|k-l|\le1}\nabla_{\mathbb{R}^4}\cdot m(D_1,D_2)((\Lambda^{-1}w_{\mathrm{high},k})(\nabla^\perp\Lambda^{-1}(B(u^{(i)}\theta^{(i)}))_l))\right)_{\mathrm{high}}\right\|_{L^\infty(0,T;\dot{B}^{-\eta}_{\infty,\infty})}\\&\le C\sum_{i=1}^2\sup_{j\le0}2^{(-\eta+1)j}\sum_{k\ge j-4}\sum_{|k-l|\le1}\|\Lambda^{-1}w_{\mathrm{high},k}\|_{L^\infty(0,T;L^\infty)}\|(B(u^{(i)}\theta^{(i)}))_l\|_{L^\infty(0,T;L^\infty)}\\&\le C\sum_{i=1}^2\|B(u^{(i)}\theta^{(i)})\|_{L^\infty(0,T;\dot{B}^0_{\infty,1})}\|w_{\mathrm{high}}\|_{L^\infty(0,T;\dot{B}^{-\eta}_{\infty,\infty})}\end{aligned}\tag{3.17}$$

and

$$(III\text{-}1)\le C\sum_{i=1}^2\|B(u^{(i)}\theta^{(i)})\|_{L^\infty(0,T;\dot{B}^0_{\infty,1})}\|w_{\mathrm{high}}\|_{L^\infty(0,T;\dot{B}^{-\eta}_{\infty,\infty})}.\tag{3.18}$$

By Lemma 2.1 (2.4), Lemma 2.4 and Lemma 2.10, we estimate $(I\text{-}2)$ as follows:

$$\begin{aligned}(I\text{-}2)&\le C\left\|\left((\nabla^\perp\Lambda^{-1}(e^{-t\Lambda}\theta_0))w_{\mathrm{low}}\right)_{\mathrm{high}}\right\|_{L^1(0,T;\dot{B}^1_{\infty,\infty})}\\&\le C\|e^{-t\Lambda}\theta_0\|_{L^1(0,T;\dot{B}^0_{\infty,1}\cap\dot{B}^1_{\infty,1})}\|w_{\mathrm{low}}\|_{L^\infty(0,T;\dot{B}^0_{\infty,1}\cap\dot{B}^1_{\infty,1})}\\&\le C\|e^{-t\Lambda}\theta_0\|_{L^1(0,T;\dot{B}^0_{\infty,1}\cap\dot{B}^1_{\infty,1})}\|w_{\mathrm{low}}\|_{L^\infty(0,T;\dot{B}^0_{\infty,1})}.\end{aligned}\tag{3.19}$$

Similarly, we obtain

$$(II\text{-}2)\le C\|e^{-t\Lambda}\theta_0\|_{L^1(0,T;\dot{B}^0_{\infty,1}\cap\dot{B}^1_{\infty,1})}\|w_{\mathrm{low}}\|_{L^\infty(0,T;\dot{B}^0_{\infty,1})}.\tag{3.20}$$

Using Lemma 2.10 and a similar argument as the estimate $(III\text{-}1b)$ (3.17) , we obtain

$$\begin{aligned}(III\text{-}2)&\le C\sum_{i=1}^2\left\|\nabla\cdot\left((\nabla^\perp\Lambda^{-1}w_{\mathrm{low}})B(u^{(i)}\theta^{(i)})+(\nabla^\perp\Lambda^{-1}B(u^{(i)}\theta^{(i)}))w_{\mathrm{low}}\right)_{\mathrm{high}}\right\|_{L^\infty(0,T;\dot{B}^{-1}_{\infty,\infty})}\\&\le C\sum_{i=1}^2\|B(u^{(i)}\theta^{(i)})\|_{L^\infty(0,T;\dot{B}^0_{\infty,1})}\|w_{\mathrm{low}}\|_{L^\infty(0,T;\dot{B}^0_{\infty,1})}.\end{aligned}\tag{3.21}$$

