

Existence and dynamical behaviour of vectorial standing waves with prescribed mass for Hartree-Fock type systems

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Abstract

In this paper, we investigate vectorial standing waves with prescribed mass for the Hartree-Fock type system (HF system) with the double coupled feature. Such system is viewed as an approximation of the Coulomb system with two particles appeared in quantum mechanics. By exploring the interaction of the double coupled terms, we prove the existence/nonexistence and symmetry of vectorial energy ground states for the corresponding stationary problem. Furthermore, we obtain the relation between vectorial energy ground states and vectorial action ground states in some cases. Finally, we establish conditions for global well-posedness and finite time blow-up to HF system with the initial data, and prove orbital stability/strong instability of standing waves.

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1 Introduction

Our starting point is the system of Hartree-Fock equations:

$$i\partial_t \psi_k + \Delta \psi_k - \left(|x|^{-1} * \sum_{j=1}^N |\psi_j|^2 \right) \psi_k + (V_{\text{ex}} \psi)_k = 0, \quad \forall k = 1, \dots, N, \quad (1)$$

where $\psi_k : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$ and $(V_{\text{ex}} \psi)_k$ is the k 'th component of *the exchange potential* defined by

$$(V_{\text{ex}} \psi)_k = \sum_{j=1}^N \psi_j(y) \int_{\mathbb{R}^3} \frac{\psi_k(y) \bar{\psi}_j(y)}{|x-y|} dy, \quad \forall k = 1, \dots, N. \quad (2)$$

This system appeared in quantum mechanics in the study of a system of N particles, which is an approximation of the Coulomb system. With respect to system (1), it has the advantage of being consistent with the Pauli principle, see e.g. [13, 34].

We note that the exchange potential (2) is the most difficult term to be treated in system (1). A very simple approximation of this potential was given by Slater [35] in the form:

$$(V_{\text{ex}} \psi)_k \approx C \left(\sum_{j=1}^N |\psi_j|^2 \right)^{1/3} \psi_k, \quad \forall k = 1, 2, \dots, N, \quad (3)$$

where $C > 0$. For more different local approximations, we refer to [21, 28] and references therein.

For the single-partical states, i.e. $N = 1$, the exchange potential (3) becomes $(V_{\text{ex}} \psi)_1 = C |\psi_1|^{2/3} \psi_1$, and then system (1) is the following Schrödinger–Poisson–Slater equation:

$$i\partial_t \psi + \Delta \psi - (|x|^{-1} * |\psi|^2) \psi + |\psi|^{2/3} \psi = 0 \quad \text{in } \mathbb{R}^3, \quad (4)$$

where we rename ψ_1 as ψ , and take $C = 1$ for simplicity. Sánchez and Soler [32] analyzed the asymptotic behaviour of solutions to equation (4). If the term $|\psi|^{2/3} \psi$ is replaced with $|\psi|^{p-2} \psi$ ($2 < p < 6$) (or, more generally, $f(\psi)$), then equation (4) becomes Schrödinger–Poisson equation (also called Schrödinger–Maxwell equation), where the existence, stability/instability and dynamics of standing waves $\psi(t, x) = e^{i\lambda t} u(x)$ has been extensively studied in recent years, see e.g. [3, 6, 7, 12, 29, 30, 33, 38, 39].

For the two-particle states, i.e. $N = 2$, system (1) becomes much more complicated. In order to further simplify it, we assume the exchange potential

$$V_{\text{ex}} \psi = C \left((|\psi_1|^{p-2} + \beta |\psi_1|^{\frac{p}{2}-2} |\psi_2|^{\frac{p}{2}}) \psi_1 \right), \quad (5)$$

where $C > 0, \beta \in \mathbb{R}$ and $2 < p < 6$. Clearly, if $p = \frac{8}{3}$, then (5) is written as

$$V_{\text{ex}}\psi = C \begin{pmatrix} (|\psi_1|^{\frac{2}{3}} + \beta |\psi_1|^{-\frac{2}{3}} |\psi_2|^{\frac{4}{3}}) \psi_1 \\ (|\psi_2|^{\frac{2}{3}} + \beta |\psi_1|^{\frac{4}{3}} |\psi_2|^{-\frac{2}{3}}) \psi_2 \end{pmatrix},$$

which can be viewed as an approximation of the exchange potential (3). Taking, for simplicity, $C = 1$, system (1) becomes the following problem, depending on the parameter $\gamma > 0$:

$$\begin{cases} i\partial_t \psi_1 + \Delta \psi_1 - \gamma \left(|x|^{-1} * \sum_{j=1}^2 |\psi_j|^2 \right) \psi_1 + |\psi_1|^{p-2} \psi_1 + \beta |\psi_2|^{\frac{p}{2}} |\psi_1|^{\frac{p}{2}-2} \psi_1 = 0 & \text{in } \mathbb{R}^3, \\ i\partial_t \psi_2 + \Delta \psi_2 - \gamma \left(|x|^{-1} * \sum_{j=1}^2 |\psi_j|^2 \right) \psi_2 + |\psi_2|^{p-2} \psi_2 + \beta |\psi_1|^{\frac{p}{2}} |\psi_2|^{\frac{p}{2}-2} \psi_2 = 0 & \text{in } \mathbb{R}^3. \end{cases} \quad (6)$$

The typical characteristic of such system lies on the presence of the double coupled terms, including a Coulomb interacting one and a cooperative pure power one.

An important topic is to establish conditions for the well-posedness and blow-up of system (6) with the initial data. When $\gamma = 0$, there has been a number of works on local existence, global existence and scattering theory, see e.g. [11, 31, 41]. However, for $\gamma > 0$, there have no any results on these aspects in the existing literature so far. We observe that the presence of the double coupled terms yields substantial difficulties and requires more detailed dispersive estimates. One objective in this paper is to investigate the global well-posedness and finite time blow-up of system (6) with the initial data.

Another interesting topic on system (6) is to study the standing waves of the form

$$\psi_1(t, x) = e^{i\lambda t} u(x) \text{ and } \psi_2(t, x) = e^{i\lambda t} v(x),$$

where $\lambda \in \mathbb{R}$ and (u, v) is a vector function to be found. Thus (u, v) must verify

$$\begin{cases} -\Delta u + \lambda u + \gamma \phi_{u,v} u = |u|^{p-2} u + \beta |v|^{\frac{p}{2}} |u|^{\frac{p}{2}-2} u & \text{in } \mathbb{R}^3, \\ -\Delta v + \lambda v + \gamma \phi_{u,v} v = |v|^{p-2} v + \beta |u|^{\frac{p}{2}} |v|^{\frac{p}{2}-2} v & \text{in } \mathbb{R}^3, \end{cases} \quad (7)$$

where $\phi_{u,v} := \int_{\mathbb{R}^3} \frac{u^2(y) + v^2(y)}{|x-y|} dy$. For a solution (u, v) of system (7), we introduce some concepts of its triviality.

Definition 1.1 *A vector function (u, v) is said to be*

- (i) *nontrivial if either $u \neq 0$ or $v \neq 0$;*
- (ii) *semitrivial if it is nontrivial but either $u = 0$ or $v = 0$;*
- (iii) *vectorial if both of u and v are not zero;*

According to the role of the frequency λ , there are two different ways to study solutions of system (7):

- (i) the frequency λ is a fixed and assigned parameter;
- (ii) the frequency λ is an unknown of the problem.

For case (i), one can see that solutions of system (7) can be obtained as critical points of the functional defined in $\mathbf{H} := H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ by

$$\begin{aligned} E_{\lambda,\gamma,\beta}(u, v) : &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx + \frac{\lambda}{2} \int_{\mathbb{R}^N} (|u|^2 + |v|^2) dx + \frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_{u,v}(u^2 + v^2) dx \\ &\quad - \frac{1}{p} \int_{\mathbb{R}^3} (|u|^p + |v|^p + 2\beta |u|^{\frac{p}{2}} |v|^{\frac{p}{2}}) dx. \end{aligned}$$

When either $3 < p < 4$ and $\beta > 0$, or $4 \leq p < 6$ and $\beta \geq 2^{2q-1} - 1$, d'Avenia, Maia and Siciliano [9] studied the existence of radial vectorial solutions to system (7) with $\lambda = 1$ via the method of Nehari-Pohozaev manifold defined in $\mathbf{H}_r := H_{rad}^1(\mathbb{R}^3) \times H_{rad}^1(\mathbb{R}^3)$ developed by Ruiz [30]. Very recently, we [37] considered another interesting case, i.e. $2 < p < 3$, and proved the existence of radial vectorial solutions for system (7) with $\lambda = 1$ when either $\gamma > 0$ small enough or $\beta > \beta(\gamma)$. In addition, by developing a novel constraint method, for $\beta > \beta(\gamma)$, we obtained a vectorial *action ground state* in $\mathbf{H}$ when either $3 \leq p < 4$ and $0 < \gamma < \gamma_0$, or $\frac{1+\sqrt{73}}{3} \leq p < 4$ and $\gamma > 0$. Here the action ground state is defined in the following sense:

Definition 1.2 *We say that (u, v) is an action ground state of system (7) if it is a solution of system (7) having least energy among all the solutions:*

$$E_{\lambda,\gamma,\beta}(u, v) = \inf \{ E_{\lambda,\gamma,\beta}(\phi_1, \phi_2) : E'_{\lambda,\gamma,\beta}(\phi_1, \phi_2) = 0 \text{ and } (\phi_1, \phi_2) \in \mathbf{H} \setminus \{(0, 0)\} \}.$$

Alternatively one can look for solutions of system (7) with λ unknown. In this case $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier and L^2 -norms of solutions are prescribed. This study seems to be particularly meaningful from the physical point of view, since solutions of system (6) with the initial data conserve their mass along time. Moreover, this study often offers a good insight of the dynamical properties of solutions for system (6), such as stability or instability. For these reasons, another objective in this paper is to focus on existence and dynamics of solutions for system (7) having prescribed mass

$$\int_{\mathbb{R}^3} (u^2 + v^2) dx = c > 0, \tag{8}$$

which has not been concerned before. From the variational point of view, solutions of problem (7)-(8) can be obtained as critical points of the energy functional $I_{\gamma,\beta} : \mathbf{H} \rightarrow \mathbb{R}$ given by

$$\begin{aligned} I_{\gamma,\beta}(u, v) : &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx + \frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_{u,v}(u^2 + v^2) dx \\ &\quad - \frac{1}{p} \int_{\mathbb{R}^3} (|u|^p + |v|^p + 2\beta |u|^{\frac{p}{2}} |v|^{\frac{p}{2}}) dx \end{aligned}$$

on the constraint

$$S_c := \left\{ (u, v) \in \mathbf{H} : \int_{\mathbb{R}^3} (u^2 + v^2) dx = c \right\}.$$

As $2 < p < 6$, it is standard that $I_{\gamma,\beta}$ is of class C^1 on S_c . We will be particularly interested in *energy ground states*, defined as follows:

Definition 1.3 We say that (u, v) is an energy ground state of system (7) if it is a solution of system (7) having minimal energy among all the solutions:

$$I'_{\gamma,\beta}|_{S_c}(u, v) = 0 \text{ and } I_{\gamma,\beta}(u, v) = \inf\{I_{\gamma,\beta}(\phi_1, \phi_2) : I'_{\gamma,\beta}|_{S(c)}(\phi_1, \phi_2) = 0 \text{ and } (\phi_1, \phi_2) \in S_c\}.$$

If $I_{\gamma,\beta}$ has a global minimizer, then this definition naturally extends the notion of ground states from linear quantum mechanics. Moreover, it allows to deal with cases when $I_{\gamma,\beta}$ is unbounded from below on S_c . We also recall the notion of stability and instability as follows:

Definition 1.4 (i) The set of the ground states $\mathcal{Z}(c)$ is orbitally stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for the initial data $(\psi_1(0), \psi_2(0)) \in \mathbf{H}$ satisfying $\inf_{(u,v) \in \mathcal{Z}(c)} \|(\psi_1(0), \psi_2(0)) - (u, v)\|_{\mathbf{H}} \leq \delta$, we have

$$\sup_{t \in (0, T_{\max})} \inf_{(u,v) \in \mathcal{Z}(c)} \|(\psi_1(t), \psi_2(t)) - (u, v)\|_{\mathbf{H}} \leq \varepsilon,$$

where $(\psi_1(t), \psi_2(t))$ is the solution to system (6) with the initial data $(\psi_1(0), \psi_2(0))$ for $t \in [0, T_{\max})$, and $T_{\max}$ denotes the maximum existence time of solution.

(ii) A standing wave $(e^{i\lambda t}u, e^{i\lambda t}v)$ is strongly unstable if for any $\varepsilon > 0$, there exists $(\psi_1(0), \psi_2(0)) \in \mathbf{H}$ such that $\|(\psi_1(0), \psi_2(0)) - (u, v)\|_{\mathbf{H}} < \varepsilon$ and the solution $(\psi_1(t), \psi_2(t))$ blows up in finite time.

We note that the definition of stability implicitly requires that system (6) has a unique global solution, at least for initial data $(\psi_1(0), \psi_2(0))$ sufficiently close to $\mathcal{Z}(c)$.

When $\gamma = 0$, system (7) is reduced to the local weakly coupled nonlinear Schrödinger system

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u + \beta|v|^{\frac{p}{2}}|u|^{\frac{p}{2}-2}u & \text{in } \mathbb{R}^3, \\ -\Delta v + \lambda v = |v|^{p-2}v + \beta|u|^{\frac{p}{2}}|v|^{\frac{p}{2}-2}v & \text{in } \mathbb{R}^3. \end{cases} \quad (9)$$

In recent years, the existence and multiplicity of solutions to system (9) with prescribed mass have attracted much attention, see e.g. [2, 8, 16, 17, 22, 26]. However, when $\gamma > 0$, due to the interaction of the double coupled terms, the situation becomes more different and complicated. Some key challenges need to be solved, such as the characterization of the geometric structure of the functional $I_{\gamma,\beta}$, and the relation between two coupled constants γ, β . In order to overcome these considerable difficulties, new ideas and techniques have been explored. More details will be discussed in the next subsection.

1.1 Main results

For sake of convenience, we set

$$A(u, v) := \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx, \quad B(u, v) := \int_{\mathbb{R}^3} \phi_{u,v}(u^2 + v^2) dx,$$

and

$$C(u, v) := \int_{\mathbb{R}^3} (|u|^p + |v|^p + 2\beta|u|^{\frac{p}{2}}|v|^{\frac{p}{2}}) dx.$$

First of all, we consider the minimization problem:

$$\sigma_{\gamma,\beta}(c) := \inf_{(u,v) \in S_c} I_{\gamma,\beta}(u, v).$$

Theorem 1.5 *Let $2 < p < 3$ and $c, \beta > 0$. Then there exists $\gamma_* := \gamma_*(c) > 0$ such that for $0 < \gamma < \gamma_*$, $\sigma_{\gamma, \beta}(c)$ admits a minimizer $(\hat{u}, \hat{v}) \in S_c$, namely, system (7) admits a vectorial energy ground state $(\hat{u}, \hat{v}) \in \mathbf{H}$. In particular, for $\frac{18}{7} < p < 3$, there exists $c_* > 0$ such that for every $c > c_*$ and $0 < \gamma < \gamma_*$, the vectorial energy ground state $(\hat{u}, \hat{v})$ is non-radial.*

For $3 \leq p < \frac{10}{3}$ and $c > 0$, we define

$$\begin{aligned} \beta_* &:= \beta_*(c, \gamma, p) \\ &= \inf_{(u, v) \in \mathbf{H} \setminus \{(0, 0)\}} \left[\frac{\frac{p}{3p-8} \left(\frac{\gamma(3p-8)}{2(10-3p)} \right)^{\frac{10-3p}{2}} c^{-2(p-3)} A(u, v)^{\frac{3p-8}{2}} B(u, v)^{\frac{10-3p}{2}} (\|u\|_2^2 + \|v\|_2^2)^{2(p-3)}}{2 \int_{\mathbb{R}^3} |u|^{\frac{p}{2}} |v|^{\frac{p}{2}} dx} \right. \\ &\quad \left. - \frac{\int_{\mathbb{R}^3} (|u|^p + |v|^p) dx}{2 \int_{\mathbb{R}^3} |u|^{\frac{p}{2}} |v|^{\frac{p}{2}} dx} \right]. \end{aligned}$$

By (32) below, β_* is well-defined. In particular, when $p = 3$, $\beta_* \geq \frac{3\sqrt{\gamma}}{2} - 1$, not depending on c .

Theorem 1.6 *Let $3 \leq p < \frac{10}{3}$ and $c, \gamma > 0$. Then we have*

- (i) **(existence)** $\sigma_{\gamma, \beta}(c)$ admits a minimizer $(\tilde{u}, \tilde{v}) \in S_c$ for $\beta > \max\{\beta_*, 0\}$. In addition, the corresponding Lagrange multiplier $\tilde{\lambda}$ is positive.
- (ii) **(non-existence)** If $\beta_* > 0$, then for $p = 3$ and $\beta < \frac{3\sqrt{\gamma}}{2} - 1$, or for $3 < p < \frac{10}{3}$ and $\beta < \beta_*$, $\sigma_{\gamma, \beta}(c)$ has no minimizer.

Remark 1.7 (a) *We give a characterisation of the quantity β_* in Theorem 1.6, which a threshold for the existence and non-existence of minimizer for $\sigma_{\gamma, \beta}(c)$.*

(b) *From Theorem 1.6, we can see that the parameter β can be any positive value if $\gamma > 0$ small. In particular, when $p = 3$, if $0 < \gamma \leq \frac{4}{9}$, then for any $\beta > 0$, the existence result still holds.*

For $2 < p < 10/3$, the main difficulty is due to the lack of compactness of the bounded minimizing sequences. Indeed, the minimizing sequence could run off to spatial infinity and/or spread uniformly in space. The usual strategy is to control the mass c to exclude both the vanishing and the dichotomy. However, in Theorems 1.5 and 1.6, we control the coupled constant either γ or β , not c , although there might be a connection between them. This leads us to have a more considered analysis in the proofs.

Due to Theorems 1.5 and 1.6, a natural question is to investigate the connection between the energy ground states and the action ground states. This correspondence has been established in [10, 19] for scalar equations and in [8] for systems. The next theorem establishes the relation between vectorial energy ground states and vectorial action ground states for system (7) when $p = 3$.

Theorem 1.8 *Let $p = 3$ and $\gamma, \lambda > 0$. Then for $\beta > \max\{\beta_*, 0\}$, system (7) has a vectorial action ground state $(u_\lambda, v_\lambda) \in \mathbf{H}$. Moreover, we have*

- (i) *There admits a unique $c = c(\lambda) > 0$ such that $\|u_\lambda\|_2^2 + \|v_\lambda\|_2^2 = c(\lambda)$.*

(ii) (u_λ, v_λ) is a vectorial action ground state of system (7) if and only if (u_λ, v_λ) is a minimizer of $\sigma_{\gamma,\beta}(c(\lambda))$. In particular, there holds

$$I_{\gamma,\beta}(u_\lambda, v_\lambda) = \sigma_{\gamma,\beta}(c(\lambda)) = \inf_{(u,v) \in S_{c(\lambda)}} I_{\gamma,\beta}(u, v).$$

Remark 1.9 Regarding on the study of vectorial action ground states to system (7), we note that the case of $p = 3$ has been considered in [37] but the coupled constant $\gamma > 0$ is required to be small. In our paper, according to Theorems 1.6 and 1.8, we can obtain a vectorial action ground state without restriction on $\gamma > 0$, which improves the result in [37].

For $\frac{10}{3} \leq p < 6$, $I_{\gamma,\beta}$ is no longer bounded from below on S_c . Then it will not be possible to find a global minimizer. In order to seek for critical points of $I_{\gamma,\beta}$ restricted to S_c , we will use the Pohozaev manifold $\mathcal{M}_{\gamma,\beta}(c)$ that contains all the critical points of $I_{\gamma,\beta}$ restricted to S_c . It is given by

$$\mathcal{M}_{\gamma,\beta}(c) := \{(u, v) \in S_c : \mathcal{P}_{\gamma,\beta}(u, v) = 0\},$$

where $\mathcal{P}_{\gamma,\beta}(u, v) = 0$ is the Pohozaev type identity corresponding to system (7). Consider the minimization problem:

$$m_{\gamma,\beta}(c) := \inf_{(u,v) \in \mathcal{M}_{\gamma,\beta}(c)} I_{\gamma,\beta}(u, v),$$

and set

$$c_* := \left(\frac{5}{3}\right)^{\frac{3}{2}} \inf_{(u,v) \in S_1} \mathcal{R}_{10/3}(u, v) \text{ and } c^* := \left(\frac{2(6-p)}{5p-12}\right)^{\frac{3p-10}{4(p-3)}} \left(\frac{3p}{5p-12}\right)^{\frac{1}{2(p-3)}} \inf_{(u,v) \in S_1} \mathcal{R}_p(u, v), \quad (10)$$

where

$$\mathcal{R}_p(u, v) := \left[\frac{A(u, v)^{3p-8} (\gamma B(u, v))^{10-3p}}{C(u, v)^2} \right]^{\frac{1}{4(p-3)}}.$$

Theorem 1.10 Let $p = \frac{10}{3}$ and $\gamma, \beta > 0$. Then for each $c_* < c < c^*$, system (7) has a vectorial energy ground state $(\bar{u}, \bar{v}) \in \mathbf{H}$. Moreover, the corresponding Lagrange multiplier $\bar{\lambda}$ is positive. While for each $0 < c < c_*$, there are no critical points of $I_{\gamma,\beta}$ restricted to S_c .

