Multiplicity results for critical fractional Ambrosetti-Prodi type system with nonlinearities interacting with the spectrum

Eduardo. H. Caqui[1], Sandra M. de S. Lima[2], and Fábio R. Pereira[3]

[1]Departamento de Ciencias Sede Breña, Universidad Privada del Norte, Av. Tingo María 1122, Cercado de Lima, Lima, Peru.
eduardocaqui11@gmail.com
[2]Departamento de Ciências Exatas, Biológicas e da Terra, INFES-UFF, Santo Antônio de Pádua - RJ, 28470-000, Brazil.
sandramsl@id.uff.br
[3]Departamento de Matemática, Instituto de Ciências Exatas, Universidade Federal de Juiz de Fora, 30161-970, Juiz de Fora - MG, Brazil.*fabio.pereira@ufjf.edu.br**

Abstract

We investigated the existence of solutions for a class of Ambrosetti-Prodi type systems involving the fractional Laplacian operator and with nonlinearities reaching critical growth and interacting, in some sense, with the spectrum of the operator. The resonant case in $\lambda_{k,s}$ for $k > 1$ is also investigated.

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1 Introduction

Let $s \in (0,1)$, $N > 2s$ and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. In this paper we study the possibility of existence of solutions for the following critical fractional system

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$$
\begin{cases}
(-\Delta)^s u = au + bv + \dfrac{\alpha}{\alpha+\beta} {u_+}^{\alpha-1} {v_+}^{\beta} + \xi_1 {u_+}^{\alpha+\beta-1} + f & \text{in } \Omega,\\
(-\Delta)^s v = bu + cv + \dfrac{\beta}{\alpha+\beta} {u_+}^{\alpha} {v_+}^{\beta-1} + \xi_2 {v_+}^{\alpha+\beta-1} + g & \text{in } \Omega,\\
\quad u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases} \tag{1.1}
$$

where

$$
(-\Delta)^s u(x) := C(N,s) \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \qquad x \in \mathbb{R}^N,
$$

is the fractional laplacian operator with $C(N,s) = \left(\int_{\mathbb{R}^N} \frac{1 - cos(\zeta_1)}{|\zeta|^{N+2s}} d\zeta \right)^{-1}$ a positive dimensional constant, $\alpha, \beta > 1$ are real constants such that the sum $\alpha + \beta$ is the fractional critical Sobolev exponent $2_s^* := \frac{2N}{N-2s}$, $\xi_1, \xi_2 \geq 0$ and the forcing terms f and g are of the form $f = t\varphi_{1,s} + f_1$ and $g = r\varphi_{1,s} + g_1$, in such a way that the pair $(t,r) \in \mathbb{R}^2$, $f_1, g_1 \in L^q(\Omega)$ for some $q > \frac{N}{2s}$ and $\int_\Omega f_1 \varphi_{1,s} dx = \int_\Omega g_1 \varphi_{1,s} dx = 0$ with $\varphi_{1,s}$ the positive eigenfunction associated with the first eigenvalue $\lambda_{1,s}$ of the operator $(-\Delta)^s$ with homogeneous Dirichlet boundary condition.

With the above decomposition, in order to state and compare our results to the scalar case, it is convenient to rewrite system (1.1) as

$$
\begin{cases}
(-\overrightarrow{\Delta})^s U = AU + \nabla F(U) + T\varphi_{1,s} + F_1 & \text{in } \Omega,\\
U = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases} \tag{1.2}
$$

where
$U = \begin{pmatrix} u \\ v \end{pmatrix}$, $(-\overrightarrow{\Delta})^s U = \begin{pmatrix} (-\Delta)^s u & 0 \\ 0 & (-\Delta)^s v \end{pmatrix}$, $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, ∇ is the gradient operator, $F(U) = \dfrac{1}{\alpha+\beta}\left({u_+}^{\alpha} {v_+}^{\beta} + \xi_1 {u_+}^{\alpha+\beta} + \xi_2 {v_+}^{\alpha+\beta} \right)$, $T = \begin{pmatrix} t \\ r \end{pmatrix}$ and $F_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$.

Let μ_1, μ_2 be real eigenvalues of the symmetric matrix A, which will assume $\mu_1 \leq \mu_2$. Thus, it is verified that $\mu_1 |U|^2 \leq (AU,U)_{\mathbb{R}^2} \leq \mu_2 |U|^2$, for all $U := (u,v) \in \mathbb{R}^2$. The interaction of these eigenvalues with the spectrum of the $(-\Delta)^s$ will play an important role in the study of existence of the solutions.

We recall that Ambrosetti and Prodi [2] in 1972, studied the following boundary value problem

$$
\begin{cases}
-\Delta u = f(u) + g(x) & \text{in } \Omega,\\
\quad u = 0 & \text{on } \partial\Omega,
\end{cases} \tag{1.3}
$$

where $g \in C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$, $f \in C^2(\mathbb{R})$ such that $f(0) = 0$, $f''(t) > 0$ for all $t \in \mathbb{R}$ and

$$
0 < \lim_{t \to -\infty} f'(t) < \lambda_1 < \lim_{t \to +\infty} f'(t) < \lambda_2,
$$

where $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \ldots$ denote the eigenvalues of $(-\Delta, H_0^1(\Omega))$. The authors showed that there exists in $C^{0,\alpha}(\overline{\Omega})$, a closed connected C^1 manifold M_1 of codimension 1 which splits the space into two connected components M_0 and M_2 such that, if $g \in M_0$, the problem (1.3) has no solution; if $g \in M_1$, the problem (1.3) has exactly one solution and if $g \in M_2$, the problem (1.3) has exactly two solutions. After the pioneering work by Ambrosetti and Prodi [2], many existence and multiplicity results have been investigated in different directions. In particular, Ruf and Srikanth (in [29]) established a multiplicity result for the local subcritical problem $-\Delta u = \lambda u + u_+^p + f(x)$ in Ω, $u = 0$ on $\partial\Omega$ provided that the non-homogeneous term f has the form $f(x) = h(x) + t\varphi_1(x)$ ($h \in L^r(\Omega)$ with $r > N$), λ is not an eigenvalue of $(-\Delta, H_0^1(\Omega))$ and $t > T$, for some sufficiently large number $T = T(h)$. Still in the local scalar case, but with nonlinearity in the critical growth ($p = 2^* - 1$), the problem above mentioned has been studied by De Figueiredo and Jianfu (in [13]), the authors proved the existence of two solutions when $N > 6$. This result was extended by Calanchi and Ruf [10] using the technique developed in [20]. Works related to this subject in the local scalar case, we recommend [4] and in the nonlocal operators situation, [3] and [19] (see references therein). For the critical system in the local operators situation, problem (1.1) was studied, for instance, in [14] and [26] when $\mu_2 < \lambda_1$ and by [24] in the uncoupled case. For the fractional subcritical system, (1.1) was studied, for instance, in [23].