From (3.14), (3.15) and (3.18)-(3.21), we obtain

$$\begin{aligned}
&\|w_{\mathrm{high}}\|_{L^\infty(0,T;\dot{B}^{-\eta}_{\infty,\infty})} \\
&\le C\exp(C\|\nabla e^{-t\Lambda}\theta_0\|_{L^1(0,T;\dot{B}^0_{\infty,1})})\Bigg(\|e^{-t\Lambda}\theta_0\|_{L^1(0,T;\dot{B}^0_{\infty,1}\cap\dot{B}^1_{\infty,1})}+\sum_{i=1}^2\|B(u^{(i)}\theta^{(i)})\|_{L^\infty(0,T;\dot{B}^0_{\infty,1})}\Bigg) \\
&\quad\times(\|w_{\mathrm{high}}\|_{L^\infty(0,T;\dot{B}^{-\eta}_{\infty,\infty})}+\|w_{\mathrm{low}}\|_{L^\infty(0,T;\dot{B}^0_{\infty,1})}).
\end{aligned} \tag{3.22}$$

Next, we consider w_{low}. A similar argument as high frequency estimate (3.11) and using the Duhamel's principle, we have

$$\begin{aligned}
&\partial_t w_{\mathrm{low},j}+\Lambda w_{\mathrm{low},j}+\nabla\phi_j*\Big((\nabla^\perp\Lambda^{-1}(e^{-t\Lambda}\theta_0))w\Big)_{\mathrm{low}} \\
&=-\nabla\phi_j*\Big((\nabla^\perp\Lambda^{-1}w)e^{-t\Lambda}\theta_0-\frac{1}{2}\sum_{i=1}^2\big((\nabla^\perp\Lambda^{-1}w)B(u^{(i)}\theta^{(i)})+(\nabla^\perp\Lambda^{-1}B(u^{(i)}\theta^{(i)}))w\big)\Big)_{\mathrm{low}}.
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
w_{\mathrm{low},j}(t)=-\int_0^t e^{-(t-s)\Lambda}\Bigg(&\nabla\phi_j*\Big((\nabla^\perp\Lambda^{-1}(e^{-t\Lambda}\theta_0))w+(\nabla^\perp\Lambda^{-1}w)e^{-t\Lambda}\theta_0 \\
&+\frac{1}{2}\sum_{i=1}^2\big((\nabla^\perp\Lambda^{-1}w)B(u^{(i)}\theta^{(i)})+(\nabla^\perp\Lambda^{-1}B(u^{(i)}\theta^{(i)}))w\big)\Big)_{\mathrm{low}}\Bigg)\,\mathrm{d}s.
\end{aligned} \tag{3.24}$$

By Lemma 2.1 (2.5), $e^{-(t-s)\Lambda}$ is a bounded operator in $\dot{B}^0_{\infty,1}$ and we obtain

$$\begin{aligned}
&\|w_{\mathrm{low}}\|_{L^\infty(0,T;\dot{B}^0_{\infty,1})} \\
&\le C\Bigg(\Big\|\nabla\cdot\Big((\nabla^\perp\Lambda^{-1}(e^{-t\Lambda}\theta_0))w_{\mathrm{high}}+(\nabla^\perp\Lambda^{-1}w_{\mathrm{high}})e^{-t\Lambda}\theta_0\Big)_{\mathrm{low}}\Big\|_{L^1(0,T;\dot{B}^0_{\infty,1})} \\
&\quad+\Big\|\nabla\cdot\Big((\nabla^\perp\Lambda^{-1}(e^{-t\Lambda}\theta_0))w_{\mathrm{low}}+(\nabla^\perp\Lambda^{-1}w_{\mathrm{low}})e^{-t\Lambda}\theta_0\Big)_{\mathrm{low}}\Big\|_{L^1(0,T;\dot{B}^0_{\infty,1})} \\
&\quad+\sum_{i=1}^2\Big\|\nabla\cdot\Big((\nabla^\perp\Lambda^{-1}w_{\mathrm{high}})B(u^{(i)}\theta^{(i)})+(\nabla^\perp\Lambda^{-1}B(u^{(i)}\theta^{(i)}))w_{\mathrm{high}}\Big)_{\mathrm{low}}\Big\|_{L^1(0,T;\dot{B}^0_{\infty,1})} \\
&\quad+\sum_{i=1}^2\Big\|\nabla\cdot\Big((\nabla^\perp\Lambda^{-1}w_{\mathrm{low}})B(u^{(i)}\theta^{(i)})+(\nabla^\perp\Lambda^{-1}B(u^{(i)}\theta^{(i)}))w_{\mathrm{low}}\Big)_{\mathrm{low}}\Big\|_{L^1(0,T;\dot{B}^0_{\infty,1})}\Bigg).
\end{aligned} \tag{3.25}$$