Theorem 1.11 Let $\frac{10}{3} < p < 4$ and $\gamma, \beta > 0$, or $4 \leq p < 6$ and $\gamma > 0, \beta > 2^{\frac{p-2}{2}}$. Then for each $0 < c < c^*$, system (7) has a vectorial energy ground state $(\bar{u}, \bar{v}) \in \mathbf{H}$. Moreover, there exists $c^* < c^*$ such that for each $0 < c < c^*$, the corresponding Lagrange multiplier $\bar{\lambda}$ is positive.

Proposition 1.12 Let $(\bar{u}, \bar{v}) \in \mathbf{H}$ be the energy ground state obtained in Theorem 1.10 or 1.11 with the positive Lagrange multiplier $\bar{\lambda}$, then $(\bar{u}, \bar{v})$ has exponential decay:

$$|\bar{u}(x)| \leq Ce^{-\zeta|x|} \text{ and } |\bar{v}(x)| \leq Ce^{-\zeta|x|} \text{ for every } x \in \mathbb{R}^3,$$

for some $C > 0, \zeta > 0$.

In our approach we do not work in the radial function space $\mathbf{H}_r$ and do not need to consider Palais-Smale sequences, so that we can avoid applying the mini-max approach based on a strong topological argument as in [2, 15, 16]. A novel strategy to recover the compactness is to take advantage of the profile decomposition of the minimizing sequence and the monotonicity of ground state energy map. It provides a unified novel approach to study both mass critical and supercritical cases, even for Schrödinger-Poisson systems.

Let us summarize the existence and non-existence results presented in our work in the following table 1.

Table 1:

p	parameters conditions	existence (yes)/non-existence (no)
$2 < p < 3$	$0 < \gamma < \gamma_*$	Yes
$3 \leq p < 10/3$	$\beta > \max\{\beta_*, 0\}$	Yes
	$\beta < \beta_*$ if $\beta_* > 0$	No
$p = 10/3$	$0 < c < c_*$	No
	$c_* < c < c^*$	Yes
$10/3 < p < 6$	$0 < c < c^*$	Yes

Next, we turn to study the dynamics of the corresponding standing waves for system (6) with the initial data.

Theorem 1.13 (Local well-posedness theory) *Let $2 < p < 6$ and the initial data $(\psi_1(0), \psi_2(0)) \in \mathbf{H}$. Then there exists $T_{\max} > 0$ such that system (6) with initial data $(\psi_1(0), \psi_2(0))$ has a unique solution $(\psi_1(t), \psi_2(t))$ with*

$$\psi_1(t), \psi_2(t) \in \mathcal{C}([0, T_{\max}); H^1(\mathbb{R}^3)).$$

Define the set of ground states $\mathcal{Z}(c)$ by

$$\mathcal{Z}(c) := \{(u, v) \in S_c : I_{\gamma, \beta}(u, v) = \sigma_{\gamma, \beta}(c)\}.$$

By Theorems 1.5 and 1.6, $\mathcal{Z}(c)$ is not empty. With the help of Theorem 1.13, the following stability holds.

Theorem 1.14 *Under the assumptions of Theorem 1.5 or 1.6, the set $\mathcal{Z}(c)$ is orbitally stable.*

Theorem 1.15 (Global well-posedness theory) *Let the initial data $(\psi_1(0), \psi_2(0)) \in \mathbf{H}$. Then system (6) with the initial data $(\psi_1(0), \psi_2(0))$ is globally well-posed in $\mathbf{H}$ if any one of the following conditions is satisfied:*

- (i) $2 < p < \frac{10}{3}$.
- (ii) $p = \frac{10}{3}$ and $\|\psi_1(0)\|_2^2 + \|\psi_2(0)\|_2^2$ is small.
- (iii) $\frac{10}{3} < p < 6$, $\mathcal{P}_{\gamma, \beta}(\psi_1(0), \psi_2(0)) > 0$ and $I_{\gamma, \beta}(\psi_1(0), \psi_2(0)) < m_{\gamma, \beta}(c)$.

Theorem 1.16 (Finite time blow-up) *Under the assumptions of Theorem 1.10 or 1.11. Let $(\psi_1(t), \psi_2(t))$ be the solution to system (6) with the initial data $(\psi_1(0), \psi_2(0))$. If*

$$(|x|\psi_1(0), |x|\psi_2(0)) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3), \mathcal{P}_{\gamma, \beta}(\psi_1(0), \psi_2(0)) < 0 \text{ and } I_{\gamma, \beta}(\psi_1(0), \psi_2(0)) < m_{\gamma, \beta}(c),$$

then $(\psi_1(t), \psi_2(t))$ blows up in finite time.

Finally, we prove that the associated standing waves are unstable.

Theorem 1.17 *Let $(\bar{u}, \bar{v}) \in \mathbf{H}$ be the energy ground state obtained in Theorem 1.10 or 1.11, then the associated standing wave $(e^{i\bar{\lambda}t}\bar{u}, e^{i\bar{\lambda}t}\bar{v})$ is strongly unstable, where $\bar{\lambda}$ is the Lagrange multiplier.*

The rest of this paper is organized as follows. After introducing some preliminary results in Section 2, we prove Theorems 1.5–1.6 in Section 3 and Theorem 1.8 in Section 4, respectively. We prove Theorems 1.10–1.11 and Proposition 1.12 in Section 5. Finally in Section 6, we study dynamics of the corresponding standing waves and prove Theorems 1.13–1.17.

2 Preliminary results

In this section, we give some related inequalities and some known results.

Lemma 2.1 (Gagliardo-Nirenberg inequality of Power type [40]) *Let $2 < p < 6$. Then there exists a sharp constant $\mathcal{S}_p > 0$ such that*

$$\|u\|_p \leq \mathcal{S}_p^{1/p} \|\nabla u\|_2^{\frac{3(p-2)}{2p}} \|u\|_2^{\frac{6-p}{2p}} \text{ for all } u \in H^1(\mathbb{R}^3), \quad (11)$$

where $\mathcal{S}_p = \frac{p}{2\|U_p\|_2^{p-2}}$, and U_p is the ground state solution of the following equation

$$-\Delta u + \frac{6-p}{3(p-2)}u = \frac{4}{3(p-2)}|u|^{p-2}u \text{ in } \mathbb{R}^3.$$

Lemma 2.2 (Gagliardo-Nirenberg inequality of Hartree type [27, 42]) *There exists the best constant $\mathcal{B} > 0$ such that*

$$\int_{\mathbb{R}^3} (|x|^{-1} * |u|^2)|u|^2 dx \leq \mathcal{B} \|\nabla u\|_2 \|u\|_2^3 \text{ for all } u \in H^1(\mathbb{R}^3), \quad (12)$$

where $\mathcal{B} = \frac{2}{\|Q\|_2^2}$ and $Q(x)$ is a positive ground state solution of the following equation

$$-\Delta u + 3u = 2(|x|^{-1} * |u|^2)u \text{ in } \mathbb{R}^3.$$

By using inequality (11), we have

$$\begin{aligned} \int_{\mathbb{R}^3} (|u|^p + |v|^p) dx &\leq \mathcal{S}_p \left(\|u\|_2^{\frac{6-p}{2}} \|\nabla u\|_2^{\frac{3(p-2)}{2}} + \|v\|_2^{\frac{6-p}{2}} \|\nabla v\|_2^{\frac{3(p-2)}{2}} \right) \\ &\leq \mathcal{S}_p \left(\|u\|_2^{\frac{6-p}{2}} + \|v\|_2^{\frac{6-p}{2}} \right) \left(\|\nabla u\|_2^{\frac{3(p-2)}{2}} + \|\nabla v\|_2^{\frac{3(p-2)}{2}} \right) \\ &\leq 4\mathcal{S}_p (\|u\|_2^2 + \|v\|_2^2)^{\frac{6-p}{4}} A(u, v)^{\frac{3(p-2)}{4}}, \end{aligned}$$

and then

$$\begin{aligned} C(u, v) &\leq (1 + \beta) \int_{\mathbb{R}^3} (|u|^p + |v|^p) dx \\ &\leq 4\mathcal{S}_p (1 + \beta) (\|u\|_2^2 + \|v\|_2^2)^{\frac{6-p}{4}} A(u, v)^{\frac{3(p-2)}{4}}. \end{aligned} \quad (13)$$

Lemma 2.3 (Hardy-Littlewood-Sobolev inequality [23]) Assume that $1 < a < b < +\infty$ is such that $\frac{1}{a} + \frac{1}{b} = \frac{5}{3}$. Then there exists $\mathcal{C}_{HLS} > 0$ such that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|f(x)||g(y)|}{|x-y|} \leq \mathcal{C}_{HLS} \|f\|_a \|g\|_b, \quad \forall f \in L^a(\mathbb{R}^3), g \in L^b(\mathbb{R}^3).$$

Lemma 2.4 Let $\beta > 0$ and $g_\beta(s) := s^{\frac{p}{2}} + (1-s)^{\frac{p}{2}} + 2\beta s^{\frac{p}{4}}(1-s)^{\frac{p}{4}}$ for $s \in [0, 1]$.

(i) If $2 < p < 4$, then there exists $s_\beta \in (0, 1)$ such that $g_\beta(s_\beta) = \max_{s \in [0, 1]} g_\beta(s) > 1$. In particular, if $\beta \geq \frac{p-2}{2}$, then $s_\beta = \frac{1}{2}$.

(ii) If $4 \leq p < 6$ and $\beta > 2^{\frac{p-2}{2}}$, then there exists $s_\beta = \frac{1}{2}$ such that $g_\beta(\frac{1}{2}) = \max_{s \in [0, 1]} g_\beta(s) > 1$.

Proof. The proof is similar to the argument in [9, Lemma 2.4], and we omit it here. ■

Lemma 2.5 Let $2 < p < 4$ and $\beta > 0$, or $4 \leq p < 6$ and $\beta > 2^{\frac{p-2}{2}}$. Then for each $\omega \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exists $s_\omega \in (0, 1)$ such that

$$I_{\gamma, \beta}(\sqrt{s_\omega}\omega, \sqrt{1-s_\omega}\omega) < I_{\gamma, \beta}(\omega, 0) = I_{\gamma, \beta}(0, \omega).$$

Proof. Let $(u, v) = (\sqrt{s_\omega}\omega, \sqrt{1-s_\omega}\omega)$ for $\omega \in H^1(\mathbb{R}^3) \setminus \{0\}$ and $s \in [0, 1]$. A direct calculation shows that

$$A(u, v) = s\|\nabla\omega\|_2^2 + (1-s)\|\nabla\omega\|_2^2 = \|\nabla\omega\|_2^2$$

and

$$B(u, v) = \int_{\mathbb{R}^3} \phi_{\sqrt{s_\omega}, \sqrt{1-s_\omega}}(s\omega^2 + (1-s)\omega^2)dx = \int_{\mathbb{R}^3} \phi_\omega \omega^2 dx.$$

Moreover, by Lemma 2.4, there exists $s_\omega \in (0, 1)$ such that

$$C(u, v) = \left[s_\omega^{\frac{p}{2}} + (1-s_\omega)^{\frac{p}{2}} + 2\beta s_\omega^{\frac{p}{4}}(1-s_\omega)^{\frac{p}{4}} \right] \int_{\mathbb{R}^3} |\omega|^p dx > \int_{\mathbb{R}^3} |\omega|^p dx.$$

Then we have

$$\begin{aligned} & I_{\gamma, \beta}(\sqrt{s_\omega}\omega, \sqrt{1-s_\omega}\omega) \\ &= \frac{1}{2}\|\nabla\omega\|_2^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_\omega \omega^2 dx - \frac{1}{p} \left[s_\omega^{\frac{p}{2}} + (1-s_\omega)^{\frac{p}{2}} + 2\beta s_\omega^{\frac{p}{4}}(1-s_\omega)^{\frac{p}{4}} \right] \int_{\mathbb{R}^3} |\omega|^p dx \\ &< \frac{1}{2}\|\nabla\omega\|_2^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_\omega \omega^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} |\omega|^p dx \\ &= I_{\gamma, \beta}(\omega, 0) = I_{\gamma, \beta}(0, \omega). \end{aligned}$$

The proof is complete. ■

Next, we show Brezis-Lieb type results for two coupled terms in the energy functional.

Lemma 2.6 (Brezis-Lieb Lemma) If $(u_n, v_n) \rightharpoonup (u, v)$ in $\mathbf{H}$, then

$$B(u_n, v_n) = B(u, v) + B(u_n - u, v_n - v) + o_n(1)$$

and

$$\int_{\mathbb{R}^3} |u_n|^{\frac{p}{2}} |v_n|^{\frac{p}{2}} dx = \int_{\mathbb{R}^3} |u|^{\frac{p}{2}} |v|^{\frac{p}{2}} dx + \int_{\mathbb{R}^3} |u_n - u|^{\frac{p}{2}} |v_n - v|^{\frac{p}{2}} dx + o_n(1),$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. We observe that

$$B(u, v) = \int_{\mathbb{R}^3} \phi_{u,v}(|u|^2 + |v|^2) dx = \int_{\mathbb{R}^3} \phi_z |z|^2 dx,$$

where $z := u + iv$ and i is the imaginary unit. The first equality can be proved by adopting the method in [27, Lemma 2.4]. The second one is a direct consequence of [4, Theorem 2] by taking $j : \mathbb{C} \rightarrow \mathbb{C}$ defined by $j(u + iv) := |u|^{\frac{p}{2}} |v|^{\frac{p}{2}}$ in that theorem. The proof is complete. ■

Finally, we give the following Pohozaev identity, and we refer to [9] for the proof.

Lemma 2.7 *Let (u, v) be a weak solution to system (7). Then it satisfies the Pohozaev identity*

$$\frac{1}{2}A(u, v) + \frac{3\lambda}{2} \int_{\mathbb{R}^N} (u^2 + v^2) dx + \frac{5\gamma}{4}B(u, v) = \frac{3}{p}C(u, v).$$

Furthermore, it holds

$$\mathcal{P}_{\gamma,\beta}(u, v) := A(u, v) + \frac{\gamma}{4}B(u, v) - \frac{3(p-2)}{2p}C(u, v) = 0. \quad (14)$$

3 The existence of global minimizer

Lemma 3.1 *Let $2 < p < \frac{10}{3}$. Then the following statements are true.*

- (i) $I_{\gamma,\beta}$ is bounded from below and coercive on S_c ;
- (ii) $\sigma_{\gamma,\beta}(c)$ satisfies the weak sub-additive inequality:

$$\sigma_{\gamma,\beta}(c) \leq \sigma_{\gamma,\beta}(c') + \sigma_{\gamma,\beta}(c - c') \quad (15)$$

for any $0 < c' < c$.

Proof. (i) For $(u, v) \in S_c$, by (13), one has

$$I_{\gamma,\beta}(u, v) \geq \frac{1}{2}A(u, v) - \frac{1}{p}C(u, v) \geq \frac{1}{2}A(u, v) - \frac{4(1+\beta)\mathcal{S}_p}{p} c^{\frac{6-p}{4}} A(u, v)^{\frac{3(p-2)}{4}},$$

which implies that $I_{\gamma,\beta}(u, v)$ is bounded from below and coercive on S_c .

(ii) Following the idea of [24]. According to the density of $C_0^\infty(\mathbb{R}^3)$ into $H^1(\mathbb{R}^3)$, for any $\varepsilon > 0$, there exist $(\varphi_1, \varphi_2), (\omega_1, \omega_2) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$ with $\|\varphi_1\|_2^2 + \|\varphi_2\|_2^2 = c'$ and $\|\omega_1\|_2^2 + \|\omega_2\|_2^2 = c - c'$ such that

$$I_{\gamma,\beta}(\varphi_1, \varphi_2) \leq \sigma_{\gamma,\beta}(c') + \frac{\varepsilon}{2} \quad \text{and} \quad I_{\gamma,\beta}(\omega_1, \omega_2) \leq \sigma_{\gamma,\beta}(c - c') + \frac{\varepsilon}{2}.$$

Denote by $(\omega_1^n, \omega_2^n) = (\omega_1(\cdot + ne_1), \omega_2(\cdot + ne_1))$, where e_1 is some given unit vector in $\mathbb{R}^3$. Since the distance between the supports of (φ_1, φ_2) and (ω_1^n, ω_2^n) is strictly positive for n large enough and goes to $+\infty$ as $n \rightarrow +\infty$, we deduce that $\text{supp } \varphi_i \cap \text{supp } \omega_i^n = \emptyset$ for $i = 1, 2$. Thus for n large enough, we have $\|\varphi_1 + \omega_1^n\|_2^2 + \|\varphi_2 + \omega_2^n\|_2^2 = c$ and

$$\begin{aligned} \sigma_{\gamma,\beta}(c) &\leq I_{\gamma,\beta}(\varphi_1 + \omega_1^n, \varphi_2 + \omega_2^n) \\ &= I_{\gamma,\beta}(\varphi_1, \varphi_2) + I_{\gamma,\beta}(\omega_1^n, \omega_2^n) \\ &\leq I_{\gamma,\beta}(\varphi_1, \varphi_2) + I_{\gamma,\beta}(\omega_1, \omega_2) \\ &\leq \sigma_{\gamma,\beta}(c') + \sigma_{\gamma,\beta}(c - c') + \varepsilon, \end{aligned}$$

where we have used the fact that $I_{\gamma,\beta}$ are translation-invariant. The proof is complete. ■

3.1 The case of $2 < p < 3$

Lemma 3.2 *Let $2 < p < 3$ and $\gamma, \beta > 0$. Then there holds $\sigma_{\gamma, \beta}(c) < 0$ for all $c > 0$.*

Proof. For fixed $(u, v) \in \mathbf{H} \setminus \{(0, 0)\}$, we set

$$(u^s(x), v^s(x)) := (s^2 u(sx), s^2 v(sx)) \text{ for } s > 0. \quad (16)$$

It is clear that $\|u^s\|_2^2 + \|v^s\|_2^2 = s(\|u\|_2^2 + \|v\|_2^2)$ and

$$I_{\gamma, \beta}(u^s, v^s) = \frac{s^3}{2} A(u, v) + \frac{s^3 \gamma}{4} B(u, v) - \frac{s^{2p-3}}{p} C(u, v).$$

Since $2p - 3 < 3$, it follows that $I_{\gamma, \beta}(u^s, v^s) < 0$ for $s > 0$ sufficiently small. This implies that there exists $c_0 > 0$ such that $\sigma_{\gamma, \beta}(c) < 0$ for $0 < c \leq c_0$. By taking $c \in (c_0, 2c_0]$ and using (15), we get

$$\sigma_{\gamma, \beta}(c) \leq \sigma_{\gamma, \beta}(c_0) + \sigma_{\gamma, \beta}(c - c_0) \leq \sigma_{\gamma, \beta}(c_0) < 0,$$

since $c - c_0 \leq c_0$. This means that $\sigma_{\gamma, \beta}(c) < 0$ for $c \in (c_0, 2c_0]$. Repeating this procedure, we have $\sigma_{\gamma, \beta}(c) < 0$ for all $c > 0$. The proof is complete. ■

Proposition 3.3 *Let $2 < p < 3$ and $\beta > 0$. Then for $\gamma > 0$ small enough, there holds*

$$\sigma_{\gamma, \beta}(c) < \sigma_{\gamma, \beta}(c') + \sigma_{\gamma, \beta}(c - c') \text{ for any } 0 < c' < c. \quad (17)$$

Proof. We follow the arguments in [6] by making some necessary modifications. By Lemma 3.2, we have proved that $\sigma_{\gamma, \beta}(c) < 0$ for all $c > 0$. Let $(\varphi, \omega) \in S_1$ and define

$$(\varphi_c, \omega_c) := (c^{\frac{2}{10-3p}} \varphi(c^{\frac{p-2}{10-3p}} x), c^{\frac{2}{10-3p}} \omega(c^{\frac{p-2}{10-3p}} x)).$$

It is obvious that $(\varphi_c, \omega_c) \in S_c$ and

$$I_{\gamma, \beta}(\varphi_c, \omega_c) = \frac{c^{\frac{6-p}{10-3p}}}{2} A(\varphi, \omega) + \frac{\gamma c^{\frac{18-5p}{10-3p}}}{4} B(\varphi, \omega) - \frac{c^{\frac{6-p}{10-3p}}}{p} C(\varphi, \omega).$$