The purpose of this work is to prove the existence of solutions for the class of nonlocal gradient systems of elliptic equations (1.1) involving critical nonlinearities on the hypothesis of an interaction of the eigenvalues μ_1, μ_2 of the matrix A with eigenvalues of the fractional Laplacian operator $(-\Delta)^s$. When $\mu_2 < \lambda_{1,s}$, this system belongs to the class of the so called Ambrosetti-Prodi type problems [2] which have been studied by several authors in the last decades with different approaches.

Problem (1.1) is an extension to systems involving fractional Laplacian operator of the equation considered in [29], [13] and [10], in which (1.1) was studied in the local operators case ($s = 1$) and nonlocal operators ($0 < s < 1$) in [3] (see [19] also) and with the particular matrix $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. In this paper, we complement the results achieved in [23], proving that the system (1.1) (or (1.2)) has at least two solutions for sufficiently large values of parameters (t, r), the first solution is negative and obtained explicitly depending on the non-homogeneous terms f and g. The second solution is obtained via the Mountain Pass Theorem when $\mu_2 < \lambda_{1,s}$, or applying the Linking Theorem in the case $\lambda_{k,s} < \mu_1 \leq \mu_2 < \lambda_{k+1,s}$ if $k \geq 1$. The resonant case $\lambda_{k,s} = \mu_1$ for $k > 1$ is also treated here.
Finally, we should point out that the corresponding local problem governed by the standard Laplacian operator can be recovered by letting $s \to 1$.

To show the existence of solution, difficulties arise when we consider fractional operators. As we know, in [10], the approximate eigenfunctions technique was used to facilitate the estimates of the energy functional associated with the local scalar problem in the space $H_0^1(\Omega)$ (for local critical systems, also see [26]). However, as noted in [22], in the nonlocal case, it is not possible to employ any more the same idea as in [10] or [26], since u and v are not orthogonal in the fractional space $X_0^s(\Omega)$ even though they have disjoint supports. On the other hand, further complications arise due to the presence of the mathematical term $F(u,v) = \dfrac{1}{\alpha+\beta}\left[{u_+}^{\alpha}{v_+}^{\beta} + \xi_1 {u_+}^{\alpha+\beta} + \xi_2 {v_+}^{\alpha+\beta}\right]$ that includes either an uncoupled or a coupled nonlinearity.

Due to these obstacles, we develop similar techniques to these known for the Laplacian operator.

It is important to point out that, with the aid of [15], our results are still valid for the general case $\nabla F(u,v)$ when F is a $(\alpha+\beta)-$homogeneous nonlinearity, which includes a larger class of functions.

The proof of the Theorem below follows arguments as in [23], so we will omit some details.

Theorem 1.1 (Existence of a negative solution) *Let $A \in M_{2\times 2}(\mathbb{R})$ be a symmetric matrix such that*

$$\det(\lambda_{j,s}\ I - A) \neq 0,\ \forall\, j = 1, 2, \dots. \tag{1.4}$$

Assume that $F_1 = (f_1, g_1) \in L^q(\Omega) \times L^q(\Omega)$ for some $q > \frac{N}{2s}$ and consider

$$\boldsymbol{R} = \left\{ (t,r) \in \mathbb{R}^2 : \begin{array}{l} br + (\lambda_{1,s} - c)t < \eta \det(\lambda_{1,s} I - A) \ \ \textit{and} \\ (\lambda_{1,s} - a)r + bt < \vartheta \det(\lambda_{1,s} I - A)\} \end{array} \right\}.$$

Then there exist $\eta, \vartheta \ll 0$ such that system (1.2) has a solution (u_T, v_T) (with $u_T < 0$ and $v_T < 0$ in Ω) for every $T \in \boldsymbol{R}$.

Remark 1.1 *Suppose that* $\det(\lambda_{1,s} I - A) > 0$ *and*

$$\lambda_{1,s} > \max\{a, c\}, \tag{1.5}$$

then the set $\boldsymbol{R}$ is a region between lines satisfying:
(i) If $b = 0$,

$$\boldsymbol{R} = (-\infty\,, \eta\, \frac{\lambda_{1,s} - c}{\det(\lambda_{1,s} I - A)}) \times (-\infty\,, \vartheta\, \frac{\lambda_{1,s} - a}{\det(\lambda_{1,s} I - A)}) \subset \mathbb{R}^2.$$

(ii) If $b > 0$,

$$\boldsymbol{R} = \left\{ (t,r) \in \mathbb{R}^2 : \begin{array}{l} r < \eta \frac{\det(\lambda_{1,s} I - A)}{b} - \frac{(\lambda_{1,s} - c)}{b} t \ \ \textit{and} \\ r < \vartheta \frac{\det(\lambda_{1,s} I - A)}{\lambda_{1,s} - a} - \frac{b}{\lambda_{1,s} - a} t \end{array} \right\}.$$

(iii) If $b < 0$,

$$\boldsymbol{R} = \left\{ (t,r) \in \mathbb{R}^2 : \begin{array}{l} r > \eta \frac{\det(\lambda_{1,s} I - A)}{b} - \frac{(\lambda_{1,s} - c)}{b} t \ \ \textit{and} \\ r < \vartheta \frac{\det(\lambda_{1,s} I - A)}{\lambda_{1,s} - a} - \frac{b}{\lambda_{1,s} - a} t \end{array} \right\}.$$

On the other hand, if $\det(\lambda_{1,s}I - A) > 0$ *and*

$$\lambda_{1,s} < \min\{a, c\}, \tag{1.6}$$

then the set $\boldsymbol{R}$ *satisfies:*
(i) If $b = 0$,

$$\boldsymbol{R} = (\eta\, \frac{\lambda_{1,s} - c}{\det(\lambda_{1,s}I - A)}\, , +\infty) \times (\vartheta\, \frac{\lambda_{1,s} - a}{\det(\lambda_{1,s}I - A)}\, , +\infty) \subset \mathbb{R}^2.$$

(ii) If $b > 0$,

$$\boldsymbol{R} = \left\{ (t,r) \in \mathbb{R}^2 : \begin{array}{l} r < \eta \frac{\det(\lambda_{1,s}I - A)}{b} - \frac{(\lambda_{1,s} - c)}{b} t \ \ and \\ r > \vartheta \frac{\det(\lambda_{1,s}I - A)}{(\lambda_{1,s} - a)} - \frac{b}{(\lambda_{1,s} - a)} t\} \end{array} \right\}.$$

(iii) If $b < 0$,

$$\boldsymbol{R} = \left\{ (t,r) \in \mathbb{R}^2 : \begin{array}{l} r > \eta \frac{\det(\lambda_{1,s}I - A)}{b} - \frac{(\lambda_{1,s} - c)}{b} t \ \ and \\ r > \vartheta \frac{\det(\lambda_{1,s}I - A)}{(\lambda_{1,s} - a)} - \frac{b}{(\lambda_{1,s} - a)} t\} \end{array} \right\}.$$

Note that, since $\det(\lambda_{1,s}\ I - A) \neq 0$, *the lines that define the region* $\boldsymbol{R}$ *are not parallel. Moreover, if* $\det(\lambda_{1,s}\ I - A) < 0$ *a similar result can be obtained as in the Remark 1.1.*

The following are the main results of the paper.

