# NEF AND EFFECTIVE CONES OF SOME QUOT SCHEMES 

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#### Abstract

Let $C$ be a smooth projective curve over $\mathbb{C}$ of genus $g(C) \geqslant 3$ (respectively, $g(C)=2$ ). Fix integers $r, k$ such that $2 \leqslant k \leqslant r-2$, (respectively, $3 \leqslant k \leqslant r-2$ ). Let $\mathcal{Q}:=\mathcal{Q}_{C / \mathbb{C}}\left(\mathcal{O}_{C}^{\oplus r}, k, d\right)$ be the Quot scheme parametrizing rank $k$ and degree $d$ quotients of the trivial bundle of rank $r$. Let $\mathcal{Q}_{L}$ denote the closed subscheme of the Quot scheme parametrizing quotients such that the quotient sheaf has determinant $L$. It is known that $\mathcal{Q}_{L}$ is an integral, normal, local complete intersection, locally factorial scheme of Picard rank 2 , when $d \gg 0$. In this article we compute the nef cone, effective cone and canonical divisor of this variety when $d \gg 0$. We show this variety is Fano iff $r=2 k+1$.


## 1. Introduction

Let $C$ be a smooth projective curve over the field of complex numbers $\mathbb{C}$. Fix integers $0<k<r$. Let $\operatorname{Quot}_{C / \mathbb{C}}\left(\mathcal{O}_{C}^{\oplus r}, k, d\right)$ denote the Quot scheme parametrizing rank $k$ and degree $d$ quotients of the trivial bundle of rank $r$. When $C \cong \mathbb{P}^{1}$, Stromme, in [Str87], proved that Quot $_{C / \mathbb{C}}\left(\mathcal{O}_{C}^{\oplus r}, k, d\right)$ is a smooth projective variety of Picard rank 2 and computed its nef cone. In [Jow12], the author computed the effective cone of Quot $_{\mathbb{P}^{1} / \mathbb{C}}\left(\mathcal{O}_{\mathbb{P}^{1}}^{\oplus n}, k, d\right)$. In [Ven11], the author determined the movable cone of $\operatorname{Quot}_{\mathbb{P}^{1} / \mathbb{C}}\left(\mathcal{O}_{\mathbb{P}^{1}}^{\oplus n}, k, d\right)$ and the stable base locus decomposition of the effective cone. In [Ito17], the author studied the birational geometry of $\operatorname{Quot}_{\mathbb{P}^{1} / \mathbb{C}}\left(\mathcal{O}_{\mathbb{P}^{1}}^{\oplus n}, k, d\right)$.

For curves of higher genus, the space Quot $_{C / \mathbb{C}}\left(\mathcal{O}_{C}^{\oplus r}, k, d\right)$ was studied in [BDW96] and it was proved that when $d \gg 0$, this Quot scheme is irreducible and generically smooth. This was generalized by Popa and Roth [PR03], who proved the same result for Quot $_{C / \mathbb{C}}(E, k, d)$, the Quot scheme parametrizing quotients of a vector bundle E. See also [Gol19], [CCH21], [CCH22], [RS24] for similar results on other variations of this Quot scheme. In [Ras24], the author generalizes the above mentioned result of Popa and Roth to the case of nodal curves.

In [GS24] it was proved that for $d \gg 0$, Quot $_{C / \mathbb{C}}(E, k, d)$ is in fact locally factorial and its Picard group was computed. Consider the determinant map

$$
\operatorname{det}: \operatorname{Quot}_{C / \mathbb{C}}(E, k, d) \rightarrow \operatorname{Pic}^{d}(C),
$$

which sends a closed point $[E \rightarrow F]$ to the determinant line bundle $[\operatorname{det}(F)]$. For $[L] \in \operatorname{Pic}^{d}(C)$ let us denote the fiber over $[L]$ by Quot $_{C / \mathbb{C}}(E, k, d)_{L}$. By definition, Quot $_{C / \mathbb{C}}(E, k, d)_{L}$ parametrizes quotients $[E \rightarrow F]$ such that rank of $F$ is $k$ and determinant of $F$ is $L$. With some mild assumptions on the genus of $C$ and $k$, it was proved in [GS24] that when $d \gg 0$ the scheme Quot ${ }_{C / \mathbb{C}}(E, k, d)_{L}$ is irreducible, locally factorial and moreover, its Picard rank is 2. See Theorem 2.3 for precise statements of the main results in [GS24].

[^0]For ease of notation, let $\mathcal{Q}_{L}$ denote $\operatorname{Quot}_{C / \mathbb{C}}\left(\mathcal{O}_{C}^{\oplus r}, k, d\right)_{L}$. Let $\mathrm{N}^{1}\left(\mathcal{Q}_{L}\right)$ denote the NeronSeveri group of $\mathcal{Q}_{L}$. Let $\operatorname{Nef}\left(\mathcal{Q}_{L}\right) \subset \mathrm{N}^{1}\left(\mathcal{Q}_{L}\right) \otimes_{\mathbb{Z}} \mathbb{R}$ denote the cone of nef divisors. Similarly, we have the cone of effective divisors, which we denote by $\operatorname{Eff}\left(\mathcal{Q}_{L}\right)$ and the movable cone of divisors which we denote $\operatorname{Mov}\left(\mathcal{Q}_{L}\right)$. In this article, we compute these cones, thereby, generalizing the results in [Str87], [Jow12], [Ven11]. Note that in the case of $C \cong P^{1}$ this scheme is just $\mathrm{Quot}_{C / \mathbb{C}}\left(\mathcal{O}_{C}^{\oplus r}, k, d\right)$. We also compute the class of the canonical divisor of $\mathcal{Q}_{L}$ in terms of $\alpha$ and $\beta_{d+g-1}$. Combining this with Theorem 4.14 we give a necessary and sufficient condition for $\mathcal{Q}_{L}$ to be Fano. This question, regarding when $\mathcal{Q}_{L}$ is Fano, was raised by Pieter Belmans in his Blog (see Fortnightly links (160)).

In the following Theorem, for the definitions of the line bundles $\alpha$ and $\beta_{d+g-1}$, see the discussion before (4.3), the discussion before Lemma 4.7 and Remark 4.8. For the definitions of the curves $D_{1}$ and $D_{2}$, see the proofs of Proposition 4.4 and Proposition 4.9.

Theorem. Assume one of the following two holds:

- $g(C) \geqslant 3$ and $2 \leqslant k \leqslant r-2$, or
- $g(C)=2$ and $3 \leqslant k \leqslant r-2$.

Let $d \gg 0$. Then we have the following results:
(A) (Theorem 4.14) $\operatorname{Pic}\left(\mathcal{Q}_{L}\right)$ is generated by the line bundles $\alpha, \beta_{d+g-1}$. Both these are globally generated, nef but not ample. In particular,

$$
\operatorname{Nef}\left(\mathcal{Q}_{L}\right)=\mathbb{R}_{\geqslant 0} \alpha+\mathbb{R}_{\geqslant 0} \beta_{d+g-1}
$$

The boundaries of the cone of effective curves are given by the classes of the curves $D_{1}$ and $D_{2}$.
(B) (Theorem 5.16) The effective cone of $\mathcal{Q}_{L}$ is given by

$$
\operatorname{Eff}\left(\mathcal{Q}_{L}\right)=\mathbb{R}_{\geqslant 0}\left(d(k+1) \alpha-k \beta_{d+g-1}\right)+\mathbb{R}_{\geqslant 0}\left(-d(r-k-1) \alpha+(r-k) \beta_{d+g-1}\right)
$$

The cone $\operatorname{Mov}\left(\mathcal{Q}_{L}\right)=\operatorname{Eff}\left(\mathcal{Q}_{L}\right)$.
(C) (Theorem 6.10) $\mathcal{Q}_{L}$ is Fano iff $r=2 k+1$.

## 2. Recollection of some results from [GS24]

Let $C$ be a smooth projective curve over the field of complex numbers $\mathbb{C}$. We shall denote the genus of $C$ by $g(C)$. Throughout this article we shall assume that $g(C) \geqslant 2$. Let $E$ be a locally free sheaf on $C$ of rank $r$ and degree $e$. In the latter sections we will be considering only the case $E=\mathcal{O}_{C}^{\oplus r}$, whence, $e=0$. Let $k$ be an integer such that $0<k<r$. Throughout this article

$$
\begin{equation*}
\operatorname{Quot}_{C / \mathbb{C}}(E, k, d) \tag{2.1}
\end{equation*}
$$

will denote the Quot scheme of quotients of $E$ of rank $k$ and degree $d$. There is a map

$$
\begin{equation*}
\operatorname{det}: \operatorname{Quot}_{C / \mathbb{C}}(E, k, d) \rightarrow \operatorname{Pic}^{d}(C) \tag{2.2}
\end{equation*}
$$

see $\left[G S 24\right.$, equation (2.5)]. This map sends a closed point $[E \rightarrow F] \in \operatorname{Quot}_{C / \mathbb{C}}(E, k, d)$ to $[\operatorname{det}(F)] \in \operatorname{Pic}^{d}(C)$. Let $L$ be a line bundle on $C$ of degree $d$ and let

$$
\operatorname{Quot}_{C / \mathbb{C}}(E, k, d)_{L}:=\operatorname{det}^{-1}([L])
$$

be the scheme theoretic fiber over the point $[L] \in \operatorname{Pic}^{d}(C)$. We recall the main results in [GS24].

Theorem 2.3. (A) [GS24, Theorem 3.3, Corollary 3.5]. First consider the case $k=r-1$. Assume $d>2 g-2+e-\mu_{\min }(E)$. There is a locally free sheaf $\mathcal{E}$ on $\operatorname{Pic}^{e-d}(C)$ such that the following holds. We have an isomorphism of schemes over $\mathrm{Pic}^{e-d}(C)$, $\mathbb{P}\left(\mathcal{E}^{\vee}\right) \xrightarrow{\sim} \operatorname{Quot}_{C / \mathbb{C}}(E, r-1, d)$. In particular, under the above assumption on $d$, the space $\operatorname{Quot}_{C / \mathbb{C}}(E, r-1, d)$ is smooth and $\operatorname{Pic}\left(\operatorname{Quot}_{C / \mathbb{C}}(E, r-1, d)\right) \cong \operatorname{Pic}\left(\operatorname{Pic}^{0}(C)\right) \times \mathbb{Z}$. Next we consider the case when $k \leqslant r-2$. There is a number $d_{0}(E, k)$, which depends only on $E$ and $k$, such that the following statements hold. Let $d \geqslant d_{0}(E, k)$.
(1) [GS24, Theorem 6.3] Then det: $\operatorname{Quot}_{C / \mathbb{C}}(E, k, d) \longrightarrow \operatorname{Pic}^{d}(C)$ is a flat map. Further, Quot $_{C / \mathbb{C}}(E, k, d)$ is local complete intersection scheme which is an integral and normal variety and is locally factorial.
(2) [GS24, Theorem 9.1] Let $k=1$ and $r \geqslant 3$ (the case $k=1$ and $r=2$ is dealt with in the case $k=r-1$ above). Then $\operatorname{Pic}\left(\operatorname{Quot}_{C / \mathbb{C}}(E, k, d)\right) \cong \operatorname{Pic}\left(\operatorname{Pic}^{d}(C)\right) \times \mathbb{Z} \times \mathbb{Z}$.
(3) [GS24, Theorem 8.7] Let $k \geqslant 2, g(C) \geqslant 2$. Then $\operatorname{Quot}_{C / \mathbb{C}}(E, k, d)_{L}$ is a local complete intersection scheme, which is also integral, normal and locally factorial.
(4) [GS24, Theorem 8.9] Assume one of the following two holds

- $k \geqslant 2$ and $g(C) \geqslant 3$, or
- $k \geqslant 3$ and $g(C)=2$.