Then we have

$$\sigma_{\gamma, \beta}(c) = c^{\frac{6-p}{10-3p}} J_1^{\gamma, c}, \quad (18)$$

where

$$J_\rho^{\gamma, c} := \inf_{(u, v) \in S_\rho} \left\{ \frac{1}{2} A(u, v) + \frac{\gamma c^{\frac{4(3-p)}{10-3p}}}{4} B(u, v) - \frac{1}{p} C(u, v) \right\} \text{ for } \rho > 0.$$

Using the same scaling argument, we also have

$$J_\rho^{\gamma, c} = \rho^{\frac{6-p}{10-3p}} J_1^{\gamma, \rho c}. \quad (19)$$

Take $\rho = c'/c$, and it follows from (18) and (19) that (17) is equivalent to

$$J_1^{\gamma, c} < J_\rho^{\gamma, c} + J_{1-\rho}^{\gamma, c}, \quad \forall \rho \in (0, 1). \quad (20)$$

Note that $\lim_{\gamma \rightarrow 0} J_1^{\gamma,c} = J_1^{0,c}$. Similar to the argument in [17], for each $\kappa > 0$, $J_\kappa^{0,c}$ satisfies the strict sub-additive inequality, that is,

$$J_\kappa^{0,c} < J_\rho^{0,c} + J_{\kappa-\rho}^{0,c}, \quad \forall \rho \in (0, \kappa). \quad (21)$$

Next, we prove that $J_1^{\gamma,c}$ satisfies (20) for $\gamma > 0$ small enough. Assume on the contrary. Then there exist two sequences $\{\gamma_n\}$ satisfying $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$ and $\{\kappa_n\} \subset (0, 1)$ such that

$$J_1^{\gamma_n,c} = J_{\kappa_n}^{\gamma_n,c} + J_{1-\kappa_n}^{\gamma_n,c}. \quad (22)$$

Without loss of generality, we assume that $\frac{1}{2} \leq \kappa_n < 1$. Moreover, we conclude that $\kappa_n \rightarrow 1$ as $n \rightarrow \infty$, otherwise we get a contradiction with (21). Now, we prove the following claim:

$$J_{\kappa_n}^{\gamma_n,c} < J_\rho^{\gamma_n,c} + J_{\kappa_n-\rho}^{\gamma_n,c}, \quad \forall \rho \in (0, \kappa_n), \quad (23)$$

for n large enough. Suppose that the claim is not true, then there exists a sequence $\{\rho_n\}$ with $\rho_n \in (\frac{1}{2}\kappa_n, \kappa_n)$ such that

$$J_{\kappa_n}^{\gamma_n,c} = J_{\rho_n}^{\gamma_n,c} + J_{\kappa_n-\rho_n}^{\gamma_n,c}. \quad (24)$$

It follows from (22) and (24) that

$$J_{\rho_n}^{\gamma_n,c} + J_{1-\rho_n}^{\gamma_n,c} \geq J_1^{\gamma_n,c} = J_{\rho_n}^{\gamma_n,c} + J_{\kappa_n-\rho_n}^{\gamma_n,c} + J_{1-\kappa_n}^{\gamma_n,c} \geq J_{\rho_n}^{\gamma_n,c} + J_{1-\rho_n}^{\gamma_n,c},$$

where we have used the fact of $J_{1-\rho_n}^{\gamma_n,c} \leq J_{1-\kappa_n}^{\gamma_n,c} + J_{1-\rho_n-(1-\kappa_n)}^{\gamma_n,c}$. So we have $J_{\rho_n}^{\gamma_n,c} + J_{1-\rho_n}^{\gamma_n,c} = J_1^{\gamma_n,c}$. Since $\rho_n \geq \frac{\kappa_n}{2}$ and $\kappa_n \geq \frac{1}{2}$, we have $\frac{1}{4} \leq \rho_n < \frac{1}{2}$. Then we may assume that $\rho_n \rightarrow \rho$, where $\frac{1}{4} \leq \rho \leq \frac{1}{2}$. Then we deduce that $J_\rho^{0,c} + J_{1-\rho}^{0,c} = J_1^{0,c}$ as $n \rightarrow \infty$, which is a contradiction with (21). Hence the strict inequality (23) holds.

For n large enough, there exists a minimizer (u_n, v_n) of $J_{\kappa_n}^{\gamma_n,c}$ such that $\{(\kappa_n)^{-1/2}(u_n, v_n)\}$ is a minimizing sequence for $J_1^{0,c}$. It follows from (21) that $(\kappa_n)^{-1/2}(u_n, v_n) \rightarrow (u_\infty, v_\infty)$ in $\mathbf{H}$, where (u_∞, v_∞) is a minimizer of $J_1^{0,c}$. The same holds for $\{u_n, v_n\}$, given that $\kappa_n \rightarrow 1$. Without loss of generality, we may assume that (u_n, v_n) and (u_∞, v_∞) satisfy Euler-Lagrange equations in $\mathbb{R}^3$, respectively,

$$\begin{cases} -\Delta u_n + \lambda_n u_n + \gamma_n c^{\frac{4(3-p)}{10-3p}} \phi_{u_n, v_n} u_n = |u_n|^{p-2} u_n + \beta |v_n|^{\frac{p}{2}} |u_n|^{\frac{p}{2}-2} u_n & \text{in } \mathbb{R}^3, \\ -\Delta v_n + \lambda_n v_n + \gamma_n c^{\frac{4(3-p)}{10-3p}} \phi_{u_n, v_n} v_n = |v_n|^{p-2} v_n + \beta |u_n|^{\frac{p}{2}} |v_n|^{\frac{p}{2}-2} v_n & \text{in } \mathbb{R}^3, \end{cases}$$

with $\|u_n\|_2^2 + \|v_n\|_2^2 = \kappa_n$, and

$$\begin{cases} -\Delta u_\infty + \lambda_\infty u_\infty = |u_\infty|^{p-2} u_\infty + \beta |v_\infty|^{\frac{p}{2}} |u_\infty|^{\frac{p}{2}-2} u_\infty & \text{in } \mathbb{R}^3, \\ -\Delta v_\infty + \lambda_\infty v_\infty = |v_\infty|^{p-2} v_\infty + \beta |u_\infty|^{\frac{p}{2}} |v_\infty|^{\frac{p}{2}-2} v_\infty & \text{in } \mathbb{R}^3, \end{cases}$$

with $\|u_\infty\|_2^2 + \|v_\infty\|_2^2 = 1$ and $\lambda_\infty > 0$. Note that

$$\frac{J_1^{\gamma_n,c} - J_{\kappa_n}^{\gamma_n,c}}{1 - \kappa_n} = \frac{J_{1-\kappa_n}^{\gamma_n,c}}{1 - \kappa_n},$$

and as $\kappa_n \rightarrow 1$, we have

$$-\lambda_\infty \geq \frac{J_1^{\gamma_n,c} - J_{\kappa_n}^{\gamma_n,c}}{1 - \kappa_n}.$$

But it follows from (19) that

$$\frac{J_{1-\kappa_n}^{\gamma_n, c}}{1-\kappa_n} = (1-\kappa_n)^{\frac{2(p-2)}{10-3p}} J_1^{\gamma_n, (1-\kappa_n)c} \rightarrow 0 \text{ as } \kappa_n \rightarrow 1.$$

Hence we arrive at a contradiction. This completes the proof. ■

Lemma 3.4 ([29, 30]) *If u is a radially symmetric function, then there holds*

$$\|u\|_p^p \leq \mathcal{C}(p, s) \|u\|_{\dot{H}^s}^{\frac{\theta p}{2-\theta}} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u(x)^2 u(y)^2}{|x-y|} dx dy \right)^{\frac{(1-\theta)p}{4-2\theta}},$$

where $\theta := \frac{6-\frac{5}{2}p}{3-ps-p}$ and $\frac{16s+2}{6s+1} < p \leq \frac{6}{3-2s}$ for $1/2 < s < 3/2$. In particular, for $\frac{18}{7} < p < 6$, there holds

$$\|u\|_p^p \leq \mathcal{C}(p) \|\nabla u\|_2^{\frac{5p-12}{3}} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u(x)^2 u(y)^2}{|x-y|} dx dy \right)^{\frac{6-p}{6}}. \quad (25)$$

Lemma 3.5 *Let $\frac{18}{7} < p < 3$. Then the energy functional $I_{\gamma, \beta}$ is bounded from below by a constant not depending on c , on $S_{r, c} := \{(u, v) \in \mathbf{H}_r : \|u\|_2^2 + \|v\|_2^2 = c\}$.*

Proof. For $(u, v) \in S_{r, c}$, it follows from (25) that

$$\begin{aligned} C(u, v) &= \int_{\mathbb{R}^3} (|u|^p + |v|^p + 2\beta |u|^{\frac{p}{2}} |v|^{\frac{p}{2}}) dx \\ &\leq (1 + \beta) \int_{\mathbb{R}^3} (|u|^p + |v|^p) dx \\ &\leq \mathcal{C}(p)(1 + \beta) \left[\|\nabla u\|_2^{\frac{5p-12}{3}} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u(x)^2 u(y)^2}{|x-y|} dx dy \right)^{\frac{6-p}{6}} \right. \\ &\quad \left. + \|\nabla v\|_2^{\frac{5p-12}{3}} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{v(x)^2 v(y)^2}{|x-y|} dx dy \right)^{\frac{6-p}{6}} \right] \\ &\leq \mathcal{C}(p)(1 + \beta) \left(\|\nabla u\|_2^{\frac{5p-12}{3}} + \|\nabla v\|_2^{\frac{5p-12}{3}} \right) \left[\left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u(x)^2 u(y)^2}{|x-y|} dx dy \right)^{\frac{6-p}{6}} \right. \\ &\quad \left. + \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{v(x)^2 v(y)^2}{|x-y|} dx dy \right)^{\frac{6-p}{6}} \right] \\ &\leq 4\mathcal{C}(p)(1 + \beta) A(u, v)^{\frac{5p-12}{6}} B(u, v)^{\frac{6-p}{6}}. \end{aligned} \quad (26)$$

Then by (26) we have

$$I_{\gamma, \beta}(u, v) \geq \frac{1}{2} A(u, v) + \frac{\gamma}{4} B(u, v) - \frac{4\mathcal{C}(p)(1 + \beta)}{p} A(u, v)^{\frac{5p-12}{6}} B(u, v)^{\frac{6-p}{6}},$$

which implies that $I_{\gamma, \beta}$ is bounded from below by a constant not depending on c . In fact, we set the function of two variables

$$f(x, y) := \frac{1}{2}x + \frac{\gamma}{4}y - \frac{4\mathcal{C}(p)(1 + \beta)}{p} x^{\frac{5p-12}{6}} y^{\frac{6-p}{6}}, \quad \forall (x, y) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

By a direct calculation, there exists stationary point $(x_0, y_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ not depending on c such that $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Furthermore, we have

$$f_{xx}(x_0, y_0) = \frac{\mathcal{C}(p)(1+\beta)(5p-12)(18-5p)}{9p} x_0^{-\frac{24-5p}{6}} y_0^{\frac{6-p}{6}}$$

and

$$f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0) = \frac{8[\mathcal{C}(p)(1+\beta)]^2(5p-12)(6-p)(3-p)}{27p^2} x_0^{-\frac{18-5p}{3}} y_0^{-\frac{p}{6}} > 0,$$

which implies that the function $f(x, y)$ achieves its minimum at point (x_0, y_0) , and so the energy functional $I_{\gamma, \beta}$ is bounded from below. This completes the proof. ■

Lemma 3.6 *Let $2 < p < 3$. Then it holds $\lim_{c \rightarrow 0} \frac{\sigma_{\gamma, \beta}(c)}{c} = 0$.*

Proof. Notice that

$$\frac{\sigma_{0, \beta}(c)}{c} \leq \frac{\sigma_{\gamma, \beta}(c)}{c} < 0,$$

where

$$\sigma_{0, \beta}(c) := \inf_{(u, v) \in S_c} I_{0, \beta}(u, v)$$

with

$$I_{0, \beta}(u, v) := \frac{1}{2}A(u, v) - \frac{1}{p}C(u, v).$$

Due to [17], there exists a minimizer (u_0, v_0) for $\sigma_{0, \beta}(c)$ satisfying

$$\begin{cases} -\Delta u_0 + \lambda_c u_0 = |u_0|^{p-2}u_0 + \beta|v_0|^{\frac{p}{2}}|u_0|^{\frac{p}{2}-2}u_0 & \text{in } \mathbb{R}^3, \\ -\Delta v_0 + \lambda_c v_0 = |v_0|^{p-2}v_0 + \beta|u_0|^{\frac{p}{2}}|v_0|^{\frac{p}{2}-2}v_0 & \text{in } \mathbb{R}^3, \end{cases} \quad (27)$$

which indicates that

$$\frac{\lambda_c}{2} = \frac{C(u_0, v_0) - A(u_0, v_0)}{2 \int_{\mathbb{R}^3} (|u_0|^2 + |v_0|^2) dx} \geq \frac{\frac{1}{p}C(u_0, v_0) - \frac{1}{2}A(u_0, v_0)}{2 \int_{\mathbb{R}^3} (|u_0|^2 + |v_0|^2) dx} = \frac{-\sigma_{0, \beta}(c)}{c} > 0,$$

where λ_c is the Lagrange multiplier associated to the minimizer (u_0, v_0) . Next, we prove that $\lim_{c \rightarrow 0} \lambda_c = 0$. Suppose on the contrary. Then we can assume that there exists a sequence $\{c_n\} \subset \mathbb{R}$ satisfying $c_n \rightarrow 0$ as $n \rightarrow \infty$ such that $\lambda_{c_n} > M$ for some $M \in (0, 1)$. Since the minimizers $(u_n, v_n) := (u_{c_n}, v_{c_n})$ satisfy system (27), we have

$$\begin{aligned} M(\|u_n\|_{H^1}^2 + \|v_n\|_{H^1}^2) &\leq \|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 + M(\|u_n\|_2^2 + \|v_n\|_2^2) \\ &\leq \|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 + \lambda_{c_n}(\|u_n\|_2^2 + \|v_n\|_2^2) \\ &= C(u_n, v_n) \\ &\leq K(p, \beta)(\|u_n\|_{H^1}^p + \|v_n\|_{H^1}^p) \end{aligned}$$

for some $K(p, \beta) > 0$. This implies that there exists $M_1 > 0$ such that $A(u_n, v_n) \geq M_1$. But as $n \rightarrow +\infty$, by (13), we get

$$0 \geq I_{0, \beta}(u_n, v_n) \geq \frac{1}{2}A(u_n, v_n) - \frac{(4+2\beta)\mathcal{S}_p}{p} c_n^{\frac{6-p}{4}} A(u_n, v_n)^{\frac{3(p-2)}{4}} \geq \frac{1}{2}M_1 - o_n(1),$$

which is a contradiction. This completes the proof. ■

Lemma 3.7 *Let $2 < p < 3$. Then it holds $\inf_{c>0} \sigma_{\gamma,\beta}(c) = -\infty$.*

Proof. Now we argue by contradiction and assume that there exists $K > 0$ such that $\sigma_{\gamma,\beta}(c) > -K$ for all $c > 0$. In this case we would have $\lim_{c \rightarrow \infty} \frac{\sigma_{\gamma,\beta}(c)}{c} = 0$. Since $\lim_{c \rightarrow 0} \frac{\sigma_{\gamma,\beta}(c)}{c} = 0$ and $\sigma_{\gamma,\beta}(c) < 0$, the function $c \rightarrow \frac{\sigma_{\gamma,\beta}(c)}{c}$ attains a global minimum, i.e. there exists c_0 such that

$$\frac{\sigma_{\gamma,\beta}(c_0)}{c_0} \leq \frac{\sigma_{\gamma,\beta}(c)}{c} \text{ for all } c > 0. \quad (28)$$

On the other hand, by Lemma 3.1, one has

$$\frac{\sigma_{\gamma,\beta}(2c_0)}{2c_0} \leq \frac{\sigma_{\gamma,\beta}(c_0) + \sigma_{\gamma,\beta}(c_0)}{2c_0} = \frac{\sigma_{\gamma,\beta}(c_0)}{c_0}. \quad (29)$$

It follows from (28) and (29) that the function $c \rightarrow \frac{\sigma_{\gamma,\beta}(c)}{c}$ attains a global minimum also at $2c_0$. Repeating such step, we deduce that

$$\frac{\sigma_{\gamma,\beta}(kc_0)}{kc_0} = \frac{\sigma_{\gamma,\beta}(c_0)}{c_0}, \quad \forall k \in \mathbb{N},$$

which implies that $\liminf_{c \rightarrow \infty} \frac{\sigma_{\gamma,\beta}(c)}{c} < 0$. This contradicts with the fact that $\lim_{c \rightarrow \infty} \frac{\sigma_{\gamma,\beta}(c)}{c} = 0$. This completes the proof. ■

Now we give the proof of Theorem 1.5: Let $\{(u_n, v_n)\} \subset S_c$ be a minimizing sequence to $\sigma_{\gamma,\beta}(c)$. Our aim is to prove that $\{(u_n, v_n)\}$ is compact in $\mathbf{H}$ up to translations. By Lemma 3.1(i), we obtain that $\{(u_n, v_n)\}$ is bounded in $\mathbf{H}$. If

$$\sup_{y \in \mathbb{R}^3} \int_{B_R(y)} (u_n^2 + v_n^2) dx = o_n(1) \text{ for some } R > 0,$$

then it follows from Vanishing lemma in [25, Lemma I.1] that $\{(u_n, v_n)\} \rightarrow (0, 0)$ in $L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ for $2 < p < 6$. This is incompatible with the fact that $I_{\gamma,\beta}(u_n, v_n) < 0$. Thus there exist a constant $D_0 > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_R(y_n)} (u_n^2 + v_n^2) dx \geq D_0 > 0.$$

Since $\{(u_n, v_n)\}$ is bounded in $\mathbf{H}$, we obtain that $\{(u_n, v_n)\}$ is weakly convergent in $\mathbf{H}$ and is locally compact in $L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ for $2 \leq p < 6$, and so

$$(u_n(x - y_n), v_n(x - y_n)) \rightharpoonup (\hat{u}, \hat{v}) \neq (0, 0) \quad \text{in } \mathbf{H}.$$

Hence, $\rho := \|\hat{u}\|_2^2 + \|\hat{v}\|_2^2 > 0$. Next, we prove that $(u, v) \in S_c$. Suppose that $\rho < c$. Since

$$\|u_n\|_2^2 + \|v_n\|_2^2 = \|u_n - \hat{u}\|_2^2 + \|v_n - \hat{v}\|_2^2 + \|\hat{u}\|_2^2 + \|\hat{v}\|_2^2 + o_n(1),$$

by Lemma 2.6 and the translational invariance, we conclude that

$$\sigma_{\gamma,\beta}(c) = \lim_{n \rightarrow \infty} I_{\gamma,\beta}(u_n, v_n) \geq I_{\gamma,\beta}(\hat{u}, \hat{v}) + \lim_{n \rightarrow \infty} I_{\gamma,\beta}(u_n - \hat{u}, v_n - \hat{v}) \geq \sigma_{\gamma,\beta}(\rho) + \sigma_{\gamma,\beta}(c - \rho),$$

which contradicts with (17). So, we have $\|\hat{u}\|_2^2 + \|\hat{v}\|_2^2 = c$ and $(\hat{u}, \hat{v}) \in S_c$. Moreover, using (13), we deduce that $(u_n, v_n) \rightarrow (\hat{u}, \hat{v})$ in $L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$. Then there holds

$$\sigma_{\gamma,\beta}(c) \leq I_{\gamma,\beta}(\hat{u}, \hat{v}) \leq \lim_{n \rightarrow \infty} I_{\gamma,\beta}(u_n, v_n) = \sigma_{\gamma,\beta}(c).$$

Hence $(\hat{u}, \hat{v}) \in S_c$ is a nontrivial minimizer of $\sigma_{\gamma,\beta}(c)$.

Next, we claim that $\hat{u} \neq 0$ and $\hat{v} \neq 0$. If not, we may assume that $\hat{v} \equiv 0$. Then by Lemma 2.5, there exists $s_\beta \in (0, 1)$ such that $(\sqrt{s_\beta}\hat{u}, \sqrt{1-s_\beta}\hat{u}) \in S_c$ and

$$I_{\gamma,\beta}(\sqrt{s_\beta}\hat{u}, \sqrt{1-s_\beta}\hat{u}) < I_{\gamma,\beta}(\hat{u}, 0) = I_{\gamma,\beta}(0, \hat{u}) = \sigma_{\gamma,\beta}(c),$$

which is a contradiction. So, $(\hat{u}, \hat{v}) \in S_c$ is a vectorial minimizer of $\sigma_{\gamma,\beta}(c)$.

Finally, we study the symmetry breaking phenomenon for minimizer $(\hat{u}, \hat{v})$ when $\frac{18}{7} < p < 3$. By Proposition 3.3 and Lemma 3.7, we know that the ground state energy $\sigma_{\gamma,\beta}(c)$ is strictly decreasing on c and $\inf_{c>0} \sigma_{\gamma,\beta}(c) = -\infty$. However, it follows Lemma 3.5 that $\inf_{c>0} \sigma_{\gamma,\beta}^r(c) > -\infty$, where $\sigma_{\gamma,\beta}^r(c) := \inf_{(u,v) \in S_{r,c}} I_{\gamma,\beta}(u, v)$. Then the symmetry breaking phenomenon occurs for $c > 0$ sufficiently large.