Theorem 1.2 *Assume that* $N > 6s$, $\xi_1, \xi_2 > 0$, $\alpha + \beta = 2_s^*$ *and that one of the following conditions hold,*

$$0 < \mu_1 \leq \mu_2 < \lambda_{1,s}, \tag{1.7}$$

$$\lambda_{k,s} < \mu_1 \leq \mu_2 < \lambda_{k+1,s},\ for\ some\ integer\ k \geq 0. \tag{1.8}$$

Then, system (1.2) has a second solution.

Remark 1.2 *It is important to note that the hypothesis (1.7) implies that the conditions (1.4) and (1.5) are verified and the hypothesis (1.8) implies in (1.4) and (1.6). In both cases,* $\det(\lambda_{1,s}\ I - A) > 0$.

Theorem 1.3 *Suppose* $N > 6s$ *and*

$$\xi_1, \xi_2 > 0 \text{ and } \lambda_{k,s} = \mu_1 \leq \mu_2 < \lambda_{k+1,s},\ for\ some\ k > 1.$$

In addition assume that

$$F_1 = (f_1, g_1) \in (Ker((-\overrightarrow{\Delta})^s - \lambda_{k,s}I))^{\perp}. \tag{1.9}$$

Then system (1.2) has a second solution.

2 Notations and preliminary stuff

For any measurable function $u : \mathbb{R}^N \to \mathbb{R}$ the Gagliardo seminorm is defined by

$$[u]_s := \Big(C(N,s)\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{1/2} = \Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 dx\Big)^{1/2}.$$

The second equality follows by [16, Proposition 3.6] when the above integrals are finite. Then, we consider the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \ [u]_s < \infty\}, \quad \|u\|_{H^s} = (\|u\|_{L^2}^2 + [u]_s^2)^{1/2},$$

which is a Hilbert space. We use the closed subspace

$$X_0^s(\Omega) := \{u \in H^s(\mathbb{R}^N) : \ u = 0 \ \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

By Theorems 6.5 and 7.1 in [16], the imbedding $X_0^s(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $r \in [1, 2_s^*]$ and compact for $r \in [1, 2_s^*)$. Due to the fractional Sobolev inequality, $X_0^s(\Omega)$ is a Hilbert space with inner product

$$\langle u, v\rangle_{X_0^s} := C(N,s)\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dx\, dy,$$

which induces the norm $\|\cdot\|_{X_0^s} = [\cdot]_s$. Observe that by Proposition 3.6 in [16], we have the following identity

$$\|u\|_{X_0^s}^2 = \frac{2}{C(N,s)}\|(-\Delta)^{\frac{s}{2}}u\|_{\mathbb{R}^N}^2, \ u \in X_0^s(\Omega).$$

Then it is proved that for $u, v \in X_0^s(\Omega)$,

$$\frac{2}{C(N,s)}\int_{\mathbb{R}^N} u(x)(-\Delta)^s v(x)dx = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dx\, dy,$$

in particular, $(-\Delta)^s$ is self-adjoint in $X_0^s(\Omega)$.

Now, we consider the Hilbert space given by the product space

$$Y(\Omega) := X_0^s(\Omega) \times X_0^s(\Omega),$$

equipped with the inner product

$$\langle (u,v), (\varphi,\psi)\rangle_Y := \langle u, \varphi\rangle_{X_0^s} + \langle v, \psi\rangle_{X_0^s}$$

and the norm

$$\|(u,v)\|_Y := (\|u\|_{X_0^s}^2 + \|v\|_{X_0^s}^2)^{1/2}.$$

The space $L^r(\Omega) \times L^r(\Omega)$ $(r > 1)$ is considered with the standard product norm

$$\|(u,v)\|_{L^r \times L^r} := (\|u\|_{L^r}^2 + \|v\|_{L^r}^2)^{1/2}.$$

Besides, we recall that

$$\mu_1|U|^2 \le (AU,U)_{\mathbb{R}^2} \le \mu_2|U|^2, \quad \text{for all } U := (u,v) \in \mathbb{R}^2, \tag{2.1}$$

where $\mu_1 \le \mu_2$ are the eigenvalues of the symmetric matrix A. In this paper, we consider the following notation for product space $\mathcal{S} \times \mathcal{S} := \mathcal{S}^2$ and

$$w^+(x) := \max\{w(x), 0\}, \ w^-(x) := \max\{-w(x), 0\}$$

for positive and negative part of a function w. Consequently we get $w = w^+ - w^-$.

Since we are wanted to obtain a solution for the problem (1.1) with critical growth, we defined S be the best constant for the Sobolev-Hardy embedding

$$X_0^s(\Omega) \hookrightarrow L^{2_s^*}(\Omega).$$

The constant

$$S = S_{\alpha+\beta}(\Omega) = \inf_{u \in X_0^s(\Omega)\setminus\{0\}} \left\{ \frac{\|u\|^2_{X_0^s}}{\left(\displaystyle\int_\Omega |u|^{2_s^*} dx\right)^{2/2_s^*}} \right\}.$$

In [11], Chen, Li and Ou prove that the best Sobolev constant $S_{\alpha+\beta} = S$ is achieved by w, where w is the unique positive solution (up to translations and dilations) of

$$(-\Delta)^s w = w^{2_s^*-1}, \quad \text{in } \mathbb{R}^N, \quad w \in L^{2_s^*}(\Omega).$$

For the case of problems involving systems, we need the following definition.

$$S_s = S_s(\alpha,\beta)(\Omega) = \inf_{(u,v)\in Y\setminus\{0\}} \frac{\|(u,v)\|_Y^2}{\left(\displaystyle\int_\Omega |u|^\alpha|v|^\beta + \xi_1|u|^{\alpha+\beta} + \xi_2|v|^{\alpha+\beta} dx\right)^{2/2_s^*}}.$$

The following result establishes a relationship between S and S_s. In local case, it was proved in [1], which the proof in our case follows arguing as was done there combined with the arguments in [17] and [18] for the nonlocal case.