We have isomorphisms

$$
\operatorname{Pic}\left(\operatorname{Quot}_{C / \mathbb{C}}(E, k, d)_{L}\right) \cong \operatorname{Pic}\left(M_{k, L}^{s}\right) \times \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}
$$

(5) [GS24, Theorem 9.1] Let $k=1$ and $r \geqslant 3$. Then $\operatorname{Pic}\left(\operatorname{Quot}_{C / \mathbb{C}}(E, k, d)_{L}\right) \cong \mathbb{Z} \times \mathbb{Z}$.

In view of point (4), it becomes a particularly interesting question to investigate the nef cone of the scheme Quot $_{C / \mathbb{C}}(E, k, d)_{L}$. The purpose of this article is to investigate this question when $E$ is the trivial bundle of rank $r$. Before we proceed we mention a few points from [GS24] which we shall use. The discussion in this paragraph assumes that $d \gg 0$. The "good locus" of the Quot scheme is defined to be the set of points (see [GS24, Definition 4.4])

$$
\operatorname{Quot}_{C / \mathbb{C}}(E, k, d)_{g}:=\left\{[E \rightarrow F] \mid H^{1}\left(E^{\vee} \otimes F\right)=0\right\}
$$

Let $A$ be a locally closed subset of the Quot scheme. Then the good locus of $A$, denoted $A_{g}$ is defined to be the subset $A \cap \mathrm{Quot}_{C / \mathbb{C}}(E, k, d)_{g}$. An important property of the good locus is that the morphism det in equation (2.2) restricted to the good locus is a smooth morphism. In particular, taking $A=\operatorname{Quot}_{C / \mathbb{C}}(E, k, d)_{L}$, we get the $\operatorname{locus~}^{Q u o t_{C / \mathbb{C}}}(E, k, d)_{g, L}$. Another subset of the Quot scheme which will be used is the locus of stable quotients, that is,

$$
\operatorname{Quot}_{C / \mathbb{C}}(E, k, d)^{s}:=\{[E \rightarrow F] \mid F \text { is stable }\} .
$$

Similarly, we define $\operatorname{Quot}_{C / \mathbb{C}}(E, k, d)_{L}^{s}$. We have inclusions

$$
\operatorname{Quot}_{C / \mathbb{C}}(E, k, d)_{L}^{s} \subset \operatorname{Quot}_{C / \mathbb{C}}(E, k, d)_{g, L} \subset \operatorname{Quot}_{C / \mathbb{C}}(E, k, d)_{L} .
$$

If $Y \subset X$ is locally closed, then we denote $\operatorname{codim}(Y, X)=\operatorname{dim}(X)-\operatorname{dim}(Y)$. In the proof of [GS24, Theorem 8.9], it is proved that

$$
\operatorname{codim}\left(\operatorname{Quot}_{C / \mathbb{C}}(E, k, d)_{L} \backslash \operatorname{Quot}_{C / \mathbb{C}}(E, k, d)_{L}^{s}, \operatorname{Quot}_{C / \mathbb{C}}(E, k, d)_{L}\right) \geqslant 2
$$

## 3. Notation

Fix a point $[L] \in \operatorname{Pic}^{d}(C)$. For the remainder of this article, we shall

- Denote by $\mathcal{Q}:=\operatorname{Quot}_{C / \mathbb{C}}\left(\mathcal{O}_{C}^{\oplus r}, k, d\right)$ and denote by $\mathcal{Q}_{L}:=\operatorname{Quot}_{C / \mathbb{C}}\left(\mathcal{O}_{C}^{\oplus r}, k, d\right)_{L}$.
- Assume one of the following two holds:
$\dagger g(C) \geqslant 3$ and $2 \leqslant k \leqslant r-2$, or
$\dagger g(C)=2$ and $3 \leqslant k \leqslant r-2$.
- If $Y \subset X$ is locally closed, then we denote $\operatorname{codim}(Y, X)=\operatorname{dim}(X)-\operatorname{dim}(Y)$.
- For a scheme $T$, we shall denote by $p_{C}: C \times T \rightarrow C$ denote the projection. The projection onto the second factor will be denoted by $p_{2}$.
- Let

$$
\begin{equation*}
0 \rightarrow \mathcal{K} \rightarrow p_{C}^{*} \mathcal{O}_{C}^{\oplus r} \rightarrow \mathcal{F} \rightarrow 0 \tag{3.1}
\end{equation*}
$$

denote the universal quotient on $C \times \mathcal{Q}$. We will abuse notation and use the same notation to denote its restriction to $C \times \mathcal{Q}_{L}$.

- We shall assume that $d \gg 0$. In particular, $d \geqslant d_{0}(E, k)$.

Remark 3.2. At several places we will use the following easy consequence of cohomology and base change. Let $f: X \rightarrow Y$ be a projective morphism and let $\mathcal{F}$ be a coherent sheaf on $X$ which is flat over $Y$. Suppose $h^{1}\left(X_{y}, \mathcal{F}_{y}\right)=0$ for all closed points $y \in Y$. Then $f_{*}(\mathcal{F})$ is locally free. Let $g: Y^{\prime} \rightarrow Y$ be a morphism and consider the Cartesian square


Then the natural map $g^{*} f_{*}(\mathcal{F}) \rightarrow f_{*}^{\prime} g^{\prime *}(\mathcal{F})$ is an isomorphism.

## 4. Nef cone of $\mathcal{Q}_{L}$

Recall the universal sequence (3.1) on $C \times \mathcal{Q}_{L}$. Applying $\wedge^{r-k}$ to the inclusion $\mathcal{K} \subset p_{C}^{*} \mathcal{O}_{C}^{\oplus r}$, we get an inclusion, which sits in a short exact sequence

$$
\begin{equation*}
0 \rightarrow \wedge^{r-k} \mathcal{K} \rightarrow \wedge^{r-k}\left(p_{C}^{*} \mathcal{O}_{C}^{\oplus r}\right) \rightarrow \mathcal{F}^{\prime} \rightarrow 0 \tag{4.1}
\end{equation*}
$$

Let $q \in \mathcal{Q}_{L}$ be a closed point. The restriction of the map in (4.1) to $C \times q$ is the same as restricting the map (3.1) to $C \times q$ and then applying $\wedge^{r-k}$. From this it easily follows that the restriction of (4.1) to $C \times q$ is an inclusion. For each point $q \in \mathcal{Q}_{L}$, the sheaf $\wedge^{r-k} \mathcal{K}_{q}:=\left.\wedge^{r-k} \mathcal{K}\right|_{C \times q} \cong L^{-1}$. Thus, it follows that the rank and degree of $\mathcal{F}_{q}^{\prime}$ are constant. As $\mathcal{Q}_{L}$ is an integral scheme, it follows that $\mathcal{F}^{\prime}$ of (4.1) is flat over $\mathcal{Q}_{L}$. The quotient

$$
\wedge^{r-k}\left(p_{C}^{*} \mathcal{O}_{C}^{\oplus r}\right) \rightarrow \mathcal{F}^{\prime}
$$

on $C \times \mathcal{Q}_{L}$ gives rise to a morphism from $\mathcal{Q}_{L}$ to the quot scheme

$$
\operatorname{Quot}_{C / \mathbb{C}}\left(\wedge^{r-k}\left(\mathcal{O}_{C}^{\oplus r}\right),\binom{r}{r-k}-1, d\right) .
$$

Moreover, for each $q$, the cokernel $\mathcal{F}_{q}^{\prime}$ has determinant $L$. It follows that the image of the composite map

$$
\mathcal{Q}_{L} \rightarrow \operatorname{Quot}_{C / \mathbb{C}}\left(\wedge^{r-k}\left(\mathcal{O}_{C}^{\oplus}\right),\binom{r}{r-k}-1, d\right) \xrightarrow{\operatorname{det}} \operatorname{Pic}^{d}(C)
$$

is, at least set theoretically, the closed point $[L]$. As $\mathcal{Q}_{L}$ is an integral scheme, it follows that the image lands in the scheme theoretic fiber Quot ${ }_{C / \mathbb{C}}\left(\wedge^{r-k}\left(\mathcal{O}_{C}^{\oplus r}\right),\binom{r}{r-k}-1, d\right)_{L}$. By (A) (that is, [GS24, Theorem 3.3, Corollary 3.5]) it follows that we get a map from

$$
\begin{equation*}
f: \mathcal{Q}_{L} \rightarrow \mathbb{P}\left(\mathcal{E}_{L^{-1}}^{\vee}\right) \tag{4.2}
\end{equation*}
$$

(note that $e=0$ ). For ease of notation we denote $\mathbb{P}\left(\mathcal{E}_{L^{-1}}^{\vee}\right)$ by $\mathbb{P}$. Define

$$
\begin{equation*}
\alpha:=f^{*} \mathcal{O}_{\mathbb{P}}(1) . \tag{4.3}
\end{equation*}
$$

Proposition 4.4. The line bundle $\alpha$ is nef but not ample.
Proof. It is clear that $\alpha$ is nef. To show it is not ample, it suffices to find a curve $D_{1} \subset \mathcal{Q}_{L}$ such that the restriction of $\alpha$ to $D_{1}$ is trivial.