3.2 The case of $3 \leq p < \frac{10}{3}$

Following the idea of Lions [24], we have the following results:

$$\begin{aligned} \int_{\mathbb{R}^3} (|u|^3 + |u|v^2)dx &= \int_{\mathbb{R}^3} (-\Delta\phi_{u,v})|u|dx \\ &= \int_{\mathbb{R}^3} \langle \nabla\phi_{u,v}, \nabla|u| \rangle dx \\ &\leq \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla\phi_{u,v}|^2 dx \right)^{\frac{1}{2}} \\ &\leq \|\nabla u\|_2 B(u, v)^{\frac{1}{2}}, \end{aligned} \tag{30}$$

and

$$\begin{aligned} \int_{\mathbb{R}^3} (|v|^3 + |v|u^2)dx &= \int_{\mathbb{R}^3} (-\Delta\phi_{u,v})|v|dx \\ &= \int_{\mathbb{R}^3} \langle \nabla\phi_{u,v}, \nabla|v| \rangle dx \\ &\leq \left(\int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla\phi_{u,v}|^2 dx \right)^{\frac{1}{2}} \\ &\leq \|\nabla v\|_2 B(u, v)^{\frac{1}{2}}. \end{aligned} \tag{31}$$

For $(u, v) \in \mathbf{H}$ and $3 \leq p < \frac{10}{3}$, by (11), (30) and (31), we have

$$\begin{aligned}
& \int_{\mathbb{R}^3} (|u|^p + |v|^p) dx \\
& \leq \left(\int_{\mathbb{R}^3} |u|^3 dx \right)^{10-3p} \left(\int_{\mathbb{R}^3} |u|^{\frac{10}{3}} dx \right)^{3(p-3)} + \left(\int_{\mathbb{R}^3} |v|^3 dx \right)^{10-3p} \left(\int_{\mathbb{R}^3} |v|^{\frac{10}{3}} dx \right)^{3(p-3)} \\
& \leq \left(\|u\|_3^{3(10-3p)} + \|v\|_3^{3(10-3p)} \right) \left(\|u\|_{10/3}^{10(p-3)} + \|v\|_{10/3}^{10(p-3)} \right) \\
& \leq 16\mathcal{S}_{10/3} (\|u\|_2^2 + \|v\|_2^2)^{2(p-3)} A(u, v)^{\frac{3p-8}{2}} B(u, v)^{\frac{10-3p}{2}},
\end{aligned}$$

which implies that

$$\begin{aligned}
& \frac{\frac{p}{3p-8} \left(\frac{\gamma(3p-8)}{2(10-3p)} \right)^{\frac{10-3p}{2}} c^{-2(p-3)} A(u, v)^{\frac{3p-8}{2}} B(u, v)^{\frac{10-3p}{2}} (\|u\|_2^2 + \|v\|_2^2)^{2(p-3)} - \int_{\mathbb{R}^3} (|u|^p + |v|^p) dx}{2 \int_{\mathbb{R}^3} |u|^{\frac{p}{2}} |v|^{\frac{p}{2}} dx} \\
& \geq \frac{\left[\frac{p}{16\mathcal{S}_{10/3}(3p-8)} \left(\frac{\gamma(3p-8)}{2(10-3p)} \right)^{\frac{10-3p}{2}} c^{-2(p-3)} - 1 \right] \int_{\mathbb{R}^3} (|u|^p + |v|^p) dx}{\int_{\mathbb{R}^3} (|u|^p + |v|^p) dx} \\
& \geq \frac{p}{16\mathcal{S}_{10/3}(3p-8)} \left(\frac{\gamma(3p-8)}{2(10-3p)} \right)^{\frac{10-3p}{2}} c^{-2(p-3)} - 1. \tag{32}
\end{aligned}$$

Thus we can define

$$\begin{aligned}
\beta_\star &= \beta_\star(c, \gamma, p) \\
&:= \inf_{(u,v) \in \mathbf{H} \setminus \{(0,0)\}} \left[\frac{\frac{p}{3p-8} \left(\frac{\gamma(3p-8)}{2(10-3p)} \right)^{\frac{10-3p}{2}} c^{-2(p-3)} A(u, v)^{\frac{3p-8}{2}} B(u, v)^{\frac{10-3p}{2}} (\|u\|_2^2 + \|v\|_2^2)^{2(p-3)}}{2 \int_{\mathbb{R}^3} |u|^{\frac{p}{2}} |v|^{\frac{p}{2}} dx} \right. \\
&\quad \left. - \frac{\int_{\mathbb{R}^3} (|u|^p + |v|^p) dx}{2 \int_{\mathbb{R}^3} |u|^{\frac{p}{2}} |v|^{\frac{p}{2}} dx} \right] > -\infty.
\end{aligned}$$

In particular, for $p = 3$, it follows from (30) and (31) that

$$\int_{\mathbb{R}^3} (|u|^3 + |v|^3) dx \leq \sqrt{2} A(u, v)^{\frac{1}{2}} B(u, v)^{\frac{1}{2}},$$

and further

$$\begin{aligned}
\beta_\star &= \inf_{(u,v) \in \mathbf{H} \setminus \{(0,0)\}} \frac{\frac{3\sqrt{2}\gamma}{2} A(u, v)^{\frac{1}{2}} B(u, v)^{\frac{1}{2}} - \int_{\mathbb{R}^3} (|u|^3 + |v|^3) dx}{2 \int_{\mathbb{R}^3} |u|^{\frac{3}{2}} |v|^{\frac{3}{2}} dx} \\
&\geq \frac{\left(\frac{3\sqrt{\gamma}}{2} - 1 \right) \int_{\mathbb{R}^3} (|u|^3 + |v|^3) dx}{2 \int_{\mathbb{R}^3} |u|^{\frac{3}{2}} |v|^{\frac{3}{2}} dx} \\
&\geq \frac{\left(\frac{3\sqrt{\gamma}}{2} - 1 \right) \int_{\mathbb{R}^3} (|u|^3 + |v|^3) dx}{\int_{\mathbb{R}^3} (|u|^3 + |v|^3) dx} \\
&= \frac{3\sqrt{\gamma}}{2} - 1.
\end{aligned}$$

Lemma 3.8 *Let $3 \leq p < \frac{10}{3}$ and $\gamma > 0$. Then for $\beta > \beta_*$, it holds $\sigma_{\gamma,\beta}(c) < 0$ for all $c > 0$.*

Proof. Let $(u, v) \in \mathbf{H} \setminus \{(0, 0)\}$ and set the scaled function

$$(\bar{u}_s, \bar{v}_s) := (s^{3/2}u(sx), s^{3/2}v(sx)) \text{ for } s > 0. \quad (33)$$

Clearly, $(\bar{u}_s, \bar{v}_s) \in S_c$. Define the function $f_\star(s)$ by

$$\begin{aligned} f_\star(s) &:= \frac{1}{s} I_{\gamma,\beta}(\bar{u}_s, \bar{v}_s) \\ &= \frac{s}{2} A(u, v) + \frac{\gamma}{4} B(u, v) - \frac{s^{\frac{3p-8}{2}}}{p} \int_{\mathbb{R}^3} (|u|^p + |v|^p + 2\beta|u|^{\frac{p}{2}}|v|^{\frac{p}{2}}) dx. \end{aligned}$$

By a direct calculation, it is easy to see that

$$\begin{aligned} \min_{s \geq 0} f_\star(s) &= f_\star(\bar{s}) \\ &= -\frac{10-3p}{2(3p-8)} \left(\frac{3p-8}{p} \right)^{\frac{2}{10-3p}} A(u, v)^{\frac{8-3p}{10-3p}} \left(\int_{\mathbb{R}^3} (|u|^p + |v|^p + 2\beta|u|^{\frac{p}{2}}|v|^{\frac{p}{2}}) dx \right)^{\frac{2}{10-3p}} \\ &\quad + \frac{\gamma}{4} B(u, v), \end{aligned}$$

where

$$\bar{s} := \left(\frac{(3p-8) \int_{\mathbb{R}^3} (|u|^p + |v|^p + 2\beta|u|^{\frac{p}{2}}|v|^{\frac{p}{2}}) dx}{pA(u, v)} \right)^{\frac{2}{3p-10}}.$$

This indicates that $\min_{s \geq 0} f_\star(s) < 0$ for $\beta > \beta_*$, and so $\sigma_{\gamma,\beta}(c) < 0$ for $\beta > \beta_*$. This completes the proof. ■

Remark 3.9 *According to the definition of β_* and the proof of Lemma 3.8, we see that β_* can be characterized by*

$$\beta_* = \inf \{ \beta > 0 : \sigma_{\gamma,\beta}(c) < 0 \} \text{ for } 3 \leq p < \frac{10}{3}.$$

Lemma 3.10 *Let $3 \leq p < \frac{10}{3}$ and $\gamma > 0$. Then for $\beta > \beta_*$, we have*

$$\sigma_{\gamma,\beta}(sc) < s\sigma_{\gamma,\beta}(c) \text{ for any } s > 1,$$

and

$$\sigma_{\gamma,\beta}(c) < \sigma_{\gamma,\beta}(c') + \sigma_{\gamma,\beta}(c - c') \text{ for all } 0 < c' < c.$$

Proof. Let $(u, v) \in S_c$. By (16) one has $\|u^s\|_2^2 + \|v^s\|_2^2 = s(\|u\|_2^2 + \|v\|_2^2)$ and

$$I_{\gamma,\beta}(u^s, v^s) = s^3 I_{\gamma,\beta}(u, v) + \frac{s^3 - s^{2p-3}}{p} C(u, v).$$

Note that $s^3 - s^{2p-3} \leq 0$ for $s > 1$. Then we get $I_{\gamma,\beta}(u^s, v^s) \leq s^3 I_{\gamma,\beta}(u, v)$, leading to $\sigma_{\gamma,\beta}(sc) \leq s^3 \sigma_{\gamma,\beta}(c)$. Furthermore, since $\sigma_{\gamma,\beta}(c) < 0$ for $\beta > \beta_*$ by Lemma 3.8, we have

$$\sigma_{\gamma,\beta}(sc) \leq s^3 \sigma_{\gamma,\beta}(c) < s\sigma_{\gamma,\beta}(c).$$

Without loss of generality, we assume that $c - c' > c'$. Then we have

$$\sigma_{\gamma,\beta}(c) < \frac{c}{c-c'}\sigma_{\gamma,\beta}(c-c') = \sigma_{\gamma,\beta}(c-c') + \frac{c'}{c-c'}\sigma_{\gamma,\beta}(c-c') < \sigma_{\gamma,\beta}(c-c') + \sigma_{\gamma,\beta}(c').$$

This completes the proof. ■

Now we give the proof of Theorem 1.6: (i) The proof of the existence of the minimizer is similar to that of Theorem 1.5, we omit it here. Next we discuss the sign of the Lagrange multiplier $\tilde{\lambda}$ associated with the minimizer $(\tilde{u}, \tilde{v}) \in S_c$. Since $\sigma_{\gamma,\beta}(c) < 0$ and

$$(5p-12)\sigma_{\gamma,\beta}(c) = (5p-12)I_{\gamma,\beta}(\tilde{u}, \tilde{v}) = 2(p-3)A(\tilde{u}, \tilde{v}) - \frac{(3p-8)\tilde{\lambda}c}{2},$$

we have $\tilde{\lambda} > 0$.

(ii) Note that

$$\frac{\sqrt{\gamma}}{2} \int_{\mathbb{R}^3} (|u|^3 + v^2|u|) dx \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\gamma}{8} \int_{\mathbb{R}^3} \phi_{u,v} (u^2 + v^2) dx \quad (34)$$

and

$$\frac{\sqrt{\gamma}}{2} \int_{\mathbb{R}^3} (u^2|v| + |v|^3) dx \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{\gamma}{8} \int_{\mathbb{R}^3} \phi_{u,v} (u^2 + v^2) dx \quad (35)$$

for all $(u, v) \in \mathbf{H}$. When $p = 3$, for any $(u, v) \in S_c$, it follows from (34) and (35) that

$$\begin{aligned} I_{\gamma,\beta}(u, v) &= \frac{1}{2}A(u, v) + \frac{\gamma}{4}B(u, v) - \frac{1}{3} \int_{\mathbb{R}^3} C(u, v) \\ &\geq \frac{\sqrt{\gamma}}{2} \int_{\mathbb{R}^3} (|u|^3 + |v|^3) dx - \frac{1+\beta}{3} \int_{\mathbb{R}^3} (|u|^3 + |v|^3) dx \\ &\geq \frac{3\sqrt{\gamma} - 2(1+\beta)}{6} \int_{\mathbb{R}^3} (|u|^3 + |v|^3) dx \\ &> 0 \text{ if } \beta < \frac{3\sqrt{\gamma}}{2} - 1. \end{aligned}$$

On the other hand, for fixed $(u, v) \in \mathbf{H} \setminus \{(0, 0)\}$, by (33) we have

$$\sigma_{\gamma,\beta}(c) \leq I_{\gamma,\beta}(u_s, v_s) = \frac{s^2}{2}A(u, v) + \frac{\gamma s}{4}B(u, v) - \frac{s^{\frac{3(p-2)}{2}}}{3}C(u, v) \text{ for } s > 0,$$

which implies that $\sigma_{\gamma,\beta}(c) \leq 0$ as $s \rightarrow 0$. So, $\sigma_{\gamma,\beta}(c)$ has no minimizer for $\beta < \frac{3\sqrt{\gamma}}{2} - 1$.

For $3 < p < \frac{10}{3}$, we follow the idea in [7]. Suppose by contradiction that there exist $c > 0$ and $\tilde{\beta} < \beta_*(c, \gamma, p)$ such that $\sigma_{\gamma,\tilde{\beta}}(c)$ has a minimizer. According to the definition of $\beta_*(c, \gamma, p)$, we find that $\beta_*(c, \gamma, p)$ is continuous and decreasing on c . Thus one can choose $0 < c < \tilde{c}$ so that

$$\tilde{\beta} < \beta_*(\tilde{c}, \gamma, p) < \beta_*(c, \gamma, p).$$

On the other hand, by the characterization of $\beta_*(c, \gamma, p)$, it follows that $\sigma_{\gamma,\tilde{\beta}}(c) = 0$. Then by Lemma 3.10, we get

$$\sigma_{\gamma,\tilde{\beta}}(\tilde{c}) < \frac{\tilde{c}}{c}\sigma_{\gamma,\tilde{\beta}}(c) = 0,$$

which implies that $\tilde{\beta} > \beta_*(\tilde{c}, \gamma, p)$. This is a contradiction. So, $\sigma_{\gamma,\beta}(c)$ has no minimizer for $\beta < \beta_*$. We complete the proof.

4 The relation between action ground states and energy ground states

Lemma 4.1 *Let $p = 3$ and $\gamma, \lambda > 0$. Then for $\beta > \max\{\beta_*, 0\}$, system (7) has a vectorial action ground state $(u_\lambda, v_\lambda) \in \mathbf{H}$.*

Proof. By Theorem 1.6, for $\beta > \max\{\beta_*, 0\}$, system (7) admits an energy ground state $(\tilde{u}, \tilde{v}) \in S_c$ with the corresponding Lagrange multiplier $\tilde{\lambda} > 0$. In other words, $(\tilde{u}, \tilde{v})$ satisfies the system:

$$\begin{cases} -\Delta u + \tilde{\lambda}u + \gamma\phi_{u,v}u = |u|^{p-2}u + \beta|v|^{\frac{p}{2}}|u|^{\frac{p}{2}-2}u & \text{in } \mathbb{R}^3, \\ -\Delta v + \tilde{\lambda}v + \gamma\phi_{u,v}v = |v|^{p-2}v + \beta|u|^{\frac{p}{2}}|v|^{\frac{p}{2}-2}v & \text{in } \mathbb{R}^3. \end{cases} \quad (36)$$

First of all, we claim that $(\tilde{u}, \tilde{v})$ is a vectorial action ground state of system (36). Indeed, let $(\phi_1, \phi_2) \in \mathbf{H}$ be a nontrivial solution of system (36). We define

$$\tilde{c} := \frac{c}{\|\phi_1\|_2^2 + \|\phi_2\|_2^2} \text{ and } (\tilde{\phi}_1, \tilde{\phi}_2) := (\tilde{c}^2\phi_1(\tilde{c}x), \tilde{c}^2\phi_2(\tilde{c}x))$$

so that $\|\tilde{\phi}_1\|_2^2 + \|\tilde{\phi}_2\|_2^2 = c$. By using the characterization of $(\tilde{u}, \tilde{v})$, one has

$$-\frac{\tilde{\lambda}c}{6} = I_{\gamma,\beta}(\tilde{u}, \tilde{v}) \leq I_{\gamma,\beta}(\tilde{\phi}_1, \tilde{\phi}_2) = \tilde{c}^3 I_{\gamma,\beta}(\phi_1, \phi_2) = -\frac{\tilde{\lambda}\tilde{c}^3}{6}(\|\phi_1\|_2^2 + \|\phi_2\|_2^2) = -\frac{\tilde{\lambda}c}{6}\tilde{c}^2,$$

which shows that $\tilde{c} \leq 1$, and hence $c \leq \frac{c}{\tilde{c}} = \|\phi_1\|_2^2 + \|\phi_2\|_2^2$. Then we get

$$E_{\tilde{\lambda},\gamma,\beta}(\tilde{u}, \tilde{v}) = \frac{\tilde{\lambda}c}{3} \leq \frac{\tilde{\lambda}}{3}(\|\phi_1\|_2^2 + \|\phi_2\|_2^2) = E_{\tilde{\lambda},\gamma,\beta}(\phi_1, \phi_2),$$

which implies that $(\tilde{u}, \tilde{v})$ is a vectorial action ground state of system (36).

Set $(u_\lambda, v_\lambda) := (\varpi^2\tilde{u}(\varpi x), \varpi^2\tilde{v}(\varpi x))$, where $\varpi := \sqrt{\frac{\tilde{\lambda}}{\lambda}}$. A direct computation shows that (u_λ, v_λ) is a vectorial solution of system (7). Next, we show that (u_λ, v_λ) is a vectorial action ground state of system (7). Indeed, let $(u, v) \in \mathbf{H}$ be any nontrivial solution of system (7) and define

$$(u^*, v^*) := (\varpi^{-2}u(x/\varpi), \varpi^{-2}v(x/\varpi)).$$

Similarly, we can prove that (u^*, v^*) is a nontrivial solution of system (36) and hence we have

$$\begin{aligned} \frac{\tilde{\lambda}}{3\varpi}(\|u_\lambda\|_2^2 + \|v_\lambda\|_2^2) &= \frac{\tilde{\lambda}}{3}(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_2^2) \\ &= E_{\tilde{\lambda},\gamma,\beta}(\tilde{u}, \tilde{v}) \\ &\leq E_{\tilde{\lambda},\gamma,\beta}(u^*, v^*) \\ &= \frac{\tilde{\lambda}}{3}(\|u^*\|_2^2 + \|v^*\|_2^2) \\ &= \frac{\tilde{\lambda}}{3\varpi}(\|u\|_2^2 + \|v\|_2^2), \end{aligned}$$

which implies that $\|u_\lambda\|_2^2 + \|v_\lambda\|_2^2 \leq \|u\|_2^2 + \|v\|_2^2$. Then we have

$$E_{\lambda,\gamma,\beta}(u_\lambda, v_\lambda) = \frac{\lambda}{3}(\|u_\lambda\|_2^2 + \|v_\lambda\|_2^2) \leq \frac{\lambda}{3}(\|u\|_2^2 + \|v\|_2^2) = E_{\lambda,\gamma,\beta}(u, v),$$

which implies that (u_λ, v_λ) is a vectorial action ground state of system (7). We complete the proof. $\blacksquare$

Next, we give a characterization of the vectorial action ground state (u_λ, v_λ) of system (7) when $p = 3$.

Proposition 4.2 *Let $p = 3$ and $\gamma, \lambda > 0$. Then for $\beta > \max\{\beta_\star, 0\}$, the following statements are true.*

(i) *For any vectorial action ground state (u, v) of system (7), there exists a unique $c = c(\lambda) > 0$ such that $\|u\|_2^2 + \|v\|_2^2 = c(\lambda)$.*

(ii) *(u_λ, v_λ) is a vectorial action ground state of system (7) if and only if (u_λ, v_λ) is a minimizer of $\sigma_{\gamma,\beta}(c(\lambda))$. In particular, we have*

$$I_{\gamma,\beta}(u_\lambda, v_\lambda) = \sigma_{\gamma,\beta}(c(\lambda)) = \inf_{(u,v) \in S_{c(\lambda)}} I_{\gamma,\beta}(u, v).$$

Proof. (i) Let (u_1, v_1) and (u_2, v_2) be two vectorial action ground states of system (7). Then we have

$$\frac{\lambda}{3}(\|u_1\|_2^2 + \|v_1\|_2^2) = I_{\gamma,\beta}(u_1, v_1) = I_{\gamma,\beta}(u_2, v_2) = \frac{\lambda}{3}(\|u_2\|_2^2 + \|v_2\|_2^2),$$

which implies the conclusion holds.