Lemma 2.1 *Let Ω be a domain (not necessarily bounded), then there exists a positive constant m such that $S_s = mS$. Moreover, if w_0 achieves S then (s_0w_0, t_0w_0) achives S_s for some positive constants s_0 and t_0.*

Remark 2.1 *The constant m of the previous lemma is given by $m = M^{-1}$, where $M = \max J(s,t)$ is attained in some (B,C) (with $B, C > 0$) of the compact set $\{(s,t) \in \mathbb{R}^2 \ : \ |s|^2 + |t|^2 = 1\}$ with*

$$J(s,t) := (|s|^\alpha|t|^\beta + \xi_1|s|^{\alpha+\beta} + \xi_2|t|^{\alpha+\beta})^{\frac{2}{\alpha+\beta}}.$$

Therefore,

$$\frac{B^2 + C^2}{(B^\alpha C^\beta + \xi_1 B^{\alpha+\beta} + \xi_2 C^{\alpha+\beta})^{\frac{2}{\alpha+\beta}}} = m.$$

2.1 An eigenvalue problem

For $\lambda \in \mathbb{R}$, we consider the problem with homogeneous Dirichlet boundary condition

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.2}$$

If (2.2) admits a weak solution $u \in X_0^s(\Omega) \setminus \{0\}$, then λ is called an eigenvalue and u a λ-eigenfunction. The set of all eigenvalues is referred as the spectrum of $(-\Delta)^s$ in $X_0^s(\Omega)$ and denoted by $\sigma((-\Delta)^s)$. Since $K = [(-\Delta)^s]^{-1}$ is a compact operator, the problem (2.2) can be written as $u = \lambda K u$ with $u \in L^2(\Omega)$, hence the following results are true (see [32], [34]).

(i) problem (2.2) admits an eigenvalue $\lambda_{1,s} = \min \sigma((-\Delta)^s) > 0$ that can be characterized as follows

$$\lambda_{1,s} = \min_{u \in X_0^s \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx}{\displaystyle\int_{\mathbb{R}^N} |u(x)|^2 dx}; \tag{2.3}$$

(ii) there exists a non-negative function $\varphi_{1,s} \in X_0^s(\Omega)$, which is an eigenfunction corresponding to $\lambda_{1,s}$, attaining the minimum in (2.3);

(iii) all $\lambda_{1,s}$-eigenfunctions are proportional, and if u is a $\lambda_{1,s}$-eigenfunction, then either $u(x) > 0$ a.e. in Ω or $u(x) < 0$ a.e. in Ω;

(iv) the set of the eigenvalues of problem (2.2) consists of a sequence $\{\lambda_{k,s}\}$ satisfying

$$0 < \lambda_{1,s} < \lambda_{2,s} \le \lambda_{3,s} \le \ldots \le \lambda_{j,s} \le \lambda_{j+1,s} \le \ldots, \;\; \lambda_{k,s} \to \infty, \text{ as } k \to \infty,$$

which is characterized by

$$\lambda_{k+1,s} = \min_{u \in \mathbb{P}_{k+1} \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy}{\displaystyle\int_{\mathbb{R}^N} |u(x)|^2 dx} \tag{2.4}$$

where

$$\mathbb{P}_{k+1} = \{u \in X_0^s(\Omega) : \;\; \langle u, \varphi_{j,s} \rangle_X = 0, \; j = 1, 2, \ldots, k\};$$

(v) if $\lambda \in \sigma((-\Delta)^s) \setminus \{\lambda_{1,s}\}$ and u is a λ-eigenfunction, then u changes sign in Ω.
(vi) Denote by $\varphi_{k,s}$ the eigenfunction associated to the eigenvalue $\lambda_{k,s}$, for each $k \in \mathbb{N}$. The sequence $\{\varphi_{k,s}\}$ is an orthonormal basis either of $L^2(\Omega)$ or of $X_0^s(\Omega)$.

Remark 2.2 *Every eigenfunction of $(-\Delta)^s$ is in $C^{0,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$ (see Theorem 1 of [32] or Proposition 2.4 of [30]).*

3 Proof of Theorem 1.1

The proof of the Theorem 1.1 needs the following lemma (see details in [23]).

Lemma 3.1 *If* (1.4) *hold and* $F_1 \in L^2(\Omega) \times L^2(\Omega)$, *then the system*

$$\begin{cases} (-\overrightarrow{\Delta})^s U = AU + F_1 & \textit{in } \Omega, \\ \quad U = 0 & \textit{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.1}$$

has a unique solution $U_0 = (u_0, v_0) \in Y(\Omega)$.

Remark 3.1 *If (1.9) holds, using the Fredholm alternative, we have that (3.1) has a unique solution.*

Remark 3.2 *If* $F_1 \in L^q(\Omega) \times L^q(\Omega)$ *with* $q > \frac{N}{2s}$, *by [[6], Theorem 3.13], we know that the solution* $U_0 = (u_0, v_0) \in C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$.
If $F_1 \in L^\infty(\Omega) \times L^\infty(\Omega)$, *by [[28], Proposition 4.6], the solution* $U_0 = (u_0, v_0) \in C^{0,s}(\overline{\Omega}) \times C^{0,s}(\overline{\Omega})$.
If $s = 1/2$ *and* $F_1 \in C_0^{0,\sigma}(\overline{\Omega}) \times C_0^{0,\sigma}(\overline{\Omega})$, *with* $0 < \sigma < 1$ *and* $N > 2s$, *then* $U_0 \in C^{1,\sigma}(\overline{\Omega}) \times C^{1,\sigma}(\overline{\Omega})$ *and* $\|U_0\|_{(C^{1,\sigma}(\overline{\Omega}))^2} \leq c\|F_1\|_{(C^{0,\sigma}(\overline{\Omega}))^2}$ *(see [9] Proposition 3.1) and if* $s > 1/2$, *arguing as in [5], we have that* $U_0 \in C^{1,2s-1}(\overline{\Omega}) \times C^{1,2s-1}(\overline{\Omega})$. *Moreover, a bootstrap argument ensures that if the function* $F_1 \in C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$ *and* $N > 2s$, *then the solution* U_0 *given by Lemma 3.1 satisfies* $\|U_0\|_{(C^{0,\sigma}(\mathbb{R}^N))^2} \leq c\|F_1\|_{(L^q(\Omega))^2}$, *where* $\sigma = min\{s, 2s - \frac{N}{q}\}$, *for some constant depending only on* N, s, q *and* Ω *(see [27] Proposition 1.4).*

We are ready to prove the existence of a negative solution for the system (1.2).

Proof of Theorem 1.1. We will prove the theorem when the conditions (1.4) and (1.6) hold (other cases ((1.4) and (1.5) or (1.9)) are analogous to this and left to the reader).