We begin with describing the map $f$ in (4.2) in some more detail. Let $p_{2}: C \times \mathcal{Q}_{L} \rightarrow \mathcal{Q}_{L}$ denote the projection. By the Seesaw Theorem, there is a line bundle $M$ on $\mathcal{Q}_{L}$ such that $\wedge^{r-k} \mathcal{K} \cong p_{C}^{*} L^{-1} \otimes p_{2}^{*} M$. Tensoring (4.1) with $p_{C}^{*} L$ we get the following exact sequence of sheaves on $C \times \mathcal{Q}_{L}$

$$
0 \rightarrow p_{2}^{*} M \rightarrow\left[\wedge^{r-k}\left(p_{C}^{*} \mathcal{O}_{C}^{\oplus r}\right)\right] \otimes p_{C}^{*} L \rightarrow \mathcal{F}^{\prime} \otimes p_{C}^{*} L \rightarrow 0
$$

Applying $p_{2 *}$ we get the following exact sequence of sheaves on $\mathcal{Q}_{L}$

$$
0 \rightarrow M \rightarrow H^{0}\left(C,\left[\wedge^{r-k}\left(\mathcal{O}_{C}^{\oplus r}\right)\right] \otimes L\right) \otimes \mathcal{O}_{\mathcal{Q}_{L}} \rightarrow p_{2 *}\left(\mathcal{F}^{\prime} \otimes p_{C}^{*} L\right) \rightarrow H^{1}\left(C, \mathcal{O}_{C}\right) \otimes M \rightarrow 0
$$

The last term on the right is 0 as the degree of $L$ is $d$, which is assumed to be very large. It follows that $p_{2 *}\left(\mathcal{F}^{\prime} \otimes p_{C}^{*} L\right)$ is locally free. Taking dual of the above sequence, we get a surjection

$$
H^{0}\left(C, \wedge^{r-k}\left[\wedge^{r-k}\left(\mathcal{O}_{C}^{\oplus r}\right)\right] \otimes L\right)^{\vee} \otimes \mathcal{O}_{\mathcal{Q}_{L}} \rightarrow M^{-1}
$$

which defines the map $f$ to

$$
\mathbb{P}\left(H^{0}\left(C, \wedge^{r-k}\left[\wedge^{r-k}\left(\mathcal{O}_{C}^{\oplus r}\right)\right] \otimes L\right)^{\vee}\right)
$$

It is clear that the pullback of $\mathcal{O}_{\mathbb{P}}(1)$ is $M^{-1}$. Thus, $\alpha=M^{-1}$.
To construct the curve $D_{1}$, fix a closed point $x \in C$ and a subsheaf $K^{\prime} \subset \mathcal{O}_{C}^{\oplus r}$ with $\operatorname{det}\left(K^{\prime}\right)=L^{-1} \otimes \mathcal{O}_{C}(x)$. Let $D_{1} \subset \mathbb{P}\left(K_{x}^{\prime}\right)$ be a line in the projective space associated to the vector space $K_{x}^{\prime}$. Let $\iota_{x}: D_{1} \rightarrow C \times D_{1}$ denote the map $t \mapsto(x, t)$. On $C \times D_{1}$ we have the surjection

$$
p_{C}^{*} K^{\prime} \rightarrow \iota_{x *} \iota_{x}^{*} p_{C}^{*} K^{\prime}=\iota_{x *}\left(K_{x}^{\prime} \otimes \mathcal{O}_{D_{1}}\right) \rightarrow \iota_{x *}\left(\mathcal{O}_{D_{1}}(1)\right) .
$$

Let $\tilde{K}_{1}$ be the kernel of this surjection. As $\iota_{x *}\left(\mathcal{O}_{D_{1}}(1)\right)$ is flat over $D_{1}$, it follows that $\tilde{K}_{1}$ is flat over $D_{1}$. We have the following commutative diagram over $C \times D_{1}$ in which all three
term sequences are exact


It follows easily that for $t \in D_{1}$, the rank and degree of $\mathcal{G}_{1, t}$ are independent of $t$. Thus, $\mathcal{G}_{1}$ is flat over $D_{1}$. Note that

$$
\begin{aligned}
\wedge^{r-k} \tilde{K}_{1} & \cong \operatorname{det}\left(p_{C}^{*} K^{\prime}\right) \otimes \operatorname{det}\left(\iota_{x *}\left(\mathcal{O}_{D_{1}}(1)\right)\right)^{-1} \\
& \cong p_{C}^{*}\left(L^{-1} \otimes \mathcal{O}_{C}(x)\right) \otimes p_{C}^{*} \mathcal{O}_{C}(-x) \\
& \cong p_{C}^{*} L^{-1}
\end{aligned}
$$

It easily follows that we get a morphism $D_{1} \rightarrow \mathcal{Q}_{L}$, which is an inclusion on closed points.
As $p_{C}^{*} L^{-1}=p_{C}^{*} L^{-1} \otimes p_{2}^{*} \mathcal{O}_{D_{1}}$, from the description of the morphism $f$, it is clear that the restriction of $f^{*}\left(\mathcal{O}_{\mathbb{P}}(1)\right)$ to $D_{1}$ is the trivial bundle. This shows that $\alpha$ is not ample.

Let $\mathcal{Q}_{L}^{s}$ denote the locus of quotients $\left[\mathcal{O}_{C}^{\oplus r} \rightarrow F\right]$ such that $F$ is a stable bundle. To construct our second nef class, we shall first construct a vector bundle on $\mathcal{Q}_{L}^{s}$. The determinant of this gives a line bundle on $\mathcal{Q}_{L}^{s}$. Under our hypothesis, it follows that $\operatorname{codim}\left(\mathcal{Q}_{L} \backslash \mathcal{Q}_{L}^{s}, \mathcal{Q}_{L}\right) \geqslant 2$ (see the discussion before equation (8.10) in [GS24]). Thus, the line bundle constructed on $\mathcal{Q}_{L}^{s}$ extends uniquely to a line bundle on $\mathcal{Q}_{L}$. We will show that this line bundle is nef.

Let $M$ be a line bundle on $C$ of degree $m$ such that

$$
\begin{equation*}
\frac{d}{k}+m>2 g-2 \tag{4.6}
\end{equation*}
$$

Recall the universal sheaf $\mathcal{F}$ from (3.1). Let $p_{2}$ denote the projection $C \times \mathcal{Q}_{L}^{s} \rightarrow \mathcal{Q}_{L}^{s}$. Consider the sheaf $p_{2 *}\left(\mathcal{F} \otimes p_{C}^{*} M\right)$ on $\mathcal{Q}_{L}^{s}$. For a point $q \in \mathcal{Q}_{L}^{s}$, we have $h^{1}\left(C, \mathcal{F}_{q} \otimes M\right)=0$ iff $\operatorname{hom}\left(\mathcal{F}_{q} \otimes M, \omega_{C}\right)=0$. Assume $\mu_{\min }\left(\mathcal{F}_{q} \otimes M\right)>\mu_{\max }\left(\omega_{C}\right)$, that is, if (4.6) holds. By [HL10, Lemma 1.3.3], it follows that $\operatorname{hom}\left(\mathcal{F}_{q} \otimes M, \omega_{C}\right)=0$. Thus, if $m$ is such that this inequality holds, then it follows easily, using cohomology and base change [Har77, Theorem 12.11], that the sheaf $p_{2 *}\left(\mathcal{F} \otimes p_{C}^{*} M\right)$ on $\mathcal{Q}_{L}^{s}$ is locally free. The determinant of this locally free sheaf gives a line bundle on $\mathcal{Q}_{L}^{s}$, which extends uniquely to a line bundle on $\mathcal{Q}_{L}$. We denote this line bundle by $\beta_{M}$.
Lemma 4.7. Let $M$ and $M^{\prime}$ have the same degree $m$ such that (4.6) holds. Then $\beta_{M}$ is numerically equivalent to $\beta_{M^{\prime}}$.

Proof. Let $\mathcal{P}$ be a Poincare bundle on $C \times \operatorname{Pic}^{m}(C)$. Let $p_{i j}$ denote the projection maps from $C \times \mathcal{Q}_{L}^{s} \times \operatorname{Pic}^{m}(C)$. Consider the sheaf $p_{23 *}\left(\mathcal{F} \otimes p_{13}^{*} \mathcal{P}\right)$ on $\mathcal{Q}_{L}^{s} \times \operatorname{Pic}^{m}(C)$. For a point $(q, M) \in \mathcal{Q}_{L}^{s} \times \operatorname{Pic}^{m}(C)$, we have $h^{1}\left(C, \mathcal{F}_{q} \otimes M\right)=0$. It follows easily that the sheaf $p_{23 *}\left(\mathcal{F} \otimes p_{13}^{*} \mathcal{P}\right)$ on $\mathcal{Q}_{L}^{s} \times \operatorname{Pic}^{m}(C)$ is locally free. Let us denote the determinant of this sheaf by $\mathcal{R}$. It can be easily seen, for example, using similar reasoning as in [GS24, Proposition
8.1], that $\mathcal{Q}_{L} \times \operatorname{Pic}^{m}(C)$ is locally factorial. It easily follows that the line bundle $\mathcal{R}$ extends uniquely to a line bundle on $\mathcal{Q}_{L} \times \operatorname{Pic}^{m}(C)$, which we continue to denote by $\mathcal{R}$.

It is also easily seen, using Remark 3.2, that the restriction of $p_{23 *}\left(\mathcal{F} \otimes p_{13}^{*} \mathcal{P}\right)$ to $\mathcal{Q}_{L}^{s} \times[M]$ equals $p_{2 *}\left(\mathcal{F} \otimes p_{C}^{*} M\right)$. Thus, it easily follows that $\mathcal{R}$ restricted to $\mathcal{Q}_{L} \times[M]$ equals $\beta_{M}$. Similarly, it follows that $\mathcal{R}$ restricted to $\mathcal{Q}_{L} \times\left[M^{\prime}\right]$ equals $\beta_{M^{\prime}}$. It easily follows that $\beta_{M}$ is numerically equivalent to $\beta_{M^{\prime}}$. This completes the proof of the Lemma.

Remark 4.8. In view of the above Lemma, when $m$ satisfies (4.6), we shall denote the corresponding numerical class by $\beta_{m}$.

It is easily checked that when $d \gg 0$ then $m=d+g-1$ satisfies (4.6).
Proposition 4.9. The class $\beta_{d+g-1}$ is globally generated and hence nef. This class is not ample.

Proof. We will show that for any point $q \in \mathcal{Q}_{L}$, there is a line bundle $M$ on $C$ of degree $d+g-1$, such that the line bundle $\beta_{M}$ on $\mathcal{Q}_{L}$ has a global section which does not vanish at $q$. This will show that $\beta_{d+g-1}$ is globally generated. As a globally generated line bundle is nef, it follows that $\beta_{d+g-1}$ is nef.