(ii) First of all, we show that if (u_λ, v_λ) is a vectorial action ground state of system (7), then (u_λ, v_λ) is a minimizer of $\sigma_{\gamma,\beta}(c(\lambda))$. To this aim, suppose by contradiction that there exists $(u_1, v_1) \in \mathbf{H}$ such that $\|u_1\|_2^2 + \|v_1\|_2^2 = c(\lambda)$ and

$$\sigma_{\gamma,\beta}(c(\lambda)) = I_{\gamma,\beta}(u_1, v_1) < I_{\gamma,\beta}(u_\lambda, v_\lambda). \quad (37)$$

Moreover, there exists a Lagrange multiplier $\lambda_1 > 0$ such that (u_1, v_1) satisfies the system:

$$\begin{cases} -\Delta u + \lambda_1 u + \gamma \phi_{u,v} u = |u|^{p-2} u + \beta |v|^{\frac{p}{2}} |u|^{\frac{p}{2}-2} u & \text{in } \mathbb{R}^3, \\ -\Delta v + \lambda_1 v + \gamma \phi_{u,v} v = |v|^{p-2} v + \beta |u|^{\frac{p}{2}} |v|^{\frac{p}{2}-2} v & \text{in } \mathbb{R}^3. \end{cases}$$

Then by (37) and item (i), we have

$$-\frac{\lambda_1}{6} c(\lambda) = I_{\gamma,\beta}(u_1, v_1) < I_{\gamma,\beta}(u_\lambda, v_\lambda) = -\frac{\lambda}{6} c(\lambda),$$

which implies that $\lambda < \lambda_1$. Set $\varpi := \sqrt{\frac{\lambda}{\lambda_1}}$ and $(\tilde{u}_1, \tilde{v}_1) := (\varpi^2 u_1(\varpi x), \varpi^2 v_1(\varpi x))$. It is clear that $(\tilde{u}_1, \tilde{v}_1)$ is a vectorial solution of system (7). Then we have

$$\frac{\lambda}{3} c(\lambda) = E_{\lambda,\gamma,\beta}(u_\lambda, v_\lambda) \leq E_{\lambda,\gamma,\beta}(\tilde{u}_1, \tilde{v}_1) = \frac{\lambda}{3}(\|\tilde{u}_1\|_2^2 + \|\tilde{v}_1\|_2^2) = \frac{\lambda \varpi}{3}(\|u_1\|_2^2 + \|v_1\|_2^2) = \frac{\lambda \varpi}{3} c(\lambda),$$

which implies that $\varpi \geq 1$. This contradicts with $\lambda < \lambda_1$. So, (u_λ, v_λ) is a minimizer of $\sigma_{\gamma,\beta}(c(\lambda))$.

Finally we show that if (u_λ, v_λ) is a minimizer of $\sigma_{\gamma,\beta}(c(\lambda))$, then (u_λ, v_λ) is a vectorial action ground state of system (7). Indeed, we note that there exists a Lagrange multiplier $\tilde{\lambda} > 0$ such that (u_λ, v_λ) satisfies the system

$$\begin{cases} -\Delta u + \tilde{\lambda}u + \gamma\phi_{u,v}u = |u|^{p-2}u + \beta|v|^{\frac{p}{2}}|u|^{\frac{p}{2}-2}u & \text{in } \mathbb{R}^3, \\ -\Delta v + \tilde{\lambda}v + \gamma\phi_{u,v}v = |v|^{p-2}v + \beta|u|^{\frac{p}{2}}|v|^{\frac{p}{2}-2}v & \text{in } \mathbb{R}^3. \end{cases} \quad (38)$$

Similar to the argument in Lemma 4.1, one can show that (u_λ, v_λ) is a vectorial action ground state of system (38). It remains to prove that $\tilde{\lambda} = \lambda$. Let (u, v) be a vectorial action ground state of system (7). It follows from item (i) that

$$-\frac{\tilde{\lambda}}{6}c(\lambda) = I_{\gamma,\beta}(u_\lambda, v_\lambda) \leq I_{\gamma,\beta}(u, v) = -\frac{\lambda}{6}c(\lambda),$$

which implies that $\lambda \leq \tilde{\lambda}$. Set $(u_\lambda^*, v_\lambda^*) := (\varpi^2 u_\lambda(\varpi x), \varpi^2 v_\lambda(\varpi x))$, where $\varpi := \sqrt{\frac{\lambda}{\tilde{\lambda}}}$. Then we have

$$\frac{\lambda}{3}c(\lambda) = E_{\lambda,\gamma,\beta}(u, v) \leq E_{\lambda,\gamma,\beta}(u_\lambda^*, v_\lambda^*) = \frac{\lambda}{3}(\|u_\lambda^*\|_2^2 + \|v_\lambda^*\|_2^2) = \frac{\lambda\varpi}{3}c(\lambda),$$

which implies that $\tilde{\lambda} \leq \lambda$. Hence, $\tilde{\lambda} = \lambda$. We complete the proof. ■

Now we give the proof of Theorem 1.8: This is an immediate consequence of Lemma 4.1 and Proposition 4.2.

5 The existence of local minimizer

For each $(u, v) \in S_1$ and $t > 0$, we set $(u, v)_t := (u_t, v_t) = (t^{3/2}u(tx), t^{3/2}v(tx))$. Then $c^{1/2}(u, v)_t \in S_c$. Define the fibering map $\Phi_{c,(u,v)}(t)$ given by

$$\Phi_{c,(u,v)}(t) := I_{\gamma,\beta}(c^{1/2}(u, v)_t) = \frac{t^2}{2}cA(u, v) + \frac{\gamma t}{4}c^2B(u, v) - \frac{t^{\frac{3p-6}{2}}}{p}c^{\frac{p}{2}}C(u, v) \text{ for } t > 0.$$

By calculating the first and second derivatives of $\Phi_{c,(u,v)}(t)$, we have

$$\Phi'_{c,(u,v)}(t) = tcA(u, v) + \frac{\gamma}{4}c^2B(u, v) - \frac{3(p-2)}{2p}c^{\frac{p}{2}}C(u, v)t^{\frac{3p-8}{2}}$$

and

$$\Phi''_{c,(u,v)}(t) = cA(u, v) - \frac{3(p-2)(3p-8)}{4p}c^{\frac{p}{2}}C(u, v)t^{\frac{3p-10}{2}}.$$

Notice that $\frac{d}{dt}I_{\gamma,\beta}(c^{1/2}(u, v)_t) = \frac{d}{dt}\Phi_{c,(u,v)}(t) = \frac{\mathcal{P}_{\gamma,\beta}((u,v)_t)}{t}$, where $\mathcal{P}_{\gamma,\beta}(u, v)$ corresponds to a Pohozaev type identity (14). We also recognize that for any $(u, v) \in S_1$, the dilated function $c^{1/2}(u, v)_t$ belongs to the constraint manifold $\mathcal{M}_{\gamma,\beta}(c)$ if and only if $t \in \mathbb{R}$ is a critical value of the fibering map $t \in (0, \infty) \mapsto \Phi_{c,(u,v)}(t)$, namely, $\Phi'_{c,(u,v)}(t) = 0$. Thus, following the idea of Soave [36] (or

Siciliano and Silva [33]), it is natural to split $\mathcal{M}_{\gamma,\beta}(c)$ into three parts corresponding to local minima, points of inflection and local maxima. Thus we define

$$\begin{aligned}\mathcal{M}_{\gamma,\beta}^+(c) &: = \{c^{1/2}(u, v) \in S_c : \Phi'_{c,(u,v)}(1) = 0, \Phi''_{c,(u,v)}(1) > 0\}; \\ \mathcal{M}_{\gamma,\beta}^-(c) &: = \{c^{1/2}(u, v) \in S_c : \Phi'_{c,(u,v)}(1) = 0, \Phi''_{c,(u,v)}(1) < 0\}; \\ \mathcal{M}_{\gamma,\beta}^0(c) &: = \{c^{1/2}(u, v) \in S_c : \Phi'_{c,(u,v)}(1) = 0, \Phi''_{c,(u,v)}(1) = 0\}.\end{aligned}$$

Furthermore, we have the following lemma.

Lemma 5.1 *If $\mathcal{M}_{\gamma,\beta}^0(c) = \emptyset$, then $\mathcal{M}_{\gamma,\beta}(c)$ is a smooth submanifold in S_c with codimension 1, and it is a natural constraint in $\mathbf{H}$.*

For $(u, v) \in \mathbf{H}$, by (12), the Cauchy-Schwarz inequality and the semigroup property of the Riesz potential, we have

$$\begin{aligned}& B(u, v) \\&= \int_{\mathbb{R}^3} \phi_{u,v}(u^2 + v^2) dx \\&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|} dx dy + 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)v^2(y)}{|x-y|} dx dy \\&\leq \mathcal{B}(\|u\|_2^3 \|\nabla u\|_2 + \|v\|_2^3 \|\nabla v\|_2) + 2 \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|} dx dy \right)^{\frac{1}{2}} \\&\leq \mathcal{B}(\|u\|_2^3 + \|v\|_2^3) (\|\nabla u\|_2 + \|\nabla v\|_2) + 2\mathcal{B}\|\nabla u\|_2^{\frac{1}{2}}\|u\|_2^{\frac{3}{2}}\|\nabla v\|_2^{\frac{1}{2}}\|v\|_2^{\frac{3}{2}} \\&\leq 4\mathcal{B}(\|u\|_2^2 + \|v\|_2^2)^{\frac{3}{2}} A(u, v)^{\frac{1}{2}} + \frac{\mathcal{B}}{2} (\|u\|_2^2 + \|v\|_2^2)^{\frac{3}{2}} A(u, v)^{\frac{1}{2}} \\&\leq \frac{9\mathcal{B}}{2} (\|u\|_2^2 + \|v\|_2^2)^{\frac{3}{2}} A(u, v)^{\frac{1}{2}}.\end{aligned}\tag{39}$$

It follows from (13) and (39) that there exists a constant $K(\gamma, \beta, p) > 0$ such that

$$\begin{aligned}\inf_{(u,v) \in S_1} \mathcal{R}_p(u, v) &= \inf_{(u,v) \in S_1} \frac{A(u, v)^{\frac{3p-8}{4(p-3)}} (\gamma B(u, v))^{\frac{10-3p}{4(p-3)}}}{C(u, v)^{\frac{1}{2(p-3)}}} \\&\geq K(\gamma, \beta, p) \text{ for } \frac{10}{3} \leq p < 6,\end{aligned}$$

which implies that c_* and c^* defined as (10) are both positive. Then we have the following two results.

Lemma 5.2 *Let $\gamma, \beta > 0$. The following statements are true.*

- (i) *If $p = 10/3$, then $\mathcal{M}_{\gamma,\beta}(c) = \mathcal{M}_{\gamma,\beta}^-(c) \neq \emptyset$ for $c > c_*$ while $\mathcal{M}_{\gamma,\beta}(c) = \emptyset$ for $0 < c < c_*$.*
- (ii) *If $10/3 < p < 6$, then $\mathcal{M}_{\gamma,\beta}(c) = \mathcal{M}_{\gamma,\beta}^-(c) \neq \emptyset$.*

Proof. (i) Since $c > c_*$, there exists $(u, v) \in S_1$ such that

$$\frac{c}{2} A(u, v) < \frac{3c^{5/3}}{10} C(u, v),$$

and so $\varphi_{c,(u,v)}$ has only one critical point at $t_c^-(u,v)$ which is a global maximum with $\varphi_{c,(u,v)}''(t_c^-(u,v)) < 0$. This implies that $\mathcal{M}_{\gamma,\beta}(c) = \mathcal{M}_{\gamma,\beta}^-(c) \neq \emptyset$. While if $0 < c < c_*$, then for any $(u,v) \in S_1$, we have

$$\frac{c}{2}A(u,v) - \frac{3c^{5/3}}{10}C(u,v) \geq 0,$$

and $\varphi_{c,(u,v)}$ is strictly increasing and has no critical points.

(ii) The proof is similar to that of item (ii), we omit it here. ■

Lemma 5.3 *Let $\gamma, \beta > 0$. The following statements are true.*

(i) *The functional $I_{\gamma,\beta}$ is bounded from below by a positive constant on $\mathcal{M}_{\gamma,\beta}(c) = \mathcal{M}_{\gamma,\beta}^-(c)$ for either $p = \frac{10}{3}$ and $c > c_*$, or $\frac{10}{3} < p < 6$ and $c > 0$.*

(ii) *For every minimizing sequence $\{u_n, v_n\} \subset \mathcal{M}_{\gamma,\beta}(c)$ of $m_{\gamma,\beta}(c)$, there holds $D_1 \leq A(u_n, v_n) \leq D_2$ for some $D_1, D_2 > 0$.*

Proof. (i) For $(u,v) \in \mathcal{M}_{\gamma,\beta}(c)$, we have

$$A(u,v) + \frac{\gamma}{4}B(u,v) - \frac{3(p-2)}{2p}C(u,v) = 0. \quad (40)$$

When $p = \frac{10}{3}$, for $c > c_*$ it follows from (40) that

$$I_{\gamma,\beta}(u,v) = \frac{3}{10}C(u,v) - \frac{1}{2}A(u,v) > 0.$$

When $\frac{10}{3} < p < 6$, by (13), one has

$$A(u,v) \geq \left[\frac{p}{3(2+\beta)(p-2)\mathcal{S}_p} \right]^{\frac{4}{3p-10}} c^{-\frac{6-p}{3p-10}},$$

and together with (40), leading to

$$\begin{aligned} I_{\gamma,\beta}(u,v) &= \frac{3p-10}{6(p-2)}A(u,v) + \frac{3p-8}{12(p-2)}\gamma B(u,v) \\ &\geq \frac{3p-10}{6(p-2)}A(u,v) \\ &> \frac{3p-10}{6(p-2)} \left[\frac{p}{3(2+\beta)(p-2)\mathcal{S}_p} \right]^{\frac{4}{3p-10}} c^{-\frac{6-p}{3p-10}} > 0. \end{aligned}$$

(ii) Let $\{(u_n, v_n)\} \subset \mathcal{M}_{\gamma,\beta}(c)$ be a minimizing sequence of $m_{\gamma,\beta}(c)$. For $\frac{10}{3} < p < 6$, $\{(u_n, v_n)\}$ is bounded in $\mathbf{H}$, since $I_{\gamma,\beta}$ is coercive on $\mathcal{M}_{\gamma,\beta}(c)$. For $p = \frac{10}{3}$, we assume on the contrary that $A(u_n, v_n) \rightarrow +\infty$ as $n \rightarrow \infty$. Since $\mathcal{P}_{\gamma,\beta}(u_n, v_n) = o_n(1)$, we have

$$m_{\gamma,\beta}(c) = I_{\gamma,\beta}(u_n, v_n) = \frac{1}{8}B(u_n, v_n), \quad (41)$$

and

$$\lim_{n \rightarrow \infty} \frac{\frac{3}{5}C(u_n, v_n)}{A(u_n, v_n)} = 1.$$

Set $\varepsilon_n := [A(u_n, v_n)]^{-1/2}$ and $(\tilde{u}_n(x), \tilde{v}_n(x)) := (\varepsilon_n^{3/2} u_n(\varepsilon_n x), \varepsilon_n^{3/2} v_n(\varepsilon_n x))$. It is clear that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and $(\tilde{u}_n(x), \tilde{v}_n(x)) \in S_c$. Moreover, we have

$$A(\tilde{u}_n, \tilde{v}_n) = 1 \quad \text{and} \quad C(\tilde{u}_n, \tilde{v}_n) \rightarrow \frac{5}{3} \text{ as } n \rightarrow \infty. \quad (42)$$

Then $\{(\tilde{u}_n, \tilde{v}_n)\}$ is bounded in $\mathbf{H}$, and from (41) it follows that

$$B(\tilde{u}_n, \tilde{v}_n) = \varepsilon_n B(u_n, v_n) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (43)$$

Let

$$\delta := \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} (\tilde{u}_n^2 + \tilde{v}_n^2) dx \geq 0.$$

If $\delta = 0$, then it follows from Vanishing lemma in [25, Lemma I.1] that $\{(\tilde{u}_n, \tilde{v}_n)\} \rightarrow (0, 0)$ in $L^{\frac{10}{3}}(\mathbb{R}^3) \times L^{\frac{10}{3}}(\mathbb{R}^3)$, which contradicts with (42). So $\delta > 0$ and there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n)} (\tilde{u}_n^2 + \tilde{v}_n^2) dx \geq \frac{\delta}{2} > 0.$$

Let $(u_n^*, v_n^*) := (\tilde{u}_n(x + y_n), \tilde{v}_n(x + y_n))$. Then we have $A(u_n^*, v_n^*) = A(\tilde{u}_n, \tilde{v}_n)$ and

$$\int_{B_1(0)} ((u_n^*)^2 + (v_n^*)^2) dx \geq \frac{\delta}{2},$$

which implies that $(u_n^*, v_n^*) \rightharpoonup (u^*, v^*) \neq (0, 0)$ in $\mathbf{H}$. By the Fatou's lemma and (43), we obtain that

$$0 < B(u^*, v^*) \leq \liminf_{n \rightarrow \infty} B(u_n^*, v_n^*) = \liminf_{n \rightarrow \infty} B(\tilde{u}_n, \tilde{v}_n) = 0,$$

which is impossible and so $\{(u_n, v_n)\}$ is bounded in $\mathbf{H}$. Moreover, it is easy to prove that the sequence $\{(u_n, v_n)\}$ can not vanish. In fact, if $\{(u_n, v_n)\}$ can vanish, then $I_{\gamma, \beta}(u_n, v_n) \rightarrow 0$ as $n \rightarrow \infty$, which contradicts with $m_{\gamma, \beta}(c) > 0$. We complete the proof. ■

Lemma 5.4 ([33, Proposition 3.1]) *Assume that $b \neq 0$, $dl - bg \neq 0$, $(be - al)/(dl - bg) > 0$, $(ag - de)(dl - bg) > 0$, $A, B, G > 0$ and $p \in (2, 6) \setminus \{3\}$. Then the following system*

$$\begin{cases} aAt + bBr + dGr^{\frac{p-2}{2}} t^{\frac{3p-8}{2}} = 0 \\ eAt + lBr + gGr^{\frac{p-2}{2}} t^{\frac{3p-8}{2}} = 0 \end{cases}$$

admits a unique positive solution (r, t) . Moreover, we have

$$r = \left(\frac{be - al}{dl - bg} \right)^{\frac{1}{2(p-3)}} \left(\frac{ag - de}{dl - bg} \right)^{\frac{3p-10}{4(p-3)}} \frac{A^{\frac{3p-8}{4(p-3)}} B^{\frac{3p-10}{4(p-3)}}}{G^{\frac{1}{2(p-2)}}}.$$

Fix $(u, v) \in S_1$ and consider the map $\hat{f}(c) := \Phi_{c, (u, v)}(t_c^-(u, v))$, where $t_c^-(u, v)$ is the unique critical point of $\Phi_{c, (u, v)}$. For simplicity, we denote $t_c = t_c^-(u, v)$. By Lemma 5.2, the map $\hat{f}$ is well-defined and

$$\hat{f}'(c) = \Phi'_{c, (u, v)}(t_c) + \frac{1}{2} \left[(t_c)^2 A(u, v) + ct_c(u) \gamma B(u, v) - c^{\frac{p-2}{2}} (t_c)^{\frac{3p-6}{2}} C(u, v) \right].$$

Thus $\hat{f}'(c) = 0$ if and only if

$$\begin{cases} A(u, v)t_c + \frac{\gamma}{4}B(u, v)c - \frac{3(p-2)}{2p}C(u, v)c^{\frac{p-2}{2}}(t_c)^{\frac{3p-8}{2}} = 0 \\ A(u, v)t_c + \gamma B(u, v)c - C(u, v)c^{\frac{p-2}{2}}(t_c)^{\frac{3p-8}{2}} = 0. \end{cases} \quad (44)$$

By Lemma 5.4, system (44) has a unique positive solution (c^*, t^*) , where c^* is as (10). In particular, when $p = 10/3$, it is clear that

$$c^* = \left(\frac{15}{7}\right)^{\frac{3}{2}} \inf_{(u,v) \in S_1} \mathcal{R}_{10/3}(u, v) = \left(\frac{9}{7}\right)^{\frac{3}{2}} c_*.$$

Lemma 5.5 *Let $\gamma, \beta > 0$. For $(u, v) \in S_1$, the following statements are true.*

- (i) *If $p = \frac{10}{3}$, then the function $\hat{f}(c)$ is decreasing on $c \in (c_*, c^*)$.*
- (ii) *If $\frac{10}{3} < p < 6$, then the function $\hat{f}(c)$ is decreasing on $c \in (0, c^*)$.*

Proof. For $(u, v) \in S_1$, by system (44), we have

$$t_c^2 A(u, v) + ct_c \gamma B(u, v) - c^{\frac{p-2}{2}} t_c^{\frac{3p-6}{2}} C(u, v) = t_c^2 h(c), \quad (45)$$

where

$$h(c) := -3A(u, v) + \frac{5p-12}{p} c^{\frac{p-2}{2}} t_c^{\frac{3p-10}{2}} C(u, v).$$

According to (45), we note that $2\hat{f}'(c) = t_c^2 h(c)$, which implies that the monotonicity of $\hat{f}(c)$ depends on the sign of $h(c)$.