By Lemma 3.1 and Remark 3.2, the system

$$\begin{cases} (-\overrightarrow{\Delta})^s U = AU + F_1 & \text{in } \Omega, \\ \quad U = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

has a unique solution $U_0 = (u_0, v_0) \in C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$.
Besides, $(w, z) = \left(\dfrac{(\lambda_{1,s} - c)t + br}{\det(\lambda_{1,s} I - A)} \varphi_{1,s}, \dfrac{bt + (\lambda_{1,s} - a)r}{\det(\lambda_{1,s} I - A)} \varphi_{1,s} \right)$ is the unique solution of the system

$$\begin{cases} (-\overrightarrow{\Delta})^s U = AU + T\varphi_{1,s} & \text{in } \Omega, \\ \quad U = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Consequently, if

$$u_T = \frac{(\lambda_{1,s} - c)t + br}{\det(\lambda_{1,s} I - A)} \varphi_{1,s} + u_0,$$
$$v_T = \frac{bt + (\lambda_{1,s} - a)r}{\det(\lambda_{1,s} I - A)} \varphi_{1,s} + v_0,$$

then $U_T = (u_T, v_T)$ is a solution of the system

$$\begin{cases} (-\overrightarrow{\Delta})^s U = AU + T\varphi_{1,s} + F_1 & \text{in } \Omega, \\ \quad U = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Clearly if u_T and v_T are negative in Ω, we deduce also that U_T is a solution of (1.2). Therefore, to conclude the proof under the conditions (1.4) and (1.6) (see Remark 1.1), it suffices to show the existence of an unbounded region $\boldsymbol{R} \subset \mathbb{R}^2$ where u_T and v_T are negative in Ω for every $T = (t, r) \in \boldsymbol{R}$.

Indeed, since $\varphi_{1,s} \in C^{0,\sigma}(\overline{\Omega})$ is strictly positive in Ω (see corollary 4.8 in [21]) and $u_0, v_0 \in C^0(\overline{\Omega})$, there exists $\eta, \vartheta \ll 0$ such that

$$\eta\varphi_{1,s} + u_0 < 0 \text{ in } \Omega,$$

$$\vartheta\varphi_{1,s} + v_0 < 0 \text{ in } \Omega.$$

Then u_T and v_T are negative in Ω for every $T = (t, r) \in \boldsymbol{R}$ and the proof of theorem is concluded. ■

4 Proof of Theorem 1.2

Let $U_T := (u_T, v_T)$ be the negative solution with $u_T, v_T < 0$ in Ω given by Theorem 1.1 for $T \in \boldsymbol{R}$. Notice that if $\overline{U} \neq (0, 0)$ is a solution of

$$\begin{cases} (-\overrightarrow{\Delta})^s U = AU + \nabla F(U + U_T) & \text{in } \Omega, \\ \quad U = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.1}$$

then $U = \overline{U} + U_T$ is a (second) solution of the system (1.2) with $\overline{U} + U_T \neq U_T$. Therefore, to prove the Theorem 1.2, we only have to show that the system (4.1) has a nonzero solution for every $T \in \boldsymbol{R}$.

Observe that the weak solutions of (4.1) are the critical points of the functional $\mathcal{I}_{\lambda,s} : Y(\Omega) \longrightarrow \mathbb{R}$ given by

$$\begin{aligned} \mathcal{I}_{\lambda,s}(U) = & \frac{C(N,s)}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2 + |v(x) - v(y)|^2}{|x - y|^{N+2s}} dxdy \\ & - \frac{1}{2} \int_\Omega (AU, U)_{\mathbb{R}^2} dx - \int_\Omega F(U + U_T) dx, \end{aligned}$$

where

$$F(U) := \frac{1}{\alpha+\beta}\left[u_+^\alpha v_+^\beta + \xi_1 {u_+}^{\alpha+\beta} + \xi_2 {v_+}^{\alpha+\beta}\right], \text{ for every } U=(u,v)\in\mathbb{R}^2$$

and that $U=0$ is a critical point for $\mathcal{I}_{\lambda,s}$ with $\mathcal{I}_{\lambda,s}(0)=0$.

Remark 4.1 *The nonlinearity F is $(\alpha+\beta)$-homogeneous, i.e.*

$$F(\lambda U) = \lambda^{\alpha+\beta}F(U),\ \forall U\in\mathbb{R}^2,\ \forall\lambda\geq 0.$$

In particular:

(i) $(\nabla F(U),U)_{\mathbb{R}^2} = uF_u(U)+vF_v(U) = (\alpha+\beta)F(U),\ \forall U=(u,v)\in\mathbb{R}^2.$

(ii) *F_u and F_v are $(\alpha+\beta-1)$-homogeneous.*

(iii) *There exists $K>0$ such that*

$$F_u(U)\leq K(|u|^{\alpha+\beta-1}+|v|^{\alpha+\beta-1}) \text{ and}$$

$$F_v(U)\leq K(|u|^{\alpha+\beta-1}+|v|^{\alpha+\beta-1}),$$

for all $U=(u,v)\in\mathbb{R}^2$.

Since $F(U)=F(u_+,v_+)$, $\forall\, U=(u,v)\in\mathbb{R}^2$, we deduce that

$$|\nabla F(U)|\leq K(u_+^{\alpha+\beta-1}+v_+^{\alpha+\beta-1})$$

for some constant $K>0$.

4.1 Geometry of the functional $\mathcal{I}_{\lambda,s}$

In this subsection, we demonstrate that the functional $\mathcal{I}_{\lambda,s}$ satisfies the geometric structure required by the Linking Theorem (see [25, Theorem 5.3]) when $\lambda_{k,s}\leq\mu_1\leq\mu_2<\lambda_{k+1,s}$, for some $k\geq 1$. In particular, if $\mu_2<\lambda_{1,s}$ holds, then the functional satisfies the conditions of the Mountain Pass Theorem.

Since $Y(\Omega)$ is a Hilbert space, consider the following orthogonal decomposition $Y(\Omega)=E_k^-\oplus E_k^+$, where

$$E_k^- = span\{(0,\varphi_{1,s}),(\varphi_{1,s},0),(0,\varphi_{2,s}),(\varphi_{2,s},0),\ldots,(0,\varphi_{k,s}),(\varphi_{k,s},0)\}$$

and $E_k^+ = (E_k^-)^\perp$, for $1\leq k\in\mathbb{N}$. Note that $E_k^+=(\mathbb{P})^2$ and $U\in Y(\Omega)$, then $U=U^-+U^+$ with $U^-\in E_k^-$ and $U^+\in E_k^+$.