Consider the action of $\mathbb{C}^{*}$ on $\mathbb{C}^{r}$ given by $t \cdot\left(a_{1}, \ldots, a_{r}\right)=\left(a_{1}, t a_{2}, \ldots, t^{r-1} a_{r}\right)$. This action gives rise to an action of $\mathbb{C}^{*}$ on $\mathcal{O}_{C}^{\oplus r}$ and so also on $\mathcal{Q}_{L}$. Indeed, this action sends an inclusion $\varphi$ to the inclusion $\varphi \circ t^{-1}$, in the following commutative diagram


Thus, given any point $q \in \mathcal{Q}_{L}$, we may find a $\mathbb{C}^{*}$ equivariant morphism $h: \mathbb{C}^{*} \rightarrow \mathcal{Q}_{L}$ such that $h(1)=q$. Note that for $t \in \mathbb{C}^{*}$, the kernel sheaf in $h(t)$ is the same as the kernel sheaf in $q$. The morphism $h$ extends to a morphism $\mathbb{C} \rightarrow \mathcal{Q}_{L}$ and the point $h(0)$ is fixed under the action of $\mathbb{C}^{*}$ on $\mathcal{Q}_{L}$. Thus, $h(0)$ is a quotient $q_{0}$ whose kernel equals

$$
K_{0}=\bigoplus_{i=1}^{r-k} \mathcal{O}_{C}\left(-D_{i}\right),
$$

where each $D_{i}$ is an effective divisor of degree $d_{i}$, and the $d_{i}$ satisfy $\sum_{i} d_{i}=d$. See, for example, [BGL94, §3]. Let $M$ be a general line bundle of degree $d+g-1$. Then $M \otimes \mathcal{O}_{C}\left(-D_{i}\right)$ is a general line bundle of degree $d-d_{i}+g-1 \geqslant g-1$. In particular, $h^{1}\left(C, M \otimes \mathcal{O}_{C}\left(-D_{i}\right)\right)=0$. It follows that $h^{1}\left(C, M \otimes K_{0}\right)=0$ and so this vanishing holds for the kernels in an open set containing $q_{0}$. In particular, it follows that

$$
\begin{equation*}
h^{1}(C, M \otimes K)=0, \tag{4.10}
\end{equation*}
$$

where $K$ is the kernel of the quotient $q$ we started with.
Since $d \gg 0$, we have $d+g-1>2 g-2$ and so $h^{1}(C, M)=0$. If $q^{\prime}$ is any point in $\mathcal{Q}_{L}$, then the cohomology long exact sequence of the short exact sequence

$$
0 \rightarrow \mathcal{K}_{q^{\prime}} \otimes M \rightarrow\left(\mathcal{O}_{C}^{\oplus r}\right) \otimes M \rightarrow \mathcal{F}_{q^{\prime}} \otimes M \rightarrow 0
$$

shows that $h^{1}\left(C, \mathcal{F}_{q^{\prime}} \otimes M\right)=0$. In particular, it follows that the sheaf $p_{2 *}\left(\mathcal{F} \otimes p_{C}^{*} M\right)$ is locally free on all of $\mathcal{Q}_{L}$. Applying $p_{2 *}\left(-\otimes p_{C}^{*} M\right)$ to (3.1), and using (4.10), it follows that on an open set containing the point $q$, the map

$$
p_{2 *}\left(p_{C}^{*}\left(\mathcal{O}_{C}^{\oplus r}\right) \otimes p_{C}^{*} M\right) \rightarrow p_{2 *}\left(\mathcal{F} \otimes p_{C}^{*} M\right)
$$

is a surjection. Applying $\wedge^{k}$ we get that the map

$$
\wedge^{k}\left(H^{0}(C, M)^{\oplus r}\right) \rightarrow \beta_{M}
$$

is surjective on an open set containing $q$. It follows that $\beta_{d+g-1}$ is globally generated and so nef.

To show that $\beta_{d+g-1}$ is not ample, we will construct a curve $D_{2} \subset \mathcal{Q}_{L}$ such that [ $D_{2}$ ]. $\left[\beta_{d+g-1}\right]=0$. Let $D_{2}$ be a line in $\mathbb{P}\left(H^{0}(C, L)^{\vee}\right)$. Then on $C \times D_{2}$ we have a short exact sequence

$$
0 \rightarrow p_{C}^{*} L^{-1} \otimes p_{2}^{*} \mathcal{O}_{D_{2}}(-1) \rightarrow p_{C}^{*} \mathcal{O}_{C} \rightarrow \mathcal{G}_{2} \rightarrow 0
$$

"Adding" to this, identity maps of the type $p_{C}^{*} \mathcal{O}_{C}^{\oplus l} \rightarrow p_{C}^{*} \mathcal{O}_{C}^{\oplus l}$, we get the quotient

$$
\begin{align*}
0 \rightarrow\left(p_{C}^{*} L^{-1} \otimes\right. & \left.p_{2}^{*} \mathcal{O}_{D_{2}}(-1)\right) \bigoplus\left(p_{C}^{*} \mathcal{O}_{C}^{\oplus(r-k-1)}\right)  \tag{4.11}\\
& \rightarrow p_{C}^{*} \mathcal{O}_{C} \bigoplus p_{C}^{*} \mathcal{O}_{C}^{\oplus(r-k-1)} \bigoplus\left(p_{C}^{*} \mathcal{O}_{C}^{\oplus k}\right) \rightarrow \mathcal{G}_{2} \bigoplus\left(p_{C}^{*} \mathcal{O}_{C}^{\oplus k}\right) \rightarrow 0
\end{align*}
$$

This defines a morphism

$$
f^{\prime}: D_{2} \rightarrow \mathcal{Q}_{L}
$$

which is clearly injective on closed points. Let $M$ be a line bundle of degree $d+g-1$ such that $h^{1}\left(C, L^{-1} \otimes M\right)=0$. Again, as $d \gg 0$, we have $h^{1}(C, M)=0$. It easily follows that for each $t \in D_{2}$,

$$
h^{1}\left(C,\left[\mathcal{G}_{2, t} \bigoplus\left(\mathcal{O}_{C}^{\oplus k}\right)\right] \otimes M\right)=0
$$

It follows easily using Remark 3.2 that $f^{\prime *} \beta_{M}=\operatorname{det}\left(p_{2 *}\left(\left(\mathcal{G} \oplus p_{C}^{*} \mathcal{O}_{C}^{\oplus k}\right) \otimes p_{C}^{*} M\right)\right.$ ). We claim that $p_{2 *}\left(\left(\mathcal{G} \bigoplus p_{C}^{*} \mathcal{O}_{C}^{\oplus k}\right) \otimes p_{C}^{*} M\right)$ is the trivial bundle. To see this, note that

$$
R^{i} p_{2 *}\left[\left\{\left(p_{C}^{*} L^{-1} \otimes p_{2}^{*} \mathcal{O}_{D_{2}}(-1)\right) \bigoplus\left(p_{C}^{*} \mathcal{O}_{C}^{\oplus(r-k-1)}\right)\right\} \otimes p_{C}^{*} M\right]=0
$$

for $i=0,1$. It follows that when we apply $p_{2 *}\left(-\otimes p_{C}^{*} M\right)$ to the sequence (4.11), we get the following exact sequence in which the first two terms are trivial bundles,

$$
0 \rightarrow H^{0}(C, M)^{\oplus(r-k-1)} \otimes \mathcal{O}_{D_{2}} \rightarrow H^{0}(C, M)^{\oplus r} \otimes \mathcal{O}_{D_{2}} \rightarrow p_{2 *}\left(\left(\mathcal{G} \bigoplus p_{C}^{*} \mathcal{O}_{C}^{\oplus k}\right) \otimes p_{C}^{*} M\right) \rightarrow 0
$$

Thus, it follows that the last bundle is also trivial, and so $f^{\prime *} \beta_{M}$ is the trivial bundle. This shows that $\left[D_{2}\right] \cdot\left[\beta_{d+g-1}\right]=0$.
Lemma 4.12. $\left[D_{2}\right] \cdot[\alpha]=\left[D_{1}\right] \cdot\left[\beta_{d+g-1}\right]=1$.
Proof. To compute $\left[D_{2}\right] \cdot[\alpha]$ we shall use the description of the map $f$ in the proof of Proposition 4.4. The kernel in the family of sheaves defining $D_{2}$ (see (4.11)) is

$$
\left(p_{C}^{*} L^{-1} \otimes p_{2}^{*} \mathcal{O}_{D_{2}}(-1)\right) \bigoplus\left(p_{C}^{*} \mathcal{O}_{C}^{\oplus(r-k-1)}\right)
$$

Taking $\wedge^{r-k}$ of this sheaf gives the line bundle $p_{C}^{*} L^{-1} \otimes p_{2}^{*} \mathcal{O}_{D_{2}}(-1)$. Thus, the pullback of $\alpha$ to $D_{2}$ is $\mathcal{O}_{D_{2}}(1)$. The degree of this line bundle is 1 . Thus, $\left[D_{2}\right] \cdot[\alpha]=1$.

Recall the family of quotients (4.5) which defines a morphism $D_{1} \rightarrow \mathcal{Q}_{L}$. Using cohomology and base change, it easily follows that the pullback of $\beta_{d+g-1}$ to $D_{1}$ is $\operatorname{det}\left(p_{2 *}\left(\mathcal{G}_{1} \otimes M\right)\right)$, where $M$ is any line bundle of degree $d+g-1$. From the short exact sequence (see (4.5))

$$
0 \rightarrow \iota_{x *}\left(\mathcal{O}_{D_{1}}(1)\right) \rightarrow \mathcal{G}_{1} \rightarrow p_{C}^{*} F^{\prime} \rightarrow 0
$$

it easily follows that $\operatorname{det}\left(p_{2 *}\left(\mathcal{G}_{1} \otimes M\right)\right)=\mathcal{O}_{D_{1}}(1)$. Thus, $\left[D_{1}\right] \cdot\left[\beta_{d+g-1}\right]=1$.
Remark 4.13. As a Corollary of $\left[D_{2}\right] \cdot[\alpha]=1$, we see that $\alpha$ and $\beta_{d+g-1}$ are not numerically equivalent. Thus, it follows that the natural map $\operatorname{Pic}\left(\mathcal{Q}_{L}\right) \rightarrow \mathrm{N}^{1}\left(\mathcal{Q}_{L}\right)$ is an isomorphism.

Putting together the above results, we have the following Theorem.
Theorem 4.14. Assume one of the following two holds:

- $g(C) \geqslant 3$ and $2 \leqslant k \leqslant r-2$, or
- $g(C)=2$ and $3 \leqslant k \leqslant r-2$.

Let $d \gg 0$. Then $\operatorname{Pic}\left(\mathcal{Q}_{L}\right)$ is generated by the line bundles $\alpha, \beta_{d+g-1}$. Both these are globally generated and so nef, but not ample. In particular, they are the boundaries of the nef cone. The boundaries of the cone of effective curves are given by the classes of $D_{1}$ and $D_{2}$.
Proof. Given an integral curve $C^{\prime} \subset \mathcal{Q}_{L}$, let us write $\left[C^{\prime}\right]=a\left[D_{1}\right]+b\left[D_{2}\right]$ in $N_{1}\left(\mathcal{Q}_{L}\right)$. Intersecting with $\alpha$ and $\beta_{d+g-1}$ we easily see that $a \geqslant 0$ and $b \geqslant 0$.