- (i) When $p = \frac{10}{3}$, we have $h(c) < 0$ for $c_* < c < c^*$. This shows that the function $\hat{f}(c)$ is decreasing on $c \in (c_*, c^*)$.
- (ii) When $\frac{10}{3} < p < 6$, since t_c is continuous and $\hat{f}$ has a unique critical point, it is sufficient to show that there exists $0 < c_1 < c^*$ such that $h(c_1) < 0$. Now we claim that

$$\lim_{c \rightarrow 0} h(c) < 0. \quad (46)$$

Indeed, if t_c is bounded from above as $c \rightarrow 0$, then (46) is obvious. If $t_c \rightarrow \infty$ as $c \rightarrow 0$, by (44), we have

$$A(u, v) - \frac{3(p-2)}{2p} c^{\frac{p-2}{2}} t_c^{\frac{3p-10}{2}} C(u, v) = o_c(1),$$

which shows that

$$\begin{aligned} h(c) &= -3A(u, v) + \frac{5p-12}{p} c^{\frac{p-2}{2}} t_c^{\frac{3p-10}{2}} C(u, v) \\ &= -3A(u, v) + \frac{5p-12}{p} \frac{2p}{3(p-2)} A(u, v) + o_c(1) \\ &= \frac{p-6}{3p-6} A(u, v) + o_c(1) \\ &< 0 \text{ as } c \rightarrow 0. \end{aligned}$$

So, the claim holds. This implies that there exists $0 < c_1 < c^*$ such that $h(c_1) < 0$. The proof is complete. ■

Lemma 5.6 *Let $\gamma, \beta > 0$. The following statements are true.*

- (i) *If $\frac{10}{3} < p < 6$, then the function $m_{\gamma, \beta}(c)$ is decreasing on $c \in (0, c^*)$.*
- (ii) *If $p = \frac{10}{3}$, then the function $m_{\gamma, \beta}(c)$ is decreasing on $c \in (c_*, c^*)$.*

Proof. (i) Fix $0 < \rho < \tau < c^*$ and let $\{(u_n, v_n)\} \subset \mathcal{M}_{\gamma, \beta}^-(\rho)$ be a minimizing sequence to $m_{\gamma, \beta}(\rho)$. By Lemma 5.5, we have $'(c) < 0$ on $c \in (0, c^*)$. Next, we further prove that $\hat{f}'(c) < -D$ for some $D > 0$. Suppose on the contrary, then there exists a sequence $\{(\bar{u}_n, \bar{v}_n)\} \subset S_1$ satisfying $\|\nabla \bar{u}_n\|_2^2 + \|\nabla \bar{v}_n\|_2^2 = 1$ and corresponding sequences $\{t_n\} \subset (0, +\infty)$, $\{c_n\} \subset [\rho, \tau]$ such that

$$\begin{cases} t_n A(\bar{u}_n, \bar{v}_n) + \frac{c_n}{4} \gamma B(\bar{u}_n, \bar{v}_n) - \frac{3(p-2)}{2p} c_n^{\frac{p-2}{2}} t_n^{\frac{3p-8}{2}} C(\bar{u}_n, \bar{v}_n) = o_n(1) \\ t_n A(\bar{u}_n, \bar{v}_n) + c_n \gamma B(\bar{u}_n, \bar{v}_n) - c_n^{\frac{p-2}{2}} t_n^{\frac{3p-8}{2}} C(\bar{u}_n, \bar{v}_n) = o_n(1). \end{cases}$$

By Lemma 5.3, we conclude that $\{t_n\}$ is bounded and

$$c_n = \left[\frac{2(6-p)}{5p-12} \right]^{\frac{3p-10}{4(p-3)}} \left(\frac{3p}{5p-12} \right)^{\frac{1}{2(p-3)}} R_p(u_n, v_n) \geq c^*,$$

which is a contradiction. Thus, the claim is proved. So by the mean value theorem, we have

$$\begin{aligned} m_{\gamma, \beta}(\tau) &\leq \Phi_{\tau, (u_n, v_n)}(t_\tau^-(u_n, v_n)) = \Phi_{\rho, (u_n, v_n)}(t_\rho^-(u_n, v_n)) + \hat{f}'(\xi)(\tau - \rho) \\ &< m_{\gamma, \beta}(\rho) - D(\tau - \rho) \\ &< m_{\gamma, \beta}(\rho), \end{aligned}$$

where $\xi \in (\rho, \tau)$. This shows that the function $m_{\gamma, \beta}(c)$ is decreasing on $c \in (0, c^*)$.

(ii) The proof is similar to that of (i), we omit it here. ■

Next, we recall the so-called profile decomposition of bounded sequences, proposed by Gérard in [14], which is crucial to recover the compactness.

Proposition 5.7 *Let $\{(u_n, v_n)\}$ be bounded in $\mathbf{H}$. Then there are sequences $\{(\bar{u}_i, \bar{v}_i)\}_{i=0}^\infty \subset \mathbf{H}$, $\{y_n^i\}_{i=0}^\infty \subset \mathbb{R}^3$ for any $n \geq 1$, such that $y_n^0 = 0$, $|y_n^i - y_n^j| \rightarrow \infty$ as $n \rightarrow \infty$ for $i \neq j$, and passing to a subsequence, the following conclusions hold for any $i \geq 0$:*

$$(u_n(\cdot + y_n^i), v_n(\cdot + y_n^i)) \rightharpoonup (\bar{u}_i, \bar{v}_i) \quad \text{as } n \rightarrow \infty, \quad (47)$$

with $\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} (|z_{1,n}^i|^q + |z_{2,n}^i|^q) dx \rightarrow 0$ as $i \rightarrow \infty$, where $2 < q < 6$, $z_{1,n}^i := u_n - \sum_{j=0}^i \bar{u}_j(\cdot - y_n^j)$

and $z_{2,n}^i := v_n - \sum_{j=0}^i \bar{v}_j(\cdot - y_n^j)$. Moreover, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = \sum_{j=0}^i \int_{\mathbb{R}^3} |\nabla \bar{u}_j|^2 dx + \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla z_{1,n}^i|^2 dx, \quad (48)$$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla v_n|^2 dx = \sum_{j=0}^i \int_{\mathbb{R}^3} |\nabla \bar{v}_j|^2 dx + \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla z_{2,n}^i|^2 dx, \quad (49)$$

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} (|u_n|^p + |v_n|^p + 2\beta |u_n|^{\frac{p}{2}} |v_n|^{\frac{p}{2}}) dx = \sum_{j=0}^{\infty} \int_{\mathbb{R}^3} (|\bar{u}_j|^p + |\bar{v}_j|^p + 2\beta |\bar{u}_j|^{\frac{p}{2}} |\bar{v}_j|^{\frac{p}{2}}) dx, \quad (50)$$

and

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \phi_{u_n, v_n} (u_n^2 + v_n^2) dx = \sum_{j=0}^{\infty} \int_{\mathbb{R}^3} \phi_{\bar{u}_j, \bar{v}_j} (\bar{u}_j^2 + \bar{v}_j^2) dx. \quad (51)$$

Proof. The proof is based on Vanishing Lemma [25, Lemma I.1] and Brezis-Lieb type results. For the proofs of (47)-(50), we refer the reader to [14, 18, 26, 43]. Here, we only need to prove the nonlocal coupled term (51) holds. Without loss of generality, we may assume that $\bar{u}_i, \bar{v}_i$ is continuous and compactly supported. For every $i \neq j$, the sequence $\{y_n^i\}_{i=0}^{\infty}$ satisfies $|y_n^i - y_n^j| \rightarrow \infty$ as $n \rightarrow \infty$, then we call this sequence $\{y_n^i\}_{i=0}^{\infty} \subset \mathbb{R}^3$ satisfying the orthogonality condition.

Since the sequence $\{(u_n, v_n)\}$ can be written, up to a subsequence, as

$$(u_n, v_n) = \left(\sum_{j=0}^i \bar{u}_j(\cdot - y_n^j) + z_{1,n}^i, \sum_{j=0}^i \bar{v}_j(\cdot - y_n^j) + z_{2,n}^i \right),$$

and $\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} (|z_{1,n}^i|^q + |z_{2,n}^i|^q) dx \rightarrow 0$ as $i \rightarrow \infty$, we claim that

$$B(u_n, v_n) \rightarrow B \left(\sum_{j=0}^i \bar{u}_j(\cdot - y_n^j), \sum_{j=0}^i \bar{v}_j(\cdot - y_n^j) \right) \text{ as } n \rightarrow \infty \text{ and } i \rightarrow \infty. \quad (52)$$

Taking $\bar{u}_{j,n} = \bar{u}_j(\cdot - y_n^j)$ and $\bar{v}_{j,n} = \bar{v}_j(\cdot - y_n^j)$, to prove (52) is equivalent to prove that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n|^2 |u_n|^2}{|x - y|} dx dy \rightarrow \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\sum_{j=0}^i \bar{u}_{j,n}|^2 |\sum_{j=0}^i \bar{u}_{j,n}|^2}{|x - y|} dx dy, \quad (53)$$

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|v_n|^2 |v_n|^2}{|x - y|} dx dy \rightarrow \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\sum_{j=0}^i \bar{v}_{j,n}|^2 |\sum_{j=0}^i \bar{v}_{j,n}|^2}{|x - y|} dx dy, \quad (54)$$

and

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n|^2 |v_n|^2}{|x - y|} dx dy \rightarrow \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\sum_{j=0}^i \bar{u}_{j,n}|^2 |\sum_{j=0}^i \bar{v}_{j,n}|^2}{|x - y|} dx dy, \quad (55)$$

as $n \rightarrow \infty$ and $i \rightarrow \infty$. Indeed, to prove (53), we only need to obtain the following estimates:

$$\lim_{i \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|z_{1,n}^i|^2 |z_{1,n}^i|^2}{|x - y|} dx dy = 0, \quad (56)$$

$$\lim_{i \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{u}_{j,n}|^2 |z_{1,n}^i|^2}{|x - y|} dx dy = 0, \quad (57)$$

$$\lim_{i \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{u}_{j,n}|^3 |z_{1,n}^i|}{|x - y|} dx dy = 0, \quad (58)$$

$$\lim_{i \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{u}_{j,n}| |z_{1,n}^i|^3}{|x-y|} dx dy = 0. \quad (59)$$

For (56), it follows from Lemma 2.3 that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|z_{1,n}^i|^2 |z_{1,n}^i|^2}{|x-y|} dx dy \leq \mathcal{C}_{HLS} \|z_{1,n}^i\|_{12/5}^2 \|z_{1,n}^i\|_{12/5}^2 \leq \mathcal{C}_{HLS} \|z_{1,n}^i\|_{12/5}^4.$$

Since $\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} (|z_{1,n}^i|^q + |z_{2,n}^i|^q) dx \rightarrow 0$ as $i \rightarrow \infty$, we have $\lim_{i \rightarrow \infty} \limsup_{n \rightarrow \infty} \|z_{1,n}^i\|_{12/5}^4 = 0$, and thus (56) is true. For (57), by Lemma 2.3 and Sobolev embedding theorem, we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{u}_{j,n}|^2 |z_{1,n}^i|^2}{|x-y|} dx dy \leq \mathcal{C}_{HLS} \|\bar{u}_{j,n}\|_{12/5}^2 \|z_{1,n}^i\|_{12/5}^2 \leq C \|\bar{u}_{j,n}\|_{H^1}^2 \|z_{1,n}^i\|_{12/5}^2,$$

which implies that (57) holds, since $\|\bar{u}_{j,n}\|_{H^1}$ is bounded. For (58) and (59), we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{u}_{j,n}|^3 |z_{1,n}^i|}{|x-y|} dx dy \leq \mathcal{C}_{HLS} \|\bar{u}_{j,n}\|_{12/5}^3 \|z_{1,n}^i\|_{12/5} \leq C \|\bar{u}_{j,n}\|_{H^1}^3 \|z_{1,n}^i\|_{12/5},$$

and

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{u}_{j,n}| |z_{1,n}^i|^3}{|x-y|} dx dy \leq \mathcal{C}_{HLS} \|\bar{u}_{j,n}\|_{12/5} \|z_{1,n}^i\|_{12/5}^3 \leq C \|\bar{u}_{j,n}\|_{H^1} \|z_{1,n}^i\|_{12/5}^3.$$

Then, similar to the arguments of (56) and (57), we can get (58) and (59). So (53) holds. Moreover, the proofs of (54) and (55) are similar, we omit it here. Hence, (52) holds.

Finally, to obtain (51), it is sufficient to prove that

$$B \left(\sum_{j=0}^{\infty} \bar{u}_j(\cdot - y_n^j), \sum_{j=0}^{\infty} \bar{v}_j(\cdot - y_n^j) \right) = \sum_{j=0}^{\infty} B(\bar{u}_j(\cdot - y_n^j), \bar{v}_j(\cdot - y_n^j)) + o_n(1).$$

By the pairwise orthogonality of the family $\{y_n^i\}_{i=0}^{\infty}$, and the following the elementary inequality [14]

$$\left| \left| \sum_{j=0}^{\infty} a_j \right|^2 - \sum_{j=0}^{\infty} |a_j|^2 \right| \leq C \sum_{j \neq k} |a_j| |a_k|,$$

we have

$$\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\sum_{j=0}^{\infty} \bar{u}_j(x - y_n^j)|^2 |\sum_{j=0}^{\infty} \bar{u}_j(y - y_n^j)|^2}{|x-y|} dx dy - \sum_{j=0}^{\infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{u}_j(x - y_n^j)|^2 |\bar{u}_j(y - y_n^j)|^2}{|x-y|} dx dy \right|$$

$$\leq \sum_{j=0}^{\infty} \sum_{k \neq j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{u}_j(x - y_n^j)| |\bar{u}_k(x - y_n^k)| |\sum_{m=0}^{\infty} \bar{u}_m(y - y_n^m)|^2}{|x-y|} dx dy \quad (60)$$

$$+ \sum_{j=0}^{\infty} \sum_{k \neq j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{u}_j(y - y_n^j)| |\bar{u}_k(y - y_n^k)| |\sum_{m=0}^{\infty} \bar{u}_m(x - y_n^m)|^2}{|x-y|} dx dy \quad (61)$$

$$+ \sum_{j=0}^{\infty} \sum_{k \neq j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{u}_j(x - y_n^j)|^2 |\bar{u}_k(y - y_n^k)|^2}{|x-y|} dx dy. \quad (62)$$

Next, we estimate (60), (61) and (62), respectively. It follows from Lemma 2.3 and orthogonality that

$$(60) \leq \sum_{j=0}^{\infty} \sum_{k \neq j} \left\| |\bar{u}_j(\cdot - y_n^j)| |\bar{u}_k(\cdot - y_n^k)| \right\|_{6/5} \left\| \left| \sum_{m=0}^{\infty} \bar{u}_m(\cdot - y_n^m) \right|^2 \right\|_{6/5} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Similarly, we get (61) $\rightarrow 0$ as $n \rightarrow \infty$. For (62), by using the transformation $\tilde{x} := x - y_n^j$ and $\tilde{y} := y - y_n^j$, we have

$$\begin{aligned} (62) &= \sum_{j=0}^{\infty} \sum_{k \neq j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{u}_j(\tilde{x})|^2 |\bar{u}_k(\tilde{y} - (y_n^k - y_n^j))|^2}{|\tilde{y} - \tilde{x}|} d\tilde{x} d\tilde{y} \\ &\leq \sum_{j=0}^{\infty} \sum_{k \neq j} \|\bar{u}_j\|_{12/5}^2 \|\bar{u}_k(\cdot - (y_n^k - y_n^j))\|_{12/5}^2 \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence, we get

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left| \sum_{j=0}^{\infty} \bar{u}_j(x - y_n^j) \right|^2 \left| \sum_{j=0}^{\infty} \bar{u}_j(y - y_n^j) \right|^2}{|x - y|} dx dy \rightarrow \sum_{j=0}^{\infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{u}_j(x - y_n^j)|^2 |\bar{u}_j(y - y_n^j)|^2}{|x - y|} dx dy$$

as $n \rightarrow \infty$. Similarly,

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left| \sum_{j=0}^{\infty} \bar{v}_j(x - y_n^j) \right|^2 \left| \sum_{j=0}^{\infty} \bar{v}_j(y - y_n^j) \right|^2}{|x - y|} dx dy \rightarrow \sum_{j=0}^{\infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{v}_j(x - y_n^j)|^2 |\bar{v}_j(y - y_n^j)|^2}{|x - y|} dx dy,$$

and

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left| \sum_{j=0}^{\infty} \bar{u}_j(x - y_n^j) \right|^2 \left| \sum_{j=0}^{\infty} \bar{v}_j(y - y_n^j) \right|^2}{|x - y|} dx dy \rightarrow \sum_{j=0}^{\infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\bar{u}_j(x - y_n^j)|^2 |\bar{v}_j(y - y_n^j)|^2}{|x - y|} dx dy$$

as $n \rightarrow \infty$. Therefore, (51) holds. We complete the proof. $\blacksquare$

Now we give the proofs of Theorems 1.10 and 1.11: Let $\{(u_n, v_n)\} \subset \mathcal{M}_{\gamma, \beta}^-(c)$ be a bounded and non-vanishing minimizing sequence to $m_{\gamma, \beta}(c)$. By applying Proposition 5.7, we can find a profile decomposition of $\{(u_n, v_n)\}$ satisfying (47)-(51). Define the index set

$$I := \{i \geq 0 : (\bar{u}_i, \bar{v}_i) \neq (0, 0)\}.$$

By using Lemma 5.3, (50) and (51), we get that $I \neq \emptyset$. Now, we claim that $\mathcal{P}_{\gamma, \beta}(\bar{u}_i, \bar{v}_i) \leq 0$ for

some $i \geq 0$. Otherwise, if $\mathcal{P}_{\gamma,\beta}(\bar{u}_i, \bar{v}_i) > 0$ for all $i \in I$. Then it follows from (50) and (51) that

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \left[\frac{3(p-2)}{2p} C(u_n, v_n) - \frac{\gamma}{4} B(u_n, v_n) \right] &= \limsup_{n \rightarrow \infty} A(u_n, v_n) \\
&\geq \sum_{i=0}^{\infty} A(\bar{u}_i, \bar{v}_i) = \sum_{i \in I} A(\bar{u}_i, \bar{v}_i) \\
&> \sum_{i=0}^{\infty} \left[\frac{3(p-2)}{2p} C(\bar{u}_i, \bar{v}_i) - \frac{\gamma}{4} B(\bar{u}_i, \bar{v}_i) \right] \\
&= \limsup_{n \rightarrow \infty} \left[\frac{3(p-2)}{2p} C(u_n, v_n) - \frac{\gamma}{4} B(u_n, v_n) \right],
\end{aligned}$$

which is a contradiction.

Let $i \in I$ and, for simplicity, let us denote $(\bar{u}, \bar{v}) := (\bar{u}_i, \bar{v}_i) \neq (0, 0)$. Now we try to prove that $(\bar{u}, \bar{v}) \in S_c$. Assume on the contrary. Then $\|\bar{u}\|_2^2 + \|\bar{v}\|_2^2 =: \rho < c < c^*$. Moreover, we claim that $\rho > c_*$ for $p = 10/3$. Indeed, since $\mathcal{P}_{\gamma,\beta}(\bar{u}, \bar{v}) \leq 0$, we have

$$A(\bar{u}, \bar{v}) < A(\bar{u}, \bar{v}) + \frac{\gamma}{4} B(\bar{u}, \bar{v}) \leq \frac{3}{5} C(\bar{u}, \bar{v}),$$

which implies that

$$\frac{A(\bar{u}, \bar{v})}{C(\bar{u}, \bar{v})} < \frac{3}{5}.$$

Set

$$(\check{u}, \check{v}) := \frac{1}{\sqrt{\rho}}(\bar{u}, \bar{v}) \in S_1.$$

Then we have

$$\begin{aligned}
\rho &> \rho \left(\frac{5}{3} \right)^{\frac{3}{2}} \left[\frac{A(\bar{u}, \bar{v})}{C(\bar{u}, \bar{v})} \right]^{\frac{3}{2}} = \left(\frac{5}{3} \right)^{\frac{3}{2}} \left[\frac{A(\check{u}, \check{v})}{C(\check{u}, \check{v})} \right]^{\frac{3}{2}} \\
&\geq \left(\frac{5}{3} \right)^{\frac{3}{2}} \inf_{(u,v) \in S_1} \left[\frac{A(u, v)}{C(u, v)} \right]^{\frac{3}{2}} = c_*.
\end{aligned}$$

Thus, the claim is true. Again using $\mathcal{P}_{\gamma,\beta}(\bar{u}, \bar{v}) \leq 0$, there exists $t^- \in (0, 1]$ such that $(\bar{u}, \bar{v})_{t^-} \in \mathcal{M}_{\gamma,\beta}^-(\rho)$. Then we have

$$\begin{aligned}
m_{\gamma,\beta}(\rho) \leq I_{\gamma,\beta}((\bar{u}, \bar{v})_{t^-}) &= \frac{3p-10}{6(p-2)} A((\bar{u}, \bar{v})_{t^-}) + \frac{3p-8}{12(p-2)} B((\bar{u}, \bar{v})_{t^-}) \\
&= (t^-)^2 \frac{3p-10}{6(p-2)} A(\bar{u}, \bar{v}) + (t^-) \frac{3p-8}{12(p-2)} B(\bar{u}, \bar{v}) \\
&\leq \frac{3p-10}{6(p-2)} A(\bar{u}, \bar{v}) + \frac{3p-8}{12(p-2)} B(\bar{u}, \bar{v}) \\
&\leq \liminf_{n \rightarrow \infty} \left[\frac{3p-10}{6(p-2)} A(u_n, v_n) + \frac{3p-8}{12(p-2)} B(u_n, v_n) \right] \\
&= m_{\gamma,\beta}(c),
\end{aligned}$$

which is a contradiction. So $\rho = c$, $t^- = 1$ and $I_{\gamma,\beta}(\bar{u}, \bar{v}) = m_{\gamma,\beta}(c)$. This implies that $(u_n(\cdot + y_n^i), v_n(\cdot + y_n^i)) \rightarrow (\bar{u}, \bar{v})$ in $\mathbf{H}$ and therefore I is a singleton set.