Therefore from the variational characterization (2.4), we have the following estimates:

$$\|U\|_Y^2\geq\lambda_{k+1,s}\|U\|^2_{L^2\times L^2},\ \text{for all } U\in E_k^+,$$

$$\|U\|_Y^2\leq\lambda_{k,s}\|U\|^2_{L^2\times L^2},\ \text{for all } U\in E_k^-.$$

Let

$$S_\rho := \partial B_\rho\cap E_k^+$$

and
$Q := \{U \in Y(\Omega) \ : \ U = W + \zeta E, \quad W \in E_k^-, \quad \|W\|_Y \le r, \quad 0 \le \zeta \le R\}$, where $E \in E_k^+$, $0 < \rho < R$ and $r > 0$ will be chosen later so that the following conditions hold:

$$\inf_{U \in S_\rho} \mathcal{I}_{\lambda,s}(U) \ge \sigma > 0,$$

$$\max_{U \in \partial Q} \mathcal{I}_{\lambda,s}(U) \le \alpha_0, \text{ with } \alpha_0 < \sigma,$$

$$\max_{U \in Q} \mathcal{I}_{\lambda,s}(U) \le \frac{s}{N} S^{\frac{N}{2s}}.$$

Proposition 4.1 *Suppose Ω is a smooth bounded domain of $\mathbb{R}^N$, $\alpha + \beta = 2_s^*$ and $\lambda_{k,s} < \mu_1 \le \mu_2 < \lambda_{k+1,s}$, for some $k \in \mathbb{N}$. Then there exists $\rho_0 > 0$ and a function $\alpha : [0, \rho_0] \to \mathbb{R}^+$ such that*

$$\mathcal{I}_{\lambda,s}(U) \ge \alpha(\rho) \ \text{ for all } \ U \in S_\rho := \partial B_\rho(0) \cap E_k^+.$$

Explicitly the maximum value of $\alpha(\rho)$ is

$$\hat{\alpha} = \frac{s}{N} S^{N/2s} (1 - \frac{\mu_2}{\lambda_{k+1,s}})^{\frac{N}{2s}} \frac{1}{(1+\xi)^{\frac{N-2s}{2s}}} \tag{4.2}$$

and this is assumed in $\hat{\rho} = S^{\frac{N}{4s}} (1 - \frac{\mu_2}{\lambda_{k+1,s}})^{\frac{N-2s}{4s}} \frac{1}{(1+\xi)^{\frac{N-2s}{4s}}}$, where S is the best constant for the embedding of X_0^s in $L^{2_s^*}$ and $\xi =: \max\{\xi_1, \xi_2\}$.

Proof Using the fact that $(A(U), U)_{\mathbb{R}^2} \le \mu_2 |u|^2$, we obtain

$$\begin{aligned} \mathcal{I}_{\lambda,s}(U) \quad &\ge \quad \frac{1}{2}||U||_Y^2 - \frac{\mu_2}{2} \int_\Omega |U|^2 dx - \frac{1}{\alpha+\beta} \int_\Omega \Big[\xi_1 (u + u_{r,t})_+^{2_s^*} + \\ & \quad + \xi_2 (v + v_{r,t})_+^{2_s^*} + (u + u_{r,t})_+^{\alpha} (v + v_{r,t})_+^{\beta} \Big] \, dx. \end{aligned}$$

Note that

$$s^\alpha t^\beta \le s^{\alpha+\beta} + t^{\alpha+\beta} \text{ for all } s, t \ge 0 \tag{4.3}$$

and

$$\int_\Omega (u + u_{r,t})_+^{2_s^*} dx \le \int_\Omega |u|^{2_s^*} dx \le S^{-2_s^*/2} ||u||_{X_0^s}^{2_s^*} = S^{-\frac{N}{N-2s}} ||u||_{X_0^s}^{2_s^*}. \tag{4.4}$$

Similarly

$$\int_\Omega (v + v_{r,t})_+^{2_s^*} dx \le S^{-\frac{N}{N-2s}} ||v||_{X_0^s}^{2_s^*}. \tag{4.5}$$

Then, by (4.3), (4.4) and (4.5), we have

$$\begin{aligned} \mathcal{I}_{\lambda,s}(U) \ge & \frac{1}{2} \left(1 - \frac{\mu_2}{\lambda_{k+1,s}} \right) ||U||_Y^2 \\ & - \left(\frac{(1+\xi_1)}{\alpha+\beta} S^{-\frac{N}{N-2s}} ||u||_{X_0^s}^{2_s^*} + \frac{(1+\xi_2)}{\alpha+\beta} S^{-\frac{N}{N-2s}} ||v||_{X_0^s}^{2_s^*} \right). \end{aligned}$$

Since $\xi =: \max\{\xi_1, \xi_2\} \geq \xi_1, \xi_2$, we obtain

$$\mathcal{I}_{\lambda,s}(U) \geq \frac{1}{2}\left(1 - \frac{\mu_2}{\lambda_{k+1,s}}\right)\rho^2 - \frac{(1+\xi)}{\alpha+\beta} S^{-\frac{N}{N-2s}} \rho^{2_s^*} =: \alpha(\rho),$$

where $\rho = ||U||_Y$. Using a standart calculus argument, we obtain that the maximum of $\alpha(\rho)$ is attained in

$$\rho_0 = \frac{1}{(1+\xi)^{\frac{N-2s}{4s}}} S^{N/4s}\left(1 - \frac{\mu_2}{\lambda_{k+1,s}}\right)^{\frac{N-2s}{4s}}.$$

So, the function $\alpha : [0, \rho_0] \to \mathbb{R}^+$ is such that $\mathcal{I}_{\lambda,s}(U) \geq \alpha(\rho)$ for all $U \in S_p$ and the maximum value is

$$\alpha(\rho_0) = \frac{s}{N} S^{N/2s}\left(1 - \frac{\mu_2}{\lambda_{k+1,s}}\right)^{\frac{N}{2s}} \frac{1}{(1+\xi)^{\frac{N-2s}{2s}}}. \tag{4.6}$$

Therefore, $\mathcal{I}_{\lambda,s}(U) \geq \alpha(\rho)$ for all $U \in S_\rho$. The proof of the proposition is complete. ■

It is well know (see [12, Theorem 1.1]) that $S = S_{\alpha+\beta}$ is achieved by

$$\widetilde{u}(x) := k(\mu^2 + |x - x_0|^2)^{-\frac{N-2s}{2}}, \tag{4.7}$$

with $k \in \mathbb{R} \setminus \{0\}, \mu > 0$ and $x_0 \in \mathbb{R}^N$ fixed constants.