Since derivatives for low frequency component can be estimated by a constant, we also have

$$
\begin{aligned}
&\|w_{\mathrm{low}}\|_{L^\infty(0,T;\dot{B}^0_{\infty,1})}\\
&\le C\left(\|e^{-t\Lambda}\theta_0\|_{L^1(0,T;\dot{B}^0_{\infty,1}\cap\dot{B}^1_{\infty,1})}+\sum_{i=1}^2\|B(u^{(i)}\theta^{(i)})\|_{L^1(0,T;\dot{B}^0_{\infty,1})}\right)\\
&\quad\times(\|w_{\mathrm{high}}\|_{L^\infty(0,T;\dot{B}^{-\eta}_{\infty,\infty})}+\|w_{\mathrm{low}}\|_{L^\infty(0,T;\dot{B}^0_{\infty,1})}).
\end{aligned}
\tag{3.26}
$$

Summing up (3.22) and (3.26), since Lemma 2.8 and Lemma 2.9, for any $\epsilon>0$, there exist $0<T_0\ll 1$ such that

$$
\|w_{\mathrm{high}}\|_{L^\infty(0,T_0;\dot{B}^{-\eta}_{\infty,\infty})}+\|w_{\mathrm{low}}\|_{L^\infty(0,T_0;\dot{B}^0_{\infty,1})}\le C\epsilon(\|w_{\mathrm{high}}\|_{L^\infty(0,T_0;\dot{B}^{-\eta}_{\infty,\infty})}+\|w_{\mathrm{low}}\|_{L^\infty(0,T_0;\dot{B}^0_{\infty,1})}).
$$

By taking ϵ sufficiently small, we obtain

$$
\|w_{\mathrm{high}}\|_{L^\infty(0,T_0;\dot{B}^{-\eta}_{\infty,\infty})}+\|w_{\mathrm{low}}\|_{L^\infty(0,T_0;\dot{B}^0_{\infty,1})}\le 0.
$$

We conclude that

$$
\theta^{(1)}=\theta^{(2)}\text{ in }\dot{B}^0_{\infty,1}\text{ for }t\in[0,T_0].
$$

We prove that $\|w\|_{\dot{B}^0_{\infty,1}}=0$ in the entire interval $[0,T]$, by contradiction argument. Set

$$
\tau^*:=\sup\{\tau\in[0,T)\mid \|\theta^{(1)}(t,\cdot)-\theta^{(2)}(t,\cdot)\|_{\dot{B}^0_{\infty,1}}=0\text{ for }t\in[0,\tau]\}.
$$

Assume that $\tau^*<T$. From the continuity in time of $\theta^{(1)}$ and $\theta^{(2)}$, we have $\theta^{(1)}(\tau^*)=\theta^{(2)}(\tau^*)$. By the same argument as local uniqueness , there exists $\delta'>0$ such that $\theta^{(1)}(s+\tau^*)=\theta^{(2)}(s+\tau^*)$ for $0\le s\le\delta'$. This is a contradiction with the definition of τ^*. Therefore $\tau^*=T$.

$\square$

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