Finally, we claim that $\bar{u} \neq 0$ and $\bar{v} \neq 0$. If not, we may assume that $\bar{v} \equiv 0$. Then by Lemma 2.5, there exists $s_\beta \in (0, 1)$ such that $(\sqrt{s_\beta}\bar{u}, \sqrt{1-s_\beta}\bar{u}) \in S_c$. If $p \in [10/3, 4)$ and $\beta > 0$, or $p \in [4, 6)$ and $\beta > 2^{(p-2)/2}$, then for any $t > 0$, we have

$$\begin{aligned} I_{\gamma,\beta}((\bar{u}, 0)_t) &= \frac{t^2}{2} \|\nabla \bar{u}\|_2^2 + \frac{\gamma t}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\bar{u}(x)^2 \bar{u}(y)^2}{|x-y|} dx dy - \frac{t^{\frac{3(p-2)}{2}}}{p} \int_{\mathbb{R}^3} |\bar{u}|^p dx \\ &> \frac{t^2}{2} A(\sqrt{s_\beta}\bar{u}, \sqrt{1-s_\beta}\bar{u}) + \frac{\gamma t}{4} B(\sqrt{s_\beta}\bar{u}, \sqrt{1-s_\beta}\bar{u}) - \frac{t^{\frac{3(p-2)}{2}}}{p} C(\sqrt{s_\beta}\bar{u}, \sqrt{1-s_\beta}\bar{u}) \\ &= I_{\gamma,\beta}((\sqrt{s_\beta}\bar{u}, \sqrt{1-s_\beta}\bar{u})_t). \end{aligned}$$

Passing to the maximum on $t > 0$, then there exists $t_\beta^- > 0$ such that $(\sqrt{s_\beta}\bar{u}, \sqrt{1-s_\beta}\bar{u})_{t_\beta^-} \in \mathcal{M}_{\gamma,\beta}^-(c)$ and

$$I_{\gamma,\beta}((\sqrt{s_\beta}\bar{u}, \sqrt{1-s_\beta}\bar{u})_{t_\beta^-}) < I_{\gamma,\beta}(\bar{u}, 0) = m_{\gamma,\beta}(c),$$

which is a contradiction. So, $(\bar{u}, \bar{v})$ is a vectorial type minimizer of $m_{\gamma,\beta}(c)$.

Finally, we determine the sign of Lagrange multiplier $\bar{\lambda}$. Since $(\bar{u}, \bar{v})$ is a weak solution of system (7), we have

$$A(\bar{u}, \bar{v}) + \bar{\lambda}c + B(\bar{u}, \bar{v}) - C(\bar{u}, \bar{v}) = 0$$

and

$$\mathcal{P}_{\gamma,\beta}(\bar{u}, \bar{v}) = A(\bar{u}, \bar{v}) + \frac{\gamma}{4} B(\bar{u}, \bar{v}) - \frac{3(p-2)}{2p} C(\bar{u}, \bar{v}) = 0.$$

For $p = \frac{10}{3}$, we deduce that

$$\bar{\lambda} = \frac{1}{c} \left[3A(\bar{u}, \bar{v}) - \frac{7}{5} C(\bar{u}, \bar{v}) \right] > 0,$$

where we have used the fact of $c < c^*$. For $\frac{10}{3} < p < 6$, it follows from (13) that

$$\begin{aligned} A(\bar{u}, \bar{v}) - \frac{6(1+\beta)(p-2)\mathcal{S}_p}{p} c^{\frac{6-p}{4}} A(\bar{u}, \bar{v})^{\frac{3(p-2)}{4}} &\leq A(\bar{u}, \bar{v}) - \frac{3(p-2)}{2p} C(\bar{u}, \bar{v}) \\ &= -\frac{\gamma}{4} B(\bar{u}, \bar{v}) \\ &< 0, \end{aligned}$$

which implies that

$$A(\bar{u}, \bar{v}) > \left[\frac{6(1+\beta)(p-2)\mathcal{S}_p}{p} \right]^{-\frac{4}{3p-10}} c^{-\frac{6-p}{3p-10}}.$$

This shows that $A(\bar{u}, \bar{v})$ is large when $c > 0$ sufficiently small. Using this fact, together with (39), we have

$$\begin{aligned} \bar{\lambda}c &= \frac{6-p}{3p-6} A(\bar{u}, \bar{v}) - \frac{5p-12}{2(3p-6)} B(\bar{u}, \bar{v}) \\ &\geq \frac{6-p}{3p-6} A(\bar{u}, \bar{v}) - \frac{9(5p-12)\mathcal{B}}{4(3p-6)} c^{\frac{3}{2}} A(\bar{u}, \bar{v})^{\frac{1}{2}} \\ &> 0 \text{ for } c > 0 \text{ sufficiently small.} \end{aligned}$$

So, $\bar{\lambda} > 0$. The proof is complete.

Proposition 5.8 *Let (u, v) be the energy ground state in Theorem 1.10 or 1.11 with the positive Lagrange multiplier λ , then (u, v) has exponential decay:*

$$|u(x)| \leq Ce^{-\zeta|x|}, |v(x)| \leq Ce^{-\zeta|x|} \text{ for every } x \in \mathbb{R}^3,$$

for some $C > 0, \zeta > 0$.

Proof. Following the idea in [1]. Set $\vartheta(r) := (\vartheta_1(r), \vartheta_2(r))$, where

$$\vartheta_1(r) = \int_{\mathbb{R}^3 \setminus B_r(0)} (|\nabla u|^2 + |u|^2) dx \quad \text{and} \quad \vartheta_2(r) = \int_{\mathbb{R}^3 \setminus B_r(0)} (|\nabla v|^2 + |v|^2) dx.$$

Let $\chi : \mathbb{R}^3 \rightarrow \mathbb{R}$ be given by

$$\chi(r) := \begin{cases} 0 & \text{if } r \leq 0, \\ r & \text{if } 0 < r < 1, \\ 1 & \text{if } r \geq 1. \end{cases}$$

Set $u^r := \chi(|x| - r)u(x)$ and $v^r := \chi(|x| - r)v(x)$ for $r \geq 0$. Then for $x \in B_{r+1}(0) \setminus B_r(0)$, we have

$$u^r(x) = (|x| - r)u(x), \quad v^r(x) = (|x| - r)v(x),$$

and

$$\nabla u^r(x) = (|x| - r)\nabla u(x) + \frac{x}{|x|}u(x), \quad \nabla v^r(x) = (|x| - r)\nabla v(x) + \frac{x}{|x|}v(x).$$

Next, we only consider one component of $\vartheta(r)$ because the other is similar. Set $\varrho := \min\{\lambda, 1\}$. Then we get

$$\begin{aligned} & \int_{\mathbb{R}^3} (\nabla u \nabla u^r + \lambda u u^r) dx \\ & \geq \varrho \vartheta_1(r+1) + \int_{B_{r+1}(0) \setminus B_r(0)} \left[(|x| - r)(|\nabla u|^2 + \lambda |u|^2) + u \frac{x}{|x|} \cdot \nabla u \right] dx \\ & \geq \varrho \vartheta_1(r+1) - \frac{1}{2} \int_{B_{r+1}(0) \setminus B_r(0)} (|\nabla u|^2 + |u|^2) dx \\ & \geq \frac{2\varrho + 1}{2} \vartheta_1(r+1) - \frac{1}{2} \vartheta_1(r). \end{aligned} \tag{63}$$

Since (u, v) solves system (6), we have

$$\begin{aligned} \int_{\mathbb{R}^3} (\nabla u \nabla u^r + \lambda u u^r) dx & \leq \int_{\mathbb{R}^3} (|u|^{p-1} u^r + \beta |v|^{\frac{p}{2}} |u|^{\frac{p}{2}-1} u^r) dx \\ & \leq \int_{\mathbb{R}^3 \setminus B_r} (|u|^p + \beta |v|^{\frac{p}{2}} |u|^{\frac{p}{2}}) dx \\ & \leq \left(1 + \frac{\beta}{2}\right) \int_{\mathbb{R}^3 \setminus B_r} (|u|^p + |v|^p) dx \\ & \leq \left(1 + \frac{\beta}{2}\right) \left[\int_{\mathbb{R}^3 \setminus B_r} (|\nabla u|^2 + |u|^2) dx \right]^{\frac{p}{2}} \left[\int_{\mathbb{R}^3 \setminus B_r} (|\nabla v|^2 + |v|^2) dx \right]^{\frac{p}{2}}. \end{aligned}$$

Using this, together with (63) one has

$$\frac{2\varrho+1}{2}|\vartheta(r+1)|_1 - \frac{1}{2}|\vartheta(r)|_1 \leq C_0(|\vartheta_1(r)|^{\frac{p}{2}} + |\vartheta_2(r)|^{\frac{p}{2}}) \leq C_1|\vartheta(r)|_1^{\frac{p}{2}}.$$

Thus we have

$$\frac{|\vartheta(r+1)|_1}{|\vartheta(r)|_1} \leq \frac{1}{2\varrho+1} \left(1 + C_2 \frac{|\vartheta(r)|_1^{\frac{p}{2}}}{|\vartheta(r)|_1} \right) = \frac{1}{2\varrho+1} \left(1 + C_2 |\vartheta(r)|_1^{\frac{p-2}{2}} \right).$$

Since $|\vartheta(r)|_1 \rightarrow 0$ as $r \rightarrow \infty$, there exists $r_0 > 0$ such that for $r > r_0 + 1$,

$$|\vartheta(r)|_1 \leq |\vartheta(\lfloor r \rfloor)|_1 = |\vartheta(r_0)|_1 \prod_{k=r_0}^{\lfloor r \rfloor - 1} \frac{|\vartheta(k+1)|_1}{|\vartheta(k)|_1} \leq (C_3 + 1)^{r_0 - r + 1} (\|u\|_{H^1}^2 + \|v\|_{H^1}^2).$$

As $|\vartheta(r)|_1 \leq (C_3 + 1)^{r_0 - r + 1} (\|u\|_{H^1}^2 + \|v\|_{H^1}^2)$ for $r \leq r_0 + 1$, then for all $r \geq 0$, we get

$$\begin{aligned} |\vartheta(r)|_1 &\leq (C_3 + 1)^{r_0 - r + 1} (\|u\|_{H^1}^2 + \|v\|_{H^1}^2) \\ &= (\|u\|_{H^1}^2 + \|v\|_{H^1}^2) (C_3 + 1)^{r_0 + 1} e^{-\ln(C_3 + 1)r}. \end{aligned}$$

For $x \in \mathbb{R}^3$ with $|x| \geq 2$, let $r = \frac{1}{2}|x|$. Then $B_1(x) \subset \mathbb{R}^3 \setminus B_r(0)$, and there holds

$$\|u\|_{H^1(B_1(x))}^2 \leq \|u\|_{H^1(\mathbb{R}^3 \setminus B_1(x))}^2 = \vartheta_1(r) \leq C_4 e^{-D_1 r},$$

where $D_1 > 0$. By the standard elliptic regularity theory, we have $u, v \in W^{2,s}(\mathbb{R}^3) \cap \mathcal{C}^2(\mathbb{R}^3)$ for $s > 2$. Thus for $x \in \mathbb{R}^3 \setminus B_2(0)$, by using the embedding $W^{2,s}(B_{1/2}(x)) \hookrightarrow L^\infty(B_{1/2}(x))$ and [1, Lemma 2.3], one has

$$|u(x)| \leq |u(x)|_{L^\infty(B_{1/2}(x))} \leq C_5 \|u\|_{W^{2,s}(B_{1/2}(x))} \leq C_6 e^{-D_2 |x|},$$

where $D_2 > 0$. Since u is continuous, we have $|u(x)| \leq C_6 e^{-D_2 |x|}$ for $x \in B_2(0)$. Therefore,

$$|u(x)| \leq C e^{-\zeta |x|} \text{ for every } x \in \mathbb{R}^3.$$

Similarly, we have

$$|v(x)| \leq C e^{-\zeta |x|} \text{ for every } x \in \mathbb{R}^3.$$

The proof is complete. ■

6 Dynamic behavior of standing waves

6.1 The local well-posedness and orbital stability

We firstly consider the local well-posedness for system (6) with the initial data. Set

$$g_1(\psi_1, \psi_2) := |\psi_1|^{p-2} \psi_1 + \beta |\psi_2|^{\frac{p}{2}} |\psi_1|^{\frac{p}{2}-2} \psi_1 - \gamma \left(|x|^{-1} * \sum_{j=1}^2 |\psi_j|^2 \right) \psi_1$$

and

$$g_2(\psi_1, \psi_2) := |\psi_2|^{p-2}\psi_2 + \beta|\psi_1|^{\frac{p}{2}}|\psi_2|^{\frac{p}{2}-2}\psi_2 - \gamma \left(|x|^{-1} * \sum_{j=1}^2 |\psi_j|^2 \right) \psi_2.$$

Then the Cauchy problem (6) is rewritten as

$$\begin{cases} i\partial_t \psi_1 + \Delta \psi_1 + g_1(\psi_1, \psi_2) = 0 & \text{in } \mathbb{R}^3, \\ i\partial_t \psi_2 + \Delta \psi_2 + g_2(\psi_1, \psi_2) = 0 & \text{in } \mathbb{R}^3, \\ \psi_1(0, x) = \psi_1(0) \in H^1(\mathbb{R}^3), \psi_2(0, x) = \psi_2(0) \in H^1(\mathbb{R}^3). \end{cases}$$

Definition 6.1 *The pair (s, r) is referred to be as an Strichartz admissible if*

$$\frac{2}{s} + \frac{3}{r} = \frac{3}{2} \quad \text{for } s, r \in [2, \infty].$$

We introduce the spaces $L_t^s([0, T]; L^r)$ and $L_t^s([0, T]; W^{1,r})$ (as $L_t^s L^r$ and $L_t^s W^{1,r}$ respectively to simplify the notation) equipped with the Strichartz norms:

$$\|w(t, x)\|_{L_t^s L^r} = \left(\int_0^T \|w(t, \cdot)\|_r^s dt \right)^{\frac{1}{s}} \quad \text{and} \quad \|w(t, x)\|_{L_t^s W^{1,r}} = \left(\int_0^T \|w(t, \cdot)\|_{W^{1,r}(\mathbb{R}^3)}^s dt \right)^{\frac{1}{s}},$$

where the function $w(t, x)$ is defined on the time-space strip $[0, T) \times \mathbb{R}^N$.

Definition 6.2 *Let $T > 0$. We say that (ψ_1, ψ_2) is an integral solution of the Cauchy problem (6.1) on the time interval $[0, T)$ if*

- (i) $\psi_1, \psi_2 \in \mathcal{C}([0, T]; H^1) \cap L_t^s([0, T]; W^{1,r})$.
- (ii) *For all $t \in [0, T)$ it holds*

$$(\psi_1(t), \psi_2(t)) = e^{it\Delta}(\psi_1(0), \psi_2(0)) + i \int_0^t e^{i(t-\rho)\Delta}(g_1(\psi_1(\rho), \psi_2(\rho)), g_2(\psi_1(\rho), \psi_2(\rho)))d\rho.$$

Let us recall the following well-known Strichartz's estimates (see [5, 20]).

Proposition 6.3 *Let $T > 0$. For every admissible pairs (s, r) and $(\tilde{s}, \tilde{r})$, there exists a constant $C > 0$ such that the following properties hold:*

- (i) *For every $\varphi \in L^2(\mathbb{R}^3)$, the function $t \mapsto e^{it\Delta}\varphi$ belongs to $L_t^s([0, T]; L^r) \cap \mathcal{C}([0, T]; L^2(\mathbb{R}^3))$ and $\|e^{it\Delta}\varphi\|_{L_t^s L^r} \leq C\|\varphi\|_{L^2}$.*
- (ii) *Let $F \in L_t^{\tilde{s}'}([0, T]; L^{\tilde{r}'}),$ where we use a prime to denote conjugate indices. Then the function $t \mapsto \Phi_F(t) := \int_0^t e^{i(t-\rho)\Delta}F(\rho)d\rho$ belongs to $L_t^s([0, T]; L^r) \cap \mathcal{C}([0, T]; L^2(\mathbb{R}^3))$ and $\|\Phi_F\|_{L_t^s L^r} \leq C\|F\|_{L_t^{\tilde{s}'} L^{\tilde{r}'}}$.*
- (iii) *For every $\varphi \in H^1(\mathbb{R}^3)$, the function $t \mapsto e^{it\Delta}\varphi$ belongs to $L_t^s([0, T]; W^{1,r}) \cap \mathcal{C}([0, T]; L^2(\mathbb{R}^3))$ and $\|e^{it\Delta}\varphi\|_{L_t^s W^{1,r}} \leq C\|\varphi\|_{H^1}$.*

We now choose the L^2 -admissible pair (s_1, r_1) and (s_2, r_2) , respectively, as $(s_1, r_1) = \left(\frac{4p}{3(p-2)}, p\right)$ and $(s_2, r_2) = \left(8, \frac{12}{5}\right)$.

Now we give the proof of Theorem 1.13: Define the solution map Ψ by

$$\Psi(\psi_1, \psi_2) := e^{it\Delta}(\psi_1(0), \psi_2(0)) + i \int_0^t e^{i(t-\rho)\Delta}(g_1(\psi_1(\rho), \psi_2(\rho)), g_2(\psi_1(\rho), \psi_2(\rho)))d\rho.$$

For $T > 0$, we define the complete metric space

$$\mathcal{B}_{R,T} := \{(\psi_1, \psi_2) \in \mathcal{H} \times \mathcal{H} \mid \vartheta(\psi_1, \psi_2) \leq R\} \text{ for some } R > 0,$$

where

$$\mathcal{H} := \mathcal{C}([0, T]; H^1) \cap L_t^{s_1}([0, T]; W^{1,r_1}) \cap L_t^{s_2}([0, T]; W^{1,r_2})$$

and

$$\vartheta(\psi_1, \psi_2) := \max \left\{ \sup_{t \in [0, T]} \|(\psi_1, \psi_2)\|_{H^1}, \|(\psi_1, \psi_2)\|_{L_t^{s_1} W^{1,r_1}}, \|(\psi_1, \psi_2)\|_{L_t^{s_2} W^{1,r_2}} \right\}$$

with the metric

$$d((\psi_1, \psi_2) - (\omega_1, \omega_2)) := \|(\psi_1, \psi_2) - (\omega_1, \omega_2)\|_{L_t^{s_1} L^{r_1} \cap L_t^{s_2} L^{r_2}}.$$

By Strichartz estimates, we have

$$\begin{aligned} \|\Psi(\psi_1, \psi_2)\|_{L_t^{\tilde{s}} W^{1,\tilde{r}}} &\lesssim \|(\psi_1(0), \psi_2(0))\|_{H^1} + \|(f_1(\psi_1, \psi_2), f_2(\psi_1, \psi_2))\|_{L_t^{s'_1} W^{1,r'_1}} \\ &\quad + \|(h_1(\psi_1, \psi_2), h_2(\psi_1, \psi_2))\|_{L_t^{s'_2} W^{1,r'_2}}, \end{aligned} \quad (64)$$

where

$$f_1 := |\psi_1|^{p-2} \psi_1 + \beta |\psi_2|^{\frac{p}{2}} |\psi_1|^{\frac{p}{2}-2} \psi_1, \quad f_2 := |\psi_2|^{p-2} \psi_2 + \beta |\psi_1|^{\frac{p}{2}} |\psi_2|^{\frac{p}{2}-2} \psi_2,$$

and

$$h_1 := \gamma \left(|x|^{-1} * \sum_{j=1}^2 |\psi_j|^2 \right) \psi_1, \quad h_2 := \gamma \left(|x|^{-1} * \sum_{j=1}^2 |\psi_j|^2 \right) \psi_2.$$

By using the Hölder inequality in time in (64), we obtain

$$\begin{aligned} \|\Psi(\psi_1, \psi_2)\|_{L_t^{\tilde{s}} W^{1,\tilde{r}}} &\lesssim \|(\psi_1(0), \psi_2(0))\|_{H^1} + T^\theta \|(f_1(\psi_1, \psi_2), f_2(\psi_1, \psi_2))\|_{L_t^{s'_1} W^{1,r'_1}} \\ &\quad + T^\delta \|(h_1(\psi_1, \psi_2), h_2(\psi_1, \psi_2))\|_{L_t^{s'_2} W^{1,r'_2}} \end{aligned}$$

where $\theta := \frac{6-p}{2p}$ and $\delta := \frac{3}{4}$.