Equivalently, we see that

$$S = \inf_{\substack{u \in X_0^s \setminus \{0\} \\ \|u\|_{L^{2_s^*}} = 1}} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy = \int_{\mathbb{R}^{2N}} \frac{|\overline{u}(x) - \overline{u}(y)|^2}{|x-y|^{N+2s}} dxdy$$

where $\overline{u}(x) = \dfrac{\widetilde{u}(x)}{||\widetilde{u}||_{L^{2_s^*}}}$. By translation, supose $x_0 = 0$ in (4.7). Then, the function $u^*(x) = \overline{u}\left(\frac{x}{S^{\frac{1}{2s}}}\right)$, $x \in \mathbb{R}^N$, is a solution for the problem

$$(-\Delta)^s u = |u|^{2_s^*-2}, \text{ em } \mathbb{R}^N \tag{4.8}$$

satisfying

$$||u^*||^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N)} = S^{\frac{N}{2s}}.$$

As in [31], for every $\varepsilon > 0$ we define the family of functions

$$U_\varepsilon(x) := \varepsilon^{-\frac{N-2s}{2}} u^*\left(\frac{x}{\varepsilon}\right), \ x \in \mathbb{R}^N,$$

then U_ε is a solution of (4.8) and verify for all $\varepsilon > 0$

$$\int_{\mathbb{R}^{2N}} \frac{|U_\varepsilon(x) - U_\varepsilon(y)|^2}{|x-y|^{N+2s}} dxdy = \int_{\mathbb{R}^{2N}} |U_\varepsilon(x)|^{2_s^*} dxdy = S^{\frac{N}{2s}}.$$

Now, take a fixed $\delta > 0$ such that $B_{4\delta} \subset \Omega$. Let $\eta \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function such that $0 \leq \eta \leq 1$ in $\mathbb{R}^N$, $\eta = 1$ in B_δ and $\eta = 0$ in $\mathbb{R}^N \setminus B_{2\delta}$, where $B_r = B_r(0)$ is the ball centered in origin and with radius $r > 0$.
Define the family of nonnegative truncated functions

$$u_\varepsilon(x) := \eta(x) U_\varepsilon(x) \;\; x \in \mathbb{R}^N, \tag{4.9}$$

and note that $u_\varepsilon \in X_0^s$.

The following Brezis-Nirenberg estimates for nonlocal setting was proved in [31] (also see [33]), which are similar to those proved for the local case in [8].

Lemma 4.1 *Suppose $s \in (0,1)$ and $N > 2s$, then for $\varepsilon > 0$ small enough, the following estimates hold true,*

$$\int_{\mathbb{R}^{2N}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{N+2s}} dxdy \leq S^{N/2s} + O(\varepsilon^{N-2s}),$$

$$\int_{\mathbb{R}^N} |u_\varepsilon(x)|^2 dx \geq \begin{cases} C_s \varepsilon^{2s} + O(\varepsilon^{N-2s}) & \text{if} \quad N > 4s, \\ C_s \varepsilon^{2s} |log \varepsilon| + O(\varepsilon^{2s}) & \text{if} \quad N = 4s, \\ C_s \varepsilon^{N-2s} + O(\varepsilon^{2s}) & \text{if} \quad 2s < N \leq 4s, \end{cases}$$

$$\int_{\mathbb{R}^N} |u_\varepsilon(x)|^{2_s^*} dx = S^{N/2s} + O(\varepsilon^N),$$

$$||u_\varepsilon||_{L^1(\mathbb{R}^N)} = O(\varepsilon^{\frac{N-2s}{2}}),$$

$$||u_\varepsilon||^{2_s^*-1}_{L^{2_s^*-1}(\mathbb{R}^N)} = O(\varepsilon^{\frac{N-2s}{2}}).$$

Denote by P_- the ortogonal projection of X_0^s in $B_k^- = span\{\varphi_1, \varphi_2, ..., \varphi_k\}$ and P_+ the orthogonal projection of X_0^s in $A_k^+ := (B_k^-)^\perp$.
Chosing depending on $\varepsilon > 0$ the vetorial function given by

$$e = \vec{e}_\varepsilon = (B(P_+ u_\varepsilon), C(P_+ u_\varepsilon)) \in E_k^+,$$

where u_ε is given in (4.9) and B and C are given by Remark 2.1.

We will denote $P_+ u_\varepsilon$ by e_ε and consequently $\vec{e}_\varepsilon = (Be_\varepsilon, Ce_\varepsilon)$.

Remark 4.2 *(i) $e_\varepsilon \in A_k^+$;*
(ii) $\langle (Be_\varepsilon, Ce_\varepsilon), (0, \varphi_j) \rangle_{L^2 \times L^2} = 0 = \langle (Be_\varepsilon, Ce_\varepsilon), (\varphi_j, 0) \rangle_{L^2 \times L^2}$, for all $j = 1, ..., k$. Then $e = \vec{e}_\varepsilon \in E_k^+$.

Hence, the following results was proved in [3], which are similar to those proved for the local case in [13].

Lemma 4.2 *For $s \in (0,1)$ and $N > 2s$, then for $\varepsilon > 0$ small enough, the following estimates hold true,*

$$||P_+u_\varepsilon||^2_{X_0^s} \leq [u_\varepsilon]_s^2 \leq S^{N/2s} + O(\varepsilon^{N-2s}),$$

$$\Big| ||P_+u_\varepsilon||^{2_s^*}_{L^{2_s^*}(\Omega)} - ||u_\varepsilon||^{2_s^*}_{L^{2_s^*}(\Omega)} \Big| \leq C\varepsilon^{N-2s},$$

$$||P_+u_\varepsilon||_{L^1(\Omega)} \leq C\varepsilon^{\frac{N-2s}{2}},$$

$$||P_+u_\varepsilon||^{2_s^*-1}_{L^{2_s^*-1}(\mathbb{R}^N)} \leq C\varepsilon^{\frac{N-2s}{2}},$$

$$|P_-u_\varepsilon(x)| \leq C\varepsilon^{\frac{N-2s}{2}}, \ for\ x \in \Omega. \tag{4.10}$$

Fix $K > 0$ and define $\Omega_{\varepsilon,K} = \{x \in \Omega : e_\varepsilon(x) = (P_+u_\varepsilon)(x) > K\}$. By (4.10) we can deduce

$$e_\varepsilon(0) = (P_+u_\varepsilon)(0) = u_\varepsilon(0) - P_-u_\varepsilon(0) \geq \frac{C_0}{\|\tilde{u}\|_{L^{2_s^*}(\mathbb{R}^N)}}\varepsilon^{-\frac{(N-2s)}{2}} - C\varepsilon^{\frac{N-2s}{2}},$$

which implies that $P_+u_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$ By the continuity of P_+u_ε, there exists $\nu > 0$ such that $B_\nu \subset \Omega_{\varepsilon,K}$. Therefore, we have the result below.

Lemma 4.3 *For $s \in (0,1)$ and $N > 2s$, we have*

$$||P_+u_\varepsilon||^{2_s^*}_{L^{2_s^*}(\Omega_{\varepsilon,K})} = ||u_\varepsilon||^{2_s^*}_{L^{2_s^*}(\Omega)} + O(\varepsilon^{N-2s}).$$

$$||P_+u_\varepsilon||^{2_s^*-1}_{L^{2_s^*-1}(\Omega_{\varepsilon,K})} = ||u_\varepsilon||^{2_s^*-1}_{L^{2_s^*-1}(\Omega)} + O(\varepsilon^{\frac{N+2s}{2}}).$$

$$||P_+u_\varepsilon||_{L^1(\Omega_{\varepsilon,K})} = ||u_\varepsilon||_{L^1(\Omega)} + O(\varepsilon^N).$$

To prove the geometric conditions of the Linking Theorem, we need two results that can be found in [13] and [14] for the case when $s = 1$. The proof is similar for $s \in (0,1)$.