## 5. Effective cone

In this section we shall determine the cone of effective divisors in the Picard group of $\mathcal{Q}_{L}$. Let

$$
\begin{equation*}
\mathcal{Q}_{L}^{\prime}=\left\{\left[\mathcal{O}_{C}^{\oplus r} \xrightarrow{q} F\right] \in \mathcal{Q}_{L}^{s} \mid \operatorname{Ker}(q) \text { is stable }\right\} . \tag{5.1}
\end{equation*}
$$

denote the open set consisting of quotients such that both $F$ and $\operatorname{Ker}(q)$ are stable.
Lemma 5.2. $\operatorname{codim}\left(\mathcal{Q}_{L} \backslash \mathcal{Q}_{L}^{\prime}, \mathcal{Q}_{L}\right) \geqslant 2$.
Proof. Let $\mathcal{Q}_{L}^{\mathrm{tf}} \subset \mathcal{Q}_{L}$ denote the locus of quotients $\left[\mathcal{O}_{C}^{\oplus r} \rightarrow F\right]$ such that $F$ is torsion free. Note that $\mathcal{Q}_{L}^{s} \subset \mathcal{Q}_{L}^{\mathrm{tf}}$. For ease of notation, let us denote the Quot scheme $\operatorname{Quot}_{C / \mathbb{C}}\left(\mathcal{O}_{C}^{\oplus r}, r-k, d\right)$ by $\widetilde{\mathcal{Q}}$. There is an isomorphism of schemes $\varphi: \mathcal{Q}_{L}^{\mathrm{tf}} \rightarrow \widetilde{\mathcal{Q}}_{L}^{\mathrm{tf}}$ which sends a quotient

$$
\left[\operatorname{Ker}(q) \subset \mathcal{O}_{C}^{\oplus r} \xrightarrow{q} F\right] \mapsto\left[F^{\vee} \subset \mathcal{O}_{C}^{\oplus r} \rightarrow \operatorname{Ker}(q)^{\vee}\right] .
$$

The open set $\mathcal{Q}_{L}^{\prime}$ is precisely the intersection $\mathcal{Q}_{L}^{s} \cap \varphi^{-1}\left(\widetilde{\mathcal{Q}}_{L}^{s}\right)$. By the discussion before equation (8.10) in [GS24], it follows that $\operatorname{codim}\left(\mathcal{Q}_{L} \backslash \mathcal{Q}_{L}^{s}, \mathcal{Q}_{L}\right) \geqslant 2$. Similarly, $\operatorname{codim}\left(\widetilde{\mathcal{Q}}_{L}^{\mathrm{tf}} \backslash \widetilde{\mathcal{Q}}_{L}^{s}, \widetilde{\mathcal{Q}}_{L}^{\mathrm{tf}}\right) \geqslant 2$. Thus, it follows that the codimension of $\mathcal{Q}_{L}^{\mathrm{tf}} \backslash \varphi^{-1}\left(\widetilde{\mathcal{Q}}_{L}^{s}\right)$ in $\mathcal{Q}_{L}^{\mathrm{tf}}$ is at least 2. The Lemma now follows easily.

We will use the above Lemma to write down another curve $D_{3} \rightarrow \mathcal{Q}_{L}$ such that the image lies in $\mathcal{Q}_{L}^{\prime}$. Let $E$ be a stable bundle of rank $r-k$ with $\operatorname{det}(E)=L^{-1}$. Consider the space $\mathbb{P}\left(\operatorname{Hom}\left(E, \mathcal{O}_{C}^{\oplus r}\right)^{\vee}\right)$. The closed points of this space are in bijection with nonzero maps $\mathcal{O}_{C}^{\oplus r} \rightarrow E^{\vee}$. The locus of points in $\mathbb{P}\left(\operatorname{Hom}\left(E, \mathcal{O}_{C}^{\oplus r}\right)^{\vee}\right)$ corresponding to non-surjective maps $\mathcal{O}_{C}^{\oplus r} \rightarrow E^{\vee}$ has codimension at least 2, see the proof in [GS24, Lemma 7.12]. Let

$$
U_{E^{\vee}} \subset \mathbb{P}\left(\operatorname{Hom}\left(\mathcal{O}_{C}^{\oplus r}, E^{\vee}\right)^{\vee}\right)=\mathbb{P}\left(\operatorname{Hom}\left(E, \mathcal{O}_{C}^{\oplus r}\right)^{\vee}\right)
$$

denote the locus parametrizing surjective maps $\mathcal{O}_{C}^{\oplus r} \rightarrow E^{\vee}$. Let $M_{r-k, L}^{s}$ denote the moduli space of stable bundles of rank $r-k$ and determinant $L$. Consider the natural map

$$
\begin{equation*}
\pi: \widetilde{\mathcal{Q}}_{L}^{s} \rightarrow M_{r-k, L}^{s}, \tag{5.3}
\end{equation*}
$$

which sends $\left[\mathcal{O}_{C}^{\oplus r} \rightarrow F\right] \mapsto[F]$. The fiber over the point $[F]$ is precisely the set $U_{F}$. Let $T$ denote the closed subset $\widetilde{\mathcal{Q}}_{L}^{s} \backslash \widetilde{\mathcal{Q}}_{L}^{\prime}$.
Lemma 5.4. For general $F \in M_{r-k, L}^{s}, \operatorname{codim}\left(T \cap U_{F}, U_{F}\right) \geqslant 2$.
Proof. If $\operatorname{codim}\left(T \cap U_{F}, U_{F}\right) \leqslant 1$ for general $[F] \in M_{r-k, L}^{s}$, then it follows that

$$
\operatorname{dim}(T)=\operatorname{dim}\left(M_{r-k, L}^{s}\right)+\operatorname{dim}\left(\pi^{-1}([F])\right)-1=\operatorname{dim}\left(\widetilde{\mathcal{Q}}_{L}^{s}\right)-1
$$

This contradicts Lemma 5.2, which says that $\operatorname{codim}\left(T, \widetilde{\mathcal{Q}}_{L}^{s}\right) \geqslant 2$.
Thus, it follows that for general $E$, the locus of points in $U_{E^{\vee}}$, such that the kernel of $\mathcal{O}_{C}^{\oplus r} \rightarrow E^{\vee}$ is not stable, has codimension $\geqslant 2$. In other words, if $U^{\prime} \subset \mathbb{P}\left(\operatorname{Hom}\left(E, \mathcal{O}_{C}^{\oplus r}\right)^{\vee}\right)$ denotes the set of points corresponding to inclusions $E \rightarrow \mathcal{O}_{C}^{\oplus r}$ such that the cokernel is torsion free and stable, then for general $E$,

$$
\operatorname{codim}\left(\mathbb{P}\left(\operatorname{Hom}\left(E, \mathcal{O}_{C}^{\oplus r}\right)^{\vee}\right) \backslash U^{\prime}, \mathbb{P}\left(\operatorname{Hom}\left(E, \mathcal{O}_{C}^{\oplus r}\right)^{\vee}\right)\right) \geqslant 2
$$

If $W \subset \mathbb{P}^{n}$ is a closed subset such that $\operatorname{codim}\left(W, \mathbb{P}^{n}\right) \geqslant 2$, then the general line in $\mathbb{P}^{n}$ does not meet $W$. This is easily seen by projecting from a point outside $W$. Thus, we can find a line $D_{3} \subset \mathbb{P}\left(\operatorname{Hom}\left(E, \mathcal{O}_{C}^{\oplus r}\right)^{\vee}\right)$, which is completely contained in $U^{\prime}$. We get a family of quotients on $C \times D_{3}$

$$
\begin{equation*}
0 \rightarrow p_{C}^{*} E \otimes p_{2}^{*} \mathcal{O}_{D_{3}}(-1) \rightarrow p_{C}^{*} \mathcal{O}_{C}^{\oplus r} \rightarrow \mathcal{G}_{3} \rightarrow 0 \tag{5.5}
\end{equation*}
$$

such that for each $t \in D_{3}$, the sheaf $\mathcal{G}_{3, t}$ is stable. The above family defines a morphism $D_{3} \rightarrow \mathcal{Q}_{L}$, which is injective on closed points. Clearly, the image of $D_{3}$ lands in $\mathcal{Q}_{L}^{\prime}$.

Lemma 5.6. $[\alpha] \cdot\left[D_{3}\right]=r-k$ and $\left[\beta_{d+g-1}\right] \cdot\left[D_{3}\right]=d(r-k-1)$.
Proof. To compute $[\alpha] \cdot\left[D_{3}\right]$ we follow the description of the map $f$ (see (4.2)) given in Proposition 4.4.

$$
\wedge^{r-k}\left(p_{C}^{*} E \otimes p_{2}^{*} \mathcal{O}_{D_{3}}(-1)\right)=p_{C}^{*} L^{-1} \otimes p_{2}^{*} \mathcal{O}_{D_{3}}(-(r-k)) .
$$

Thus, it follows that $[\alpha] \cdot\left[D_{3}\right]=r-k$.
Let $M$ be a line bundle of degree $d+g-1$. By Serre duality, we have, $h^{1}(C, E \otimes M)=$ $h^{0}\left(E^{\vee} \otimes M^{\vee} \otimes \omega_{C}\right)$. Note $E$ is stable and

$$
\mu\left(E^{\vee} \otimes M^{\vee} \otimes \omega_{C}\right)=\frac{d}{r-k}-(d+g-1)+2 g-2
$$

The slope is $<0$ for $d \gg 0$ as $r-k \geqslant 2$. Thus, $h^{0}\left(E^{\vee} \otimes M^{\vee} \otimes \omega_{C}\right)=h^{1}(C, E \otimes M)=0$. Thus, applying $p_{2 *}\left(-\otimes p_{C}^{*} M\right)$ to (5.5), we get the short exact sequence

$$
0 \rightarrow H^{0}(C, E \otimes M) \otimes \mathcal{O}_{D_{3}}(-1) \rightarrow H^{0}(C, M) \otimes \mathcal{O}_{D_{3}} \rightarrow p_{2 *}\left(\mathcal{G}_{3} \otimes p_{C}^{*} M\right) \rightarrow 0
$$

It follows that

$$
\begin{aligned}
{\left[D_{3}\right] \cdot\left[\beta_{d+g-1}\right] } & =\operatorname{deg}\left(p_{2 *}\left(\mathcal{G}_{3} \otimes p_{C}^{*} M\right)\right) \\
& =h^{0}(C, E \otimes M)=\chi(E \otimes M) \\
& =d(r-k-1)
\end{aligned}
$$