Using Hölder and Sobolev inequalities, we have

$$\|f_1(\psi_1, \psi_2)\|_{L_t^{s'_1} W^{1,r'_1}} \lesssim \|\psi_1\|_{L_t^{s'_1} W^{r_1}}^{p-1} + 2\|\psi_2\|_{L_t^{s'_1} W^{r_1}}^{\frac{p}{2}} \|\psi_1\|_{L_t^{s'_1} W^{r_1}}^{\frac{p-2}{2}}.$$

Similarly, we get

$$\|f_2(\psi_1, \psi_2)\|_{L_t^{s'_1} W^{1,r'_1}} \lesssim \|\psi_2\|_{L_t^{s'_1} W^{r_1}}^{p-1} + 2\|\psi_1\|_{L_t^{s'_1} W^{r_1}}^{\frac{p}{2}} \|\psi_2\|_{L_t^{s'_1} W^{r_1}}^{\frac{p-2}{2}}.$$

Moreover, it follows from Hölder, Hardy-Littlewood-Sobolev and Sobolev inequalities that

$$\begin{aligned} &\|\nabla h_1(\psi_1, \psi_2)\|_{L_t^{s'_2} L^{r'_2}} \\ &\lesssim \| |x|^{-1} * (|\psi_1| \nabla \psi_1) \|_{L_t^{s'_2} L^6} \|\psi_1\|_{L_t^\infty L^{\frac{12}{5}}} + \| |x|^{-1} * |\psi_1|^2 \|_{L_t^\infty L^6} \|\nabla \psi_1\|_{L_t^{s'_2} L^{\frac{12}{5}}} \\ &\quad + \| |x|^{-1} * (|\psi_2| \nabla \psi_2) \|_{L_t^{s'_2} L^6} \|\psi_1\|_{L_t^\infty L^{\frac{12}{5}}} + \| |x|^{-1} * |\psi_2|^2 \|_{L_t^\infty L^6} \|\nabla \psi_1\|_{L_t^{s'_2} L^{\frac{12}{5}}} \\ &\lesssim \| |\psi_1| \nabla \psi_1 \|_{L_t^{s'_2} L^{\frac{6}{5}}} \|\psi_1\|_{L_t^\infty L^{r_2}} + \|\psi_1\|_{L_t^\infty L^{r_2}}^2 \|\nabla \psi_1\|_{L_t^{s'_2} L^{r_2}} \\ &\quad + \| |\psi_2| \nabla \psi_2 \|_{L_t^{s'_2} L^{\frac{6}{5}}} \|\psi_1\|_{L_t^\infty L^{r_2}} + \|\psi_2\|_{L_t^\infty L^{r_2}}^2 \|\nabla \psi_1\|_{L_t^{s'_2} L^{r_2}} \\ &\lesssim \|\nabla \psi_1\|_{L_t^{s'_2} L^{r_2}} \|\psi_1\|_{L_t^\infty H^1} \|\psi_1\|_{L_t^\infty H^1} + \|\psi_1\|_{L_t^\infty H^1}^2 \|\nabla \psi_1\|_{L_t^{s'_2} L^{r_2}} \\ &\quad + \|\nabla \psi_2\|_{L_t^{s'_2} L^{r_2}} \|\psi_2\|_{L_t^\infty H^1} \|\psi_1\|_{L_t^\infty H^1} + \|\psi_2\|_{L_t^\infty H^1}^2 \|\nabla \psi_1\|_{L_t^{s'_2} L^{r_2}}, \end{aligned}$$

and

$$\begin{aligned} \|h_1(\psi_1, \psi_2)\|_{L_t^{s_2} W^{1, r'_2}} &\lesssim \|\psi_1\|_{L_t^\infty H^1}^2 \|\psi_1\|_{L_t^{s_2} W^{r_2}} + \|\psi_2\|_{L_t^{s_2} W^{r_2}} \|\psi_2\|_{L_t^\infty H^1} \|\psi_1\|_{L_t^\infty H^1} \\ &\quad + \|\psi_2\|_{L_t^\infty H^1}^2 \|\psi_1\|_{L_t^{s_2} W^{r_2}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|h_2(\psi_1, \psi_2)\|_{L_t^{s_2} W^{1, r'_2}} &\lesssim \|\psi_2\|_{L_t^\infty H^1}^2 \|\psi_2\|_{L_t^{s_2} W^{r_2}} + \|\psi_1\|_{L_t^{s_2} W^{r_2}} \|\psi_1\|_{L_t^\infty H^1} \|\psi_2\|_{L_t^\infty H^1} \\ &\quad + \|\psi_1\|_{L_t^\infty H^1}^2 \|\psi_2\|_{L_t^{s_2} W^{r_2}}. \end{aligned}$$

So, for $(\psi_1, \psi_2) \in \mathcal{B}_{R, T}$, we get

$$\|\Psi(\psi_1, \psi_2)\|_{L_t^{\tilde{s}} W^{1, \tilde{r}}} \leq C\|(\psi_1(0), \psi_2(0))\|_{\mathbf{H}} + CT^\theta R^{p-1} + CT^\delta R^3.$$

Now we choose $R = 2C\|(\psi_1(0), \psi_2(0))\|_{\mathbf{H}} > 0$ and $0 < T < 1$ such that $CT^\theta R^{p-2} + CT^\delta R^2 \leq \frac{1}{2}$. This implies that $\mathcal{B}_{R, T}$ is an invariant set of Ψ . Furthermore, for $(\psi_1, \psi_2), (\omega_1, \omega_2) \in \mathcal{B}_{R, T}$, by using the same argument, we get

$$d(\Psi(\psi_1, \psi_2) - \Psi(\omega_1, \omega_2)) \lesssim (T^\theta + T^\delta)d((\psi_1, \psi_2) - (\omega_1, \omega_2)),$$

which leads to

$$d(\Psi(\psi_1, \psi_2) - \Psi(\omega_1, \omega_2)) \leq \frac{1}{2}d((\psi_1, \psi_2) - (\omega_1, \omega_2)).$$

This implies that Ψ is a contraction map. In particular, Ψ has a unique fixed point in this space. Moreover, by the classical arguments in [5], the solution (ψ_1, ψ_2) satisfies the mass conservation law

$$\|\psi_1(t)\|_2^2 + \|\psi_2(t)\|_2^2 = \|\psi_1(0)\|_2^2 + \|\psi_2(0)\|_2^2$$

and the energy conservation law

$$\begin{aligned} I_{\gamma, \beta}(\psi_1(t), \psi_2(t)) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \psi_1(t)|^2 + |\nabla \psi_2(t)|^2) dx + \frac{\gamma}{4} \int_{\mathbb{R}^3} \phi_{\psi_1(t), \psi_2(t)} (|\psi_1(t)|^2 + |\psi_2(t)|^2) dx \\ &\quad - \frac{1}{p} \int_{\mathbb{R}^3} (|\psi_1(t)|^p + |\psi_2(t)|^p + 2\beta |\psi_1(t)|^{\frac{p}{2}} |\psi_2(t)|^{\frac{p}{2}}) dx \\ &= I_{\gamma, \beta}(\psi_1(0), \psi_2(0)). \end{aligned}$$

We give the proof of Theorem 1.14: Since the proof is similar to the classical arguments in [24], we only give a sketch. Assume on the contrary. Then there exist $\varepsilon_0 > 0$, $\{\psi_1^n(0), \psi_2^n(0)\}$ and $\{t_n\} \subset \mathbb{R}^+$ such that

$$\inf_{(u, v) \in \mathcal{Z}(c)} \|(\psi_1^n(0), \psi_2^n(0)) - (u, v)\|_{H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} \rightarrow 0$$

and

$$\inf_{(u, v) \in \mathcal{Z}(c)} \|(\psi_1^n(t_n), \psi_2^n(t_n)) - (u, v)\|_{H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} \geq \varepsilon_0. \quad (65)$$

According to mass and energy conservation laws, we have

$$\sum_{i=1}^2 \|\psi_i^n(t_n)\|_2^2 = \sum_{i=1}^2 \|\psi_i^n(0)\|_2^2 \quad \text{and} \quad I_{\gamma,\beta}(\psi_1^n(t_n), \psi_2^n(t_n)) = I_{\gamma,\beta}(\psi_1^n(0), \psi_2^n(0)) \quad \text{for } i = 1, 2.$$

Define

$$\tilde{\psi}_i^n = \frac{\psi_i^n(t_n)}{\sum_{i=1}^2 \|\psi_i^n(t_n)\|_2^2} c \quad \text{for } i = 1, 2.$$

A direct calculation shows that

$$\sum_{i=1}^2 \|\tilde{\psi}_i^n\|_2^2 = c \quad \text{and} \quad I_{\gamma,\beta}(\tilde{\psi}_1^n, \tilde{\psi}_2^n) = \sigma_{\gamma,\beta}(c) + o_n(1),$$

which indicates that $(\tilde{\psi}_1^n, \tilde{\psi}_2^n)$ is a minimizing sequence for $\sigma_{\gamma,\beta}(c)$. By Theorems 1.5 and 1.6, it is precompact in $\mathbf{H}$, which is a contradiction with (65). Therefore, the set $\mathcal{Z}(c)$ is orbitally stable.

6.2 The global well-posedness

We give the proof of Theorem 1.15: Since system (6) with the initial data is locally well-posed by Theorem 1.13, we obtain that either $T_{\max} = +\infty$, or $T_{\max} < +\infty$ and $\lim_{t \rightarrow T_{\max}^-} A(\psi_1(t), \psi_2(t)) \rightarrow +\infty$. Then the global existence boils down to obtaining a priori bound on $A(\psi_1(t), \psi_2(t))$.

(i) $2 < p \leq \frac{10}{3}$. By mass and energy conservation laws and (13), we have

$$\begin{aligned} & A(\psi_1(t), \psi_2(t)) \\ & \leq 2I_{\gamma,\beta}(\psi_1(0), \psi_2(0)) + \frac{2}{p} C(\psi_1(t), \psi_2(t)) \\ & \leq 2I_{\gamma,\beta}(\psi_1(0), \psi_2(0)) + 4(1 + \beta) \mathcal{S}_p \left(\|\psi_1(0)\|_2^2 + \|\psi_2(0)\|_2^2 \right)^{\frac{6-p}{4}} (A(\psi_1(t), \psi_2(t)))^{\frac{3(p-2)}{4}}. \end{aligned}$$

If $2 < p < \frac{10}{3}$, then there exists a constant $M > 0$ independent of t such that $A(\psi_1(t), \psi_2(t)) \leq M$. If $p = \frac{10}{3}$, then also there exists a constant $M > 0$ independent of t such that $A(\psi_1(t), \psi_2(t)) \leq M$ provided that $\|\psi_1(0)\|_2^2 + \|\psi_2(0)\|_2^2$ is small.

(ii) $10/3 < p < 6$. If by contradiction $T_{\max} < +\infty$, we have $\lim_{t \rightarrow T_{\max}^-} A(\psi_1(t), \psi_2(t)) \rightarrow +\infty$. Since

$$I_{\gamma,\beta}(\psi_1, \psi_2) - \frac{2}{3(p-2)} \mathcal{P}_{\gamma,\beta}(\psi_1, \psi_2) = \frac{3p-10}{6(p-2)} A(\psi_1, \psi_2) + \frac{3p-8}{12(p-2)} B(\psi_1, \psi_2),$$

together with the energy conservation law, we have $\lim_{t \rightarrow T_{\max}^-} \mathcal{P}_{\gamma,\beta}(\psi_1, \psi_2) = -\infty$. Then by the continuity, there exists $t_0 \in (0, T_{\max})$ such that $\mathcal{P}_{\gamma,\beta}(\psi_1(t_0), \psi_2(t_0)) = 0$, where we have used the assumption of $\mathcal{P}_{\gamma,\beta}(\psi_1(0), \psi_2(0)) > 0$. Using again the energy conservation law and the assumption of $I_{\gamma,\beta}(\psi_1(0), \psi_2(0)) < m_{\gamma,\beta}(c)$, we deduce that

$$m_{\gamma,\beta}(c) > I_{\gamma,\beta}(\psi_1(0), \psi_2(0)) = I_{\gamma,\beta}(\psi_1(t_0), \psi_2(t_0)) \geq \inf_{(u,v) \in \mathcal{M}_{\gamma,\beta}(c)} I_{\gamma,\beta}(u, v) = m_{\gamma,\beta}(c),$$

which is a contradiction. So, $T_{\max} = +\infty$.

6.3 Finite time blow-up and strong unstability

Lemma 6.4 *Assume that the conditions of Theorem 1.10 or Theorem 1.11 hold. Let $(\psi_1(0), \psi_2(0)) \in S_c$ such that $I_{\gamma,\beta}(\psi_1(0), \psi_2(0)) < m_{\gamma,\beta}(c)$. Then*

$$\Phi_{(\psi_1(0), \psi_2(0))}(t) := \frac{t^2}{2}A(\psi_1(0), \psi_2(0)) + \frac{\gamma t}{4}B(\psi_1(0), \psi_2(0)) - \frac{t^{\frac{3(p-2)}{2}}}{p}C(\psi_1(0), \psi_2(0))$$

has a unique global maximum point $t^-(\psi_1(0), \psi_2(0))$. Furthermore, if

$$0 < t^-(\psi_1(0), \psi_2(0)) < 1 \text{ and } (|x|\psi_1(0), |x|\psi_2(0)) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3),$$

then the solution $(\psi_1(t), \psi_2(t))$ of system (6) with initial datum $(\psi_1(0), \psi_2(0))$ blows-up in finite time.

Proof. Let $(\psi_1(0), \psi_2(0)) \in S_c$. Since $\frac{10}{3} \leq p < 6$, it is easy to prove that $\Phi_{(\psi_1(0), \psi_2(0))}(t)$ has a unique global maximum point $t^-(\psi_1(0), \psi_2(0))$ and $\Phi_{(\psi_1(0), \psi_2(0))}(t)$ is strict decreasing and concave on $(t^-(\psi_1(0), \psi_2(0)), +\infty)$. We claim that if $t^-(\psi_1(0), \psi_2(0)) \in (0, 1)$, then

$$\mathcal{P}_{\gamma,\beta}(\psi_1(0), \psi_2(0)) \leq I_{\gamma,\beta}(\psi_1(0), \psi_2(0)) - m_{\gamma,\beta}(c). \quad (66)$$

The fact of $t^-(\psi_1(0), \psi_2(0)) \in (0, 1)$ implies that $\mathcal{P}_{\gamma,\beta}(\psi_1(0), \psi_2(0)) < 0$. Then there holds

$$\begin{aligned} I_{\gamma,\beta}(\psi_1(0), \psi_2(0)) &= \Phi_{(\psi_1(0), \psi_2(0))}(1) \\ &\geq \Phi_{(\psi_1(0), \psi_2(0))}(t^-(\psi_1(0), \psi_2(0))) - (t^-(\psi_1(0), \psi_2(0)) - 1)\Phi'_{(\psi_1(0), \psi_2(0))}(1) \\ &= I_{\gamma,\beta}((\psi_1(0), \psi_2(0))_{t^-(\psi_1(0), \psi_2(0))}) - |\mathcal{P}_{\gamma,\beta}(\psi_1(0), \psi_2(0))|(1 - t^-(\psi_1(0), \psi_2(0))) \\ &\geq m_{\gamma,\beta}(c) - |\mathcal{P}_{\gamma,\beta}(\psi_1(0), \psi_2(0))| \\ &= m_{\gamma,\beta}(c) + \mathcal{P}_{\gamma,\beta}(\psi_1(0), \psi_2(0)). \end{aligned}$$

This proves the claim.

Now let us consider the solution (ψ_1, ψ_2) with initial data $(\psi_1(0), \psi_2(0))$. Similar to the argument in Soave [36, Lemma 5.3], the map $(\psi_1(0), \psi_2(0)) \mapsto t^-(\psi_1(0), \psi_2(0))$ is continuous, and together with $0 < t^-(\psi_1(0), \psi_2(0)) < 1$, we have $t^-(\psi_1(\tau), \psi_2(\tau)) < 1$ for every $|\tau|$ small, say $|\tau| < \bar{\tau}$ for some $\bar{\tau} > 0$. By (66) and the assumption of $I_{\gamma,\beta}(\psi_1(0), \psi_2(0)) < m_{\gamma,\beta}(c)$, we have

$$\begin{aligned} \mathcal{P}_{\gamma,\beta}(\psi_1(\tau), \psi_2(\tau)) &\leq I_{\gamma,\beta}(\psi_1(\tau), \psi_2(\tau)) - m_{\gamma,\beta}(c) \\ &= I_{\gamma,\beta}(\psi_1(0), \psi_2(0)) - m_{\gamma,\beta}(c) := -\eta < 0, \end{aligned}$$

for every such τ , and hence $t^-(\psi_1(\bar{\tau}), \psi_2(\bar{\tau})) < 1$. Applying the continuity argument yields

$$\mathcal{P}_{\gamma,\beta}(\psi_1(t), \psi_2(t)) \leq -\eta \text{ for } t \in (T_{\min}, T_{\max}).$$

Since $(|x|\psi_1(0), |x|\psi_2(0)) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, by the Virial identity [5, Proposition 6.5.1], the function

$$\bar{f}(t) := \sum_{i=1}^2 \int_{\mathbb{R}^3} |x|^2 |\psi_i(t, x)|^2 dx$$

is of class C^2 . We now use the method established by Cazenave [5]. By multiplying the first equation in system (6) by $i|x|^2\bar{\psi}_1$ and the second equation in system (6) by $i|x|^2\bar{\psi}_2$, taking the real part and integrating over $\mathbb{R}^3$, we have

$$\bar{f}'(t) = 2\operatorname{Re} \sum_{i=1}^2 \int_{\mathbb{R}^3} |x|^2 \bar{\psi}_i \partial_t \psi_i dx = 4\operatorname{Im} \sum_{i=1}^2 \int_{\mathbb{R}^3} \bar{\psi}_i x \cdot \nabla \psi_i dx,$$

and

$$\bar{f}''(t) = -4\operatorname{Im} \sum_{i=1}^2 \int_{\mathbb{R}^3} (2x \cdot \nabla \psi_i + 3\bar{\psi}_i) \psi_i dx = 8\mathcal{P}_{\gamma,\beta}(\psi_1(t), \psi_2(t)) \leq -8\eta < 0$$

for $t \in (T_{\min}, T_{\max})$. Integrating twice in time gives $0 \leq \bar{f}(t) \leq \bar{f}(0) + \bar{f}'(0)t - 4\eta t^2$. Since the right hand side becomes negative for t sufficiently large, it is necessary that both $T_{\min}$ and $T_{\max}$ are bounded, which in turn implies finite time blow-up. The proof is complete. ■

We give the proofs of Theorems 1.16 and 1.17: Theorem 1.16 directly follows from Lemma 6.4.

Next, we prove Theorem 1.17. Let $(\bar{u}, \bar{v})$ be an energy ground state obtained in Theorem 1.10 or 1.11. Let $(\psi_1(t), \psi_2(t))$ be the solution to system (6) with the initial data $(\bar{u}_s, \bar{v}_s)$, where $(\bar{u}_s, \bar{v}_s) = (s^{3/2}\bar{u}(sx), s^{3/2}\bar{v}(sx))$ with $s > 1$. Then $(\bar{u}_s, \bar{v}_s) \rightarrow (\bar{u}, \bar{v})$ in $\mathbf{H}$ as $s \searrow 1$, and so it is sufficient to prove that $(\psi_1(t), \psi_2(t))$ blows up in finite time. Clearly, $t^-(\bar{u}_s, \bar{v}_s) = \frac{1}{s} < 1$, and by the definition of $m_{\gamma,\beta}(c)$ one has $I_{\gamma,\beta}(\bar{u}_s, \bar{v}_s) < I_{\gamma,\beta}(\bar{u}, \bar{v}) = m_{\gamma,\beta}(c)$. Therefore, it follows from Lemma 6.4 that the solution $(\psi_1(t), \psi_2(t))$ blows up in finite time and the associated standing wave is orbitally unstable.

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