This completes the proof of the Lemma.
Consider the map $\pi: \mathcal{Q}_{L}^{s} \rightarrow M_{k, L}^{s}$, where $M_{k, L}^{s}$ denotes the moduli space parametrizing stable bundles of rank $k$ and determinant $L$. The map $\pi$ sends a quotient $\left[\mathcal{O}_{C}^{\oplus r} \rightarrow F\right]$ to $[F]$. The fiber of $\pi$ over the point $[F]$ is the subset $U_{F} \subset \mathbb{P}\left(\operatorname{Hom}\left(\mathcal{O}_{C}^{\oplus r}, F\right)^{\vee}\right)$ corresponding to surjective maps. Let $U_{F}^{\prime} \subset U_{F}$ be the subset corresponding to maps for which the kernel is also a stable bundle. Arguing as in the construction of the curve $D_{3}$, we see that for a general stable bundle $F$, one has

$$
\operatorname{codim}\left(\mathbb{P}\left(\operatorname{Hom}\left(\mathcal{O}_{C}^{\oplus r}, F\right)^{\vee}\right) \backslash U_{F}^{\prime}, \mathbb{P}\left(\operatorname{Hom}\left(\mathcal{O}_{C}^{\oplus r}, F\right)^{\vee}\right)\right) \geqslant 2
$$

Thus, taking $F$ general stable and taking $D_{4}$ to be a general line in $\mathbb{P}\left(\operatorname{Hom}\left(\mathcal{O}_{C}^{\oplus r}, F\right)^{\vee}\right)$, to get a family

$$
\begin{equation*}
0 \rightarrow \mathcal{K}_{4} \rightarrow p_{C}^{*} \mathcal{O}_{C}^{\oplus r} \rightarrow p_{C}^{*} F \otimes p_{2}^{*} \mathcal{O}_{D_{4}}(1) \rightarrow 0 \tag{5.7}
\end{equation*}
$$

on $C \times D_{4}$. Again, this family has the property that for each $t \in D_{4}$, the sheaf $\mathcal{K}_{4, t}$ is stable.
Lemma 5.8. $[\alpha] \cdot\left[D_{4}\right]=k$ and $\left[\beta_{d+g-1}\right] \cdot\left[D_{4}\right]=d(k+1)$.
Proof. Note that $\wedge^{r-k} \mathcal{K}_{4}=\left(p_{C}^{*} L \otimes p_{2}^{*} \mathcal{O}_{D_{4}}(k)\right)^{-1}$. Using the description in Proposition 4.4, it follows that $[\alpha] \cdot\left[D_{4}\right]=k$. Let $M$ be a line bundle of degree $d+g-1$. Since

$$
p_{2 *}\left(p_{C}^{*}(F \otimes M) \otimes p_{2}^{*} \mathcal{O}_{D_{4}}(1)\right)=H^{0}(C, F \otimes M) \otimes \mathcal{O}_{D_{4}}(1)
$$

and $H^{1}(C, F \otimes M)=0$, it follows that $\left[\beta_{d+g-1}\right] \cdot\left[D_{4}\right]=\chi(F \otimes M)=d(k+1)$.
Lemma 5.9. Let $a$ and $b$ be integers such that $a \alpha+b \beta_{d+g-1}$ is an effective divisor. Then $a k+b d(k+1) \geqslant 0$.

Proof. Let $Y \subset \mathcal{Q}_{L}$ be an effective divisor. Then $Y$ cannot contain all the fibers of the map $\pi: \mathcal{Q}_{L}^{s} \rightarrow M_{k, L}^{s}$. Thus, for general $F$, the intersection $Y \cap U_{F}^{\prime} \varsubsetneqq U_{F}^{\prime}$. In particular, $Y$ does not contain the general line in $U_{F}^{\prime}$, that is, $Y$ does not contain $D_{4}$. Thus, $[Y] \cdot\left[D_{4}\right] \geqslant 0$. Letting the class of $Y$ to be $a \alpha+b \beta_{d+g-1}$, the Lemma follows easily.

Let $\operatorname{Pic}\left(M_{k, L}\right)=\operatorname{Pic}\left(M_{k, L}^{s}\right)=\mathbb{Z}[\Theta]$, where $\Theta$ is the unique ample generator.
Lemma 5.10. Let $\lambda_{0}:=\operatorname{gcd}(k, d(k+1))=\operatorname{gcd}(k, d)$. Then $\pi^{*} \Theta=\frac{1}{\lambda_{0}}\left(d(k+1) \alpha-k \beta_{d+g-1}\right)$.
Proof. Let us write $\pi^{*} \Theta=a \alpha+b \beta_{d+g-1}$. Clearly, $\left[\pi^{*} \Theta\right] \cdot\left[D_{4}\right]=0$ as $\pi\left(D_{4}\right)=[F]$. This gives

$$
a k+b d(k+1)=0
$$

Thus, $\pi^{*} \Theta=\lambda^{\prime}\left(-d(k+1) \alpha+k \beta_{d+g-1}\right)$, for some rational number $\lambda^{\prime}$. We claim that $\lambda^{\prime} \neq 0$. This is clear from the point (4), which says that the pullback of $\Theta$ is not trivial.

Note that $\left[\pi\left(D_{3}\right)\right] \cdot[\Theta] \geqslant 0$ and so we get $\left[\pi^{*} \Theta\right] \cdot\left[D_{3}\right] \geqslant 0$. Using Lemma 5.6 we get

$$
\begin{aligned}
{\left[\pi^{*} \Theta\right] \cdot\left[D_{3}\right] } & =\lambda^{\prime}\left(-d(k+1) \alpha+k \beta_{d+g-1}\right) \cdot\left[D_{3}\right] \\
& =\lambda^{\prime}(-d(k+1)(r-k)+k d(r-k-1)) \\
& =\lambda^{\prime}(-d r)
\end{aligned}
$$

The condition that $\left[\pi^{*} \Theta\right] \cdot\left[D_{3}\right] \geqslant 0$ now forces that $\lambda^{\prime}<0$. Thus, we get that $\pi^{*} \Theta=$ $\lambda\left(d(k+1) \alpha-k \beta_{d+g-1}\right)$, where $\lambda>0$ is a rational number.

We can determine the precise value of $\lambda$ as follows. First we need to recall some facts from [GS24]. In the proof of [GS24, Theorem 8.9], it is shown that there is a commutative diagram

in which the rows are exact sequences. We recall the map $\theta: \mathcal{Q}_{L}^{s} \rightarrow M_{k, L \otimes \mathcal{O}_{C}(k n P)}^{s}$ is defined as $\theta\left(\left[\mathcal{O}_{C}^{\oplus r} \rightarrow F\right]\right)=\left[F \otimes \mathcal{O}_{C}(n P)\right]$. The existence of the above diagram is proved using the same method as described in the second paragraph of the proof of [GS24, Theorem 7.17]. Fix a point $P \in C$ and let $n \gg 0$ be a fixed integer, as in the discussion in the beginning of $\S 7$ in [GS24]. Consider the isomorphism $\delta: M_{k, L}^{s} \rightarrow M_{k, L \otimes \mathcal{O}_{C}(k n P)}^{s}$ given by $[F] \mapsto\left[F \otimes \mathcal{O}_{C}(n P)\right]$. Then $\theta=\delta \circ \pi$ and so $\theta^{-1}\left(\left[F \otimes \mathcal{O}_{C}(n P)\right]\right)=\pi^{-1}([F])=U_{F}$. Thus, the lower row in the above diagram is identified with

$$
1 \rightarrow \operatorname{Pic}\left(M_{k, L}^{s}\right) \rightarrow \operatorname{Pic}\left(\mathcal{Q}_{L}^{s}\right) \rightarrow \operatorname{Pic}\left(U_{F}\right) \cong \mathbb{Z}
$$

Given an element $\gamma \in \operatorname{Pic}\left(\mathcal{Q}_{L}^{s}\right)$, its image in $\operatorname{Pic}\left(U_{F}\right) \cong \mathbb{Z}$ is the intersection of a general line in $U_{F}$ with $\gamma$, that is, $[\gamma] \cdot\left[D_{4}\right]$. Thus, to compute $\pi^{*} \Theta$ in terms of $\alpha$ and $\beta_{d+g-1}$, we need to compute the generator of the kernel of the map $\mathbb{Z}^{\oplus 2} \xrightarrow{\varphi} \mathbb{Z}$ which sends $\varphi(1,0)=k$ and $\varphi(0,1)=d(k+1)$. Let $\lambda_{0}:=\operatorname{gcd}(k, d(k+1))=\operatorname{gcd}(k, d)$. It is easily checked that this generator, that is, $\pi^{*} \Theta$, equals

$$
\pi^{*} \Theta=\frac{d(k+1)}{\lambda_{0}} \alpha-\frac{k}{\lambda_{0}} \beta .
$$

This completes the proof of the Lemma.
Remark 5.11. As a corollary, we also get the following. Since $\operatorname{Pic}\left(\mathcal{Q}_{L}^{s}\right)$ is generated by $\alpha$ and $\beta_{d+g-1}$, it follows that the image of the restriction map $\operatorname{Pic}\left(\mathcal{Q}_{L}^{s}\right) \rightarrow \operatorname{Pic}\left(U_{F}\right)$ is generated by $\operatorname{gcd}(k, d(k+1))=\operatorname{gcd}(k, d)$.

As a corollary of Lemma 5.9 and Lemma 5.10 we get the following Corollary.
Corollary 5.12. The class $\pi^{*} \Theta$ is a boundary of the effective cone.
Proof. It is clear that $\pi^{*} \Theta$ is an effective divisor. By Lemma 5.9, if $a \alpha+b \beta_{d+g-1}$ is effective, then $a$ and $b$ satisfy the inequality $a k+b d(k+1) \geqslant 0$. It is clear that the coefficients $a$ and $b$ in $\pi^{*} \Theta$ satisfy the equality $a k+b d(k+1)=0$. Thus, the Corollary follows.

Recall the space $\mathcal{Q}_{L}^{\prime}$ defined before Lemma 5.2. On $\mathcal{Q}_{L}^{\prime}$ we have the map $\pi^{\prime}: \mathcal{Q}_{L}^{\prime} \rightarrow M_{r-k, L}^{s}$, which sends a quotient $\left[\mathcal{O}_{C}^{\oplus r} \xrightarrow{q} F\right]$ to $[\operatorname{Ker}(q)]$. Arguing as in Lemma 5.9, we have the following Lemma.

Lemma 5.13. Let $a$ and $b$ be integers such that $a \alpha+b \beta_{d+g-1}$ is an effective divisor. Then $a(r-k)+b d(r-k-1) \geqslant 0$.

Proof. The proof is the same as that of Lemma 5.9, except that we use the map $\pi^{\prime}$ now. In this case we will have the condition $[Y] \cdot\left[D_{3}\right] \geqslant 0$. The Lemma easily follows using Lemma 5.6.

Let $\operatorname{Pic}\left(M_{r-k, L}\right)=\operatorname{Pic}\left(M_{r-k, L}^{s}\right)=\mathbb{Z}\left[\Theta^{\prime}\right]$, where $\Theta^{\prime}$ is the unique ample generator. Similar to Lemma 5.10, we have the following Lemma.

Lemma 5.14. Let $\lambda_{1}:=\operatorname{gcd}(r-k, d(r-k-1))=\operatorname{gcd}(r-k, d)$. Then

$$
\pi^{\prime *} \Theta^{\prime}=\frac{1}{\lambda_{1}}\left(d(r-k-1) \alpha-(r-k) \beta_{d+g-1}\right) .
$$

Proof. Let us write $\pi^{\prime *} \Theta^{\prime}=a \alpha+b \beta_{d+g-1}$. Recall the curve $D_{3}$ defined using the family (5.5). Clearly, $\left[\pi^{\prime *} \Theta^{\prime}\right] \cdot\left[D_{3}\right]=0$ as $\pi^{\prime}\left(D_{3}\right)=[E]$. This gives

$$
a(r-k)+b d(r-k-1)=0 .
$$

Thus, $\pi^{\prime *} \Theta^{\prime}=\lambda^{\prime}\left(-d(r-k-1) \alpha+(r-k) \beta_{d+g-1}\right)$, for some rational number $\lambda^{\prime}$. As in Lemma 5.10, we have that $\lambda^{\prime} \neq 0$. Note that $\left[\pi^{\prime}\left(D_{4}\right)\right] \cdot\left[\Theta^{\prime}\right] \geqslant 0$ and so we get $\left[\pi^{\prime *} \Theta^{\prime}\right] \cdot\left[D_{4}\right] \geqslant 0$. Using Lemma 5.8 we get

$$
\begin{aligned}
{\left[\pi^{\prime *} \Theta^{\prime}\right] \cdot\left[D_{4}\right] } & =\lambda^{\prime}\left(-d(r-k-1) \alpha+(r-k) \beta_{d+g-1}\right) \cdot\left[D_{4}\right] \\
& =\lambda^{\prime}(-d(r-k-1) k+(r-k) d(k+1)) \\
& =\lambda^{\prime}(d r)
\end{aligned}
$$

The condition that $\left[\pi^{\prime *} \Theta^{\prime}\right] \cdot\left[D_{4}\right] \geqslant 0$ now forces that $\lambda^{\prime}>0$. Thus, we get that $\pi^{\prime *} \Theta^{\prime}=$ $\lambda\left(-d(r-k-1) \alpha+(r-k) \beta_{d+g-1}\right)$, where $\lambda>0$ is a rational number. Arguing as in Lemma 5.10, we get

$$
\pi^{\prime *} \Theta^{\prime}=\frac{-d(r-k-1)}{\lambda_{1}} \alpha+\frac{r-k}{\lambda_{1}} \beta_{d+g-1}
$$

This completes the proof of the Lemma.
As a corollary of Lemma 5.13 and Lemma 5.14 we get the following Corollary.
Corollary 5.15. The class $\pi^{\prime *} \Theta^{\prime}$ is a boundary of the effective cone.
Thus, combining the above results, we have the following Theorem.
Theorem 5.16. Assume one of the following two holds:

- $g(C) \geqslant 3$ and $2 \leqslant k \leqslant r-2$, or
- $g(C)=2$ and $3 \leqslant k \leqslant r-2$.

Let $d \gg 0$. The effective cone of $\mathcal{Q}_{L}$ is spanned by non-negative linear combinations of the classes $d(k+1) \alpha-k \beta_{d+g-1}$ and $-d(r-k-1) \alpha+(r-k) \beta_{d+g-1}$. Further, $\operatorname{Mov}\left(\mathcal{Q}_{L}\right)=\operatorname{Eff}\left(\mathcal{Q}_{L}\right)$.

Proof. Clearly, $\operatorname{Mov}\left(\mathcal{Q}_{L}\right) \subset \operatorname{Eff}\left(\mathcal{Q}_{L}\right)$. Since the boundaries of $\operatorname{Eff}\left(\mathcal{Q}_{L}\right)$, namely $\pi^{*} \Theta$ and $\pi^{\prime *} \Theta^{\prime}$, define morphisms on the open subset $\mathcal{Q}_{L}^{\prime}$, whose complement in $\mathcal{Q}_{L}$ has codimension $\geqslant 2$, it follows that these boundaries are in $\operatorname{Mov}\left(\mathcal{Q}_{L}\right)$. Thus, equality follows.

## 6. Canonical divisor

In this section we shall determine the canonical divisor of $\mathcal{Q}_{L}$ in terms of $\alpha$ and $\beta$.
Let $\omega_{C}$ denote the canonical divisor of $C$. Consider the open subset $\mathcal{Q}_{g} \subset \mathcal{Q}$ consisting of quotients $\left[\mathcal{O}_{C}^{\oplus r} \rightarrow F\right]$ for which $h^{1}(C, F)=0$. If $h^{1}(C, F)=0$, then applying $\operatorname{Hom}(-, F)$ to the short exact sequence $0 \rightarrow K \rightarrow \mathcal{O}_{C}^{\oplus r} \rightarrow F \rightarrow 0$, it follows that $\operatorname{ext}^{1}(K, F)=0$. Thus, $\mathcal{Q}_{g}$ is contained in the smooth locus of $\mathcal{Q}$. Recall that $\mathcal{Q}^{s}$ denoted the open subset consisting of quotients $\left[\mathcal{O}_{C}^{\oplus r} \rightarrow F\right]$ for which $F$ is stable. Clearly, $\mathcal{Q}^{s} \subset \mathcal{Q}_{g}$ as $d \gg 0$. Using Lemma 2.7 and equation (6.4) in [GS24], it follows that the morphism det : $\mathcal{Q}_{g} \rightarrow \operatorname{Pic}^{d}(C)$ is a smooth morphism. It follows that the locus $\mathcal{Q}_{g, L}=\mathcal{Q}_{L} \cap \mathcal{Q}_{g}$ is is contained in the smooth locus of $\mathcal{Q}_{L}$. As $\operatorname{codim}\left(\mathcal{Q}_{L} \backslash \mathcal{Q}_{L}^{s}, \mathcal{Q}_{L}\right) \geqslant 2$, it follows that $\operatorname{codim}\left(\mathcal{Q}_{L} \backslash \mathcal{Q}_{g, L}, \mathcal{Q}_{L}\right) \geqslant 2$. Thus, to determine the canonical divisor of $\mathcal{Q}_{L}$, it suffices to determine the canonical divisor of $\mathcal{Q}_{g, L}$. As the morphism det is smooth on $\mathcal{Q}_{g}$, and the canonical divisor of $\operatorname{Pic}^{d}(C)$ is trivial, it follows easily from the exact sequence (det being the morphism in (2.2))

$$
0 \rightarrow \operatorname{det}^{*} \Omega_{\operatorname{Pic}^{d}(C)} \rightarrow \Omega_{\mathcal{Q}_{g}} \rightarrow \Omega_{\operatorname{det}} \rightarrow 0
$$

that

$$
\begin{equation*}
\operatorname{det}\left(\Omega_{\mathcal{Q}_{g, L}}\right)=\operatorname{det}\left(\left.\Omega_{\operatorname{det}}\right|_{\mathcal{Q}_{g, L}}\right)=\left.\operatorname{det}\left(\Omega_{\mathcal{Q}_{g}}\right)\right|_{\mathcal{Q}_{g, L}} . \tag{6.1}
\end{equation*}
$$

Recall the universal sequence (3.1) on $C \times \mathcal{Q}$. Using the same method as in [Str87, Theorem 7.1], we may show that the tangent bundle on $\mathcal{Q}_{g}$ equals $p_{2 *}\left(\mathcal{K}^{\vee} \otimes \mathcal{F}\right)$. It easily follows that

$$
\begin{equation*}
\left.\Omega_{\mathcal{Q}_{g}}\right|_{\mathcal{Q}_{g, L}}=\left(p_{2 *}\left(\mathcal{K}^{\vee} \otimes \mathcal{F}\right)\right)^{\vee}, \tag{6.2}
\end{equation*}
$$

where we use the same notation to denote the restriction of the sheaves $\mathcal{K}, \mathcal{F}$ to $C \times \mathcal{Q}_{g, L}$. Thus, the canonical divisor of $\mathcal{Q}_{g, L}$ equals the determinant of the locally free sheaf ( $p_{2 *}\left(\mathcal{K}^{\vee} \otimes\right.$ $\mathcal{F}))^{\vee}$.

To compute the canonical divisor in terms of $\alpha$ and $\beta$, we need two curves in $\mathcal{Q}_{g, L}$. One of these is the curve $D_{1}$, given by the family (4.5). Let us check that the image of $D_{1}$ is contained in $\mathcal{Q}_{g, L}$. Recall from [PR03, Corollary 6.3] that when $d \gg 0$, and $K^{\prime}$ is a general stable bundle, then the cokernel of the general inclusion $K^{\prime} \rightarrow \mathcal{O}_{C}^{\oplus r}$ is a stable bundle. In particular, we may assume that both $K^{\prime}$ and $F^{\prime}$ in (4.5) are stable bundles. For each $t \in D_{1}$, the quotient $\mathcal{G}_{1, t} \cong F^{\prime} \oplus \mathbb{C}_{x}$ and so

$$
h^{1}\left(C, \mathcal{G}_{1, t}\right)=0 .
$$

It follows that the image of $D_{1}$ is contained in $\mathcal{Q}_{g, L}$.
Our second curve is the curve $D_{3}$ given by the family (5.5). Recall the space $\mathcal{Q}_{L}^{\prime}$ from (5.1). We had seen that the image of $D_{3} \rightarrow \mathcal{Q}_{L}$ is contained in $\mathcal{Q}_{L}^{\prime}$. Also note that $\mathcal{Q}_{L}^{\prime} \subset \mathcal{Q}_{g, L}$. Thus, the curves $D_{1}$ and $D_{3}$ are contained in $\mathcal{Q}_{g, L}$. Next we will compute the degree of the line bundle $\operatorname{det}\left(p_{2 *}\left(\mathcal{K}^{\vee} \otimes \mathcal{F}\right)\right)$ restricted to $D_{1}$ and $D_{3}$.
Lemma 6.3. The degree of $\operatorname{det}\left(p_{2 *}\left(\mathcal{K}^{\vee} \otimes \mathcal{F}\right)\right)$ restricted to $D_{1}$ is $r-2 k$.

Proof. As $\operatorname{ext}^{1}(K, F)=0$ for a point $\left[K \subset \mathcal{O}_{C}^{\oplus r} \rightarrow F\right] \in \mathcal{Q}_{g}$, it follows easily that (see (4.5))

$$
\left.p_{2 *}\left(\mathcal{K}^{\vee} \otimes \mathcal{F}\right)\right|_{D_{1}}=p_{2 *}\left(\tilde{K}_{1}^{\vee} \otimes \mathcal{G}_{1}\right)
$$

From (4.5) it follows that we have the following short exact sequence on $C \times D_{1}$

$$
0 \rightarrow \iota_{x *}\left(\mathcal{O}_{D_{1}}(1)\right) \rightarrow \mathcal{G}_{1} \rightarrow p_{C}^{*} F^{\prime} \rightarrow 0
$$

For ease of notation, let $\mathcal{T}$ denote the sheaf $\iota_{x *}\left(\mathcal{O}_{D_{1}}(1)\right)$. Applying $p_{2 *}\left(\tilde{K}_{1}^{\vee} \otimes-\right)$ we get the short exact sequence

$$
\begin{equation*}
0 \rightarrow p_{2 *}\left(\tilde{K}_{1}^{\vee} \otimes \mathcal{T}\right) \rightarrow p_{2 *}\left(\tilde{K}_{1}^{\vee} \otimes \mathcal{G}_{1}\right) \rightarrow p_{2 *}\left(\tilde{K}_{1}^{\vee} \otimes p_{C}^{*} F^{\prime}\right) \rightarrow 0 \tag{6.4}
\end{equation*}
$$

Let us first compute determinant of the sheaf $p_{2 *}\left(\tilde{K}_{1}^{\vee} \otimes \mathcal{T}\right)$. Apply $\mathscr{H}$ om $(-, \mathcal{T})$ to the short exact sequence $0 \rightarrow \tilde{K}_{1} \rightarrow p_{C}^{*} K^{\prime} \rightarrow \mathcal{T} \rightarrow 0$ (see (4.5)) yields the long exact sequence

$$
\begin{equation*}
0 \rightarrow \mathscr{H} \operatorname{om}(\mathcal{T}, \mathcal{T}) \rightarrow \mathcal{T}^{\oplus(r-k)} \rightarrow \tilde{K}_{1}^{\vee} \otimes \mathcal{T} \rightarrow \mathscr{E} x t^{1}(\mathcal{T}, \mathcal{T}) \rightarrow 0 \tag{6.5}
\end{equation*}
$$

Applying $\mathscr{H} \operatorname{om}(-, \mathcal{T})$ to the short exact sequence

$$
\begin{equation*}
0 \rightarrow p_{C}^{*} \mathcal{O}_{C}(-x) \otimes p_{2}^{*} \mathcal{O}_{D_{1}}(1) \rightarrow p_{2}^{*} \mathcal{O}_{D_{1}}(1) \rightarrow \mathcal{T} \rightarrow 0 \tag{6.6}
\end{equation*}
$$

one easily checks that the sheaves $\mathscr{H} \operatorname{om}(\mathcal{T}, \mathcal{T})$ and $\mathscr{E} x t^{1}(\mathcal{T}, \mathcal{T})$ are isomorphic to $\iota_{x *}\left(\mathcal{O}_{D_{1}}\right)$. As all the sheaves in (6.5) are coherent over $D_{1}$, applying $p_{2 *}$ we get the following exact sequence of sheaves on $D_{1}$

$$
0 \rightarrow \mathcal{O}_{D_{1}} \rightarrow \mathcal{T}^{\oplus(r-k)} \rightarrow p_{2 *}\left(\tilde{K}_{1}^{\vee} \otimes \mathcal{T}\right) \rightarrow \mathcal{O}_{D_{1}} \rightarrow 0
$$

From this it follows that

$$
\begin{equation*}
\operatorname{det}\left(p_{2 *}\left(\tilde{K}_{1}^{\vee} \otimes \mathcal{T}\right)\right) \cong \mathcal{O}_{D_{1}}(r-k) \tag{6.7}
\end{equation*}
$$

Next let us compute the determinant of the sheaf $p_{2 *}\left(\tilde{K}_{1}^{\vee} \otimes p_{C}^{*} F^{\prime}\right)$. For this we apply $\mathscr{H}$ om $\left(-, p_{C}^{*} F^{\prime}\right)$ to the short exact sequence

$$
0 \rightarrow \tilde{K}_{1} \rightarrow p_{C}^{*} K^{\prime} \rightarrow \mathcal{T} \rightarrow 0
$$

We get the following long exact sequence on $C \times D_{1}$

$$
\begin{equation*}
0 \rightarrow p_{C}^{*} K^{\prime \vee} \otimes p_{C}^{*} F^{\prime} \rightarrow \tilde{K}_{1}^{\vee} \otimes p_{C}^{*} F^{\prime} \rightarrow \mathscr{E} x t^{1}\left(\mathcal{T}, p_{C}^{*} F^{\prime}\right) \rightarrow 0 \tag{6.8}
\end{equation*}
$$

The last term equals

$$
\begin{aligned}
\mathscr{E} x t^{1}\left(\mathcal{T}, p_{C}^{*} F^{\prime}\right) & \cong \mathscr{E} x t^{1}\left(\mathcal{T} \otimes p_{C}^{*} F^{\prime}, \mathcal{O}_{C \times D_{1}}\right) \\
& \cong \mathscr{E} x t^{1}\left(\mathcal{T}, \mathcal{O}_{C \times D_{1}}\right)^{\oplus k}
\end{aligned}
$$

Applying $\mathscr{H} \operatorname{om}\left(-, \mathcal{O}_{C \times D_{1}}\right)$ to (6.6) one easily sees that $\mathscr{E} x t^{1}\left(\mathcal{T}, \mathcal{O}_{C \times D_{1}}\right) \cong \iota_{x *}\left(\mathcal{O}_{D_{1}}(-1)\right)$. Note that $h^{1}\left(K^{\prime N} \otimes F^{\prime}\right)=0$ as both $K^{\prime}$ and $F^{\prime}$ are stable. Applying $p_{2 *}$ to (6.8), we get the following exact sequence

$$
0 \rightarrow \operatorname{Hom}\left(K^{\prime}, F^{\prime}\right) \rightarrow p_{2 *}\left(\tilde{K}_{1}^{\vee} \otimes p_{C}^{*} F^{\prime}\right) \rightarrow \mathcal{O}_{D_{1}}(-1)^{\oplus k} \rightarrow 0
$$

It follows that $\operatorname{det}\left(p_{2 *}\left(\tilde{K}_{1}^{\vee} \otimes p_{C}^{*} F^{\prime}\right)\right) \cong \mathcal{O}_{D_{1}}(-k)$. Using this and equation (6.7) in (6.4), we $\operatorname{get} \operatorname{det}\left(p_{2 *}\left(\tilde{K}_{1}^{V} \otimes \mathcal{G}_{1}\right)\right)=\mathcal{O}_{D_{1}}(r-2 k)$.
Lemma 6.9. The degree of $\operatorname{det}\left(p_{2 *}\left(\mathcal{K}^{\vee} \otimes \mathcal{F}\right)\right)$ restricted to $D_{3}$ is $r(d+(r-k)(1-g))$.

Proof. Recall from (5.5) the family parameterized by $D_{3}$. It follows that

$$
\left.p_{2 *}\left(\mathcal{K}^{\vee} \otimes \mathcal{F}\right)\right|_{D_{3}}=p_{2 *}\left(p_{C}^{*} E^{\vee} \otimes \mathcal{O}_{D_{3}}(1) \otimes \mathcal{G}_{3}\right)
$$

Note that as $E^{\vee}$ is a stable bundle of degree $d \gg 0$, we have $H^{1}\left(C, E^{\vee}\right)=0$. Tensoring (5.5) with $p_{C}^{*} E^{\vee} \otimes p_{2}^{*} \mathcal{O}_{D_{3}}(1)$ and applying $p_{2 *}$ yields the long exact sequence

$$
\begin{aligned}
& 0 \rightarrow \operatorname{Hom}(E, E) \otimes \mathcal{O}_{D_{3}} \rightarrow\left[H^{0}\left(C, E^{\vee}\right) \otimes \mathcal{O}_{D_{3}}(1)\right]^{\oplus r} \rightarrow \\
& p_{2 *}\left(p_{C}^{*} E^{\vee} \otimes \mathcal{O}_{D_{3}}(1) \otimes \mathcal{G}_{3}\right) \rightarrow \operatorname{Ext}^{1}(E, E) \otimes \mathcal{O}_{D_{3}} \rightarrow 0 .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\operatorname{det}\left(p_{2 *}\left(p_{C}^{*} E^{\vee} \otimes \mathcal{O}_{D_{3}}(1) \otimes \mathcal{G}_{3}\right)\right) & \cong \mathcal{O}_{D_{3}}\left(r h^{0}\left(C, E^{\vee}\right)\right) \\
& =\mathcal{O}_{D_{3}}\left(r \chi\left(E^{\vee}\right)\right) \\
& =\mathcal{O}_{D_{3}}(r(d+(r-k)(1-g)))
\end{aligned}
$$

This completes the proof of the Lemma.
Theorem 6.10. Assume one of the following two holds:

- $g(C) \geqslant 3$ and $2 \leqslant k \leqslant r-2$, or
- $g(C)=2$ and $3 \leqslant k \leqslant r-2$.

Let $d \gg 0$. Let $\omega_{\mathcal{Q}_{L}}$ denote the canonical divisor of $\mathcal{Q}_{L}$. Then

$$
\omega_{\mathcal{Q}_{L}}=[d(r-2 k-2)+r(g-1)] \alpha+(2 k-r) \beta_{d+g-1} .
$$

In particular, $\mathcal{Q}_{L}$ is Fano iff $r=2 k+1$.
Proof. Let us write $\omega_{\mathcal{Q}_{L}}=a \alpha+b \beta_{d+g-1}$. Recall $\omega_{\mathcal{Q}_{L}}=\operatorname{det}\left(p_{2 *}\left(\mathcal{K}^{\vee} \otimes \mathcal{F}\right)\right)^{\vee}$. It follows from Lemma 6.3 and Lemma 6.9 that

$$
\begin{aligned}
& {\left[\omega_{\mathcal{Q}_{L}}\right] \cdot\left[D_{1}\right]=2 k-r} \\
& {\left[\omega_{\mathcal{Q}_{L}}\right] \cdot\left[D_{3}\right]=-r(d+(r-k)(1-g))}
\end{aligned}
$$

The proof of Proposition 4.4 shows that $[\alpha] \cdot\left[D_{1}\right]=0$. Using Lemma 4.12 and Lemma 5.6, we get the following two equations in $a$ and $b$

$$
\begin{aligned}
b & =2 k-r \\
a(r-k)+b d(r-k-1) & =-r(d+(r-k)(1-g))
\end{aligned}
$$

One easily computes that $a=d(r-2 k-2)+r(g-1)$. Thus,

$$
\omega_{\mathcal{Q}_{L}}=[d(r-2 k-2)+r(g-1)] \alpha+(2 k-r) \beta_{d+g-1} .
$$

For $\mathcal{Q}_{L}$ to be Fano, we need that $d(r-2 k-2)+r(g-1)<0$ and $2 k-r<0$. Since $d \gg 0$, this happens iff $r-2 k-2<0$ and $2 k-r<0$, that is, iff $2 k<r<2 k+2$, that is, iff $r=2 k+1$.

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