Lemma 4.4 *Given $u, v \in L^p(\Omega)$ with $2 \leq p \leq 2_s^*$ and $u + v > 0$ a.e. on a measurable subset $\Sigma \subset \Omega$, it holds*

$$\Big| \int_\Sigma (u+v)^p dx - \int_\Sigma |u|^p dx - \int_\Sigma |v|^p dx \Big| \leq C \int_\Sigma (|u|^{p-1}|v| + |u||v|^{p-1})\, dx,$$

with a constant $C > 0$ depending only on p.

Lemma 4.5 *Given $(a,b),(u,v)\in L^p(\Omega)\times L^q(\Omega)$ with $p,q\geq 2$ and $p+q\leq 2^*_s$. If $a+b,\ u+v>0$ a.e. on a measurable subset $\Sigma\subset\Omega$ and $H(x,y)=|x|^p|y|^q$, then*

$$\begin{aligned}
&\Big|\int_\Sigma H(a+u,b+v)dx-\int_\Sigma H(u,v)dx-\int_\Sigma H(a,b)dx\Big|\\
\leq C\Big[&\int_\Sigma (|a|^{p-1}|b|^q|u|+|a|^{p-1}|v|^q|u|+|u|^{p-1}|b|^q|a|+|u|^{p-1}|v|^q|a|)\,dx\\
&+\int_\Sigma (|a|^{p-1}|v|^{q-1}|b||u|+|u|^p|b|^q+|u|^p|v|^{q-1}|b|)\,dx\\
&+\int_\Sigma (|a|^p|b|^{q-1}|v|+|a|^p|v|^q+|u|^{p-1}|b|^{q-1}|a||v|)\,dx\\
&+\int_\Sigma (|b|^{q-1}|u|^p|v|+|v|^{q-1}|a|^p|b|)\,dx\Big],
\end{aligned}\tag{4.11}$$

where the constant $C>0$ depending only on $p+q$.

Proof Let us define

$$h(\zeta):=\int_\Sigma [H(a+\zeta u,b+\zeta v)-H(\zeta u,\zeta v)]\,dx.$$

Employing the Fundamental Theorem of the Calculus, $|h(1)-h(0)|=\int_0^1 h'(\zeta)\,d\zeta$, and consequently

$$\begin{aligned}
&\int_\Sigma [H(a+u,b+v)-H(a,b)]\,dx\\
&\leq \int_0^1\int_\Sigma |((\nabla H(a+\zeta u,b+\zeta v)-\nabla H(\zeta u,\zeta v))\ ,\ (u,v))_{\mathbb{R}^2}|\,dxd\zeta.
\end{aligned}\tag{4.12}$$

Using the Mean Value Theorem to the function $\nabla H(x,y)$, there exist $\theta_1,\theta_2\in(0,1)$ such that

$$\begin{aligned}
&\nabla H(a+\zeta u,b+\zeta v)-\nabla H(\zeta u,\zeta v)\\
&=\Big(p|a+\zeta u|^{p-2}(a+\zeta u)|b+\zeta v|^q-p|\zeta u|^{p-2}(\zeta u)|\zeta v|^q\ ,\\
&\quad q|a+\zeta u|^p|b+\zeta v|^{q-2}(b+\zeta v)-q|\zeta u|^p|\zeta v|^{q-2}(\zeta v)\Big)\\
&=\Big(p(p-1)|(1-\theta_1)a+\zeta u|^{p-2}|(1-\theta_1)b+\zeta v|^q a\\
&+pq|(1-\theta_1)a+\zeta u)|^{p-2}((1-\theta_1)a+\zeta u)|(1-\theta_1)b+\zeta v|^{q-2}(1-\theta_1)b+\zeta v)b\ ,\\
&pq|(1-\theta_2)a+\zeta u)|^{p-2}((1-\theta_2)a+\zeta u)|(1-\theta_2)b+\zeta v|^{q-2}(1-\theta_2)b+\zeta v)a\\
&+q(q-1)|(1-\theta_2)b+\zeta v|^{q-2}|(1-\theta_2)a+\zeta u|^p b\Big).
\end{aligned}\tag{4.13}$$

Inequality (4.11) follows by substituting (4.13) in (4.12) and making some forward estimations. ■

The following inequality which is a direct consequence of Young Inequality, is essential for the proof of Theorem 1.2.

Lemma 4.6 *If $\alpha, \beta > 1$, $\alpha + \beta = 2_s^*$ and $\alpha > \dfrac{2_s^* - 1}{2}$, there is $p > 2$ such that, each $\varepsilon > 0$, the following inequality holds*

$$|s|^\alpha |t|^\beta \leq C_\varepsilon |s|^{2_s^* - 1} + C\varepsilon^p |t|^{\beta p}$$

where C_ε and C are positive constants.

Finally we need

Lemma 4.7 *Suppose A, B, C and θ positive numbers. Consider the function $\Phi_\varepsilon(s) = \frac{1}{2}s^2 A - \frac{1}{2_s^*}s^{2_s^*}B + s^{2_s^*}\varepsilon^\theta C$ with $s > 0$. Then $s_\varepsilon = \left(\frac{A}{B - 2_s^* \varepsilon^\theta C}\right)^{\frac{1}{2_s^* - 2}}$ is the maximum point of Φ_ε and*

$$\Phi_\varepsilon(s) \leq \Phi_\varepsilon(s_\varepsilon) = \frac{s}{N}\left(\frac{A^N}{B^{N-2s}}\right)^{\frac{1}{2s}} + O(\varepsilon^\theta).$$

Lemma 4.8 *If $\lambda_{k,s} < \mu_1 \leq \mu_2 < \lambda_{k+1,s}$, there are constants $r_0, R_0 > 0$ and $\varepsilon_0 > 0$ such that, for $r > r_0$, $R > R_0$ and $0 < \varepsilon \leq \varepsilon_0$, we have*

$$\mathcal{I}_{\lambda,s}|_{\partial Q} < \hat{\alpha},$$

with $\hat{\alpha} > 0$ as in Proposition 4.1.

Proof Let $\partial Q = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 = \overline{B_R} \cap E_k^-,$$

$$\Gamma_2 = \{U \in Y : U = W + s\vec{e}_\varepsilon \text{ with } W \in E_k^-,\ ||W||_Y = r,\ 0 \leq s \leq R\},$$

$$\Gamma_3 = \{U \in Y : U = W + R\vec{e}_\varepsilon \text{ with } W \in E_k^- \cap B_r(0)\}.$$

We will show that for each Γ_i we have $\mathcal{I}_{\lambda,s}|_{\Gamma_i} < \hat{\alpha}$, for all $i = 1, 2, 3$.
(i) For all $U \in \Gamma_1 (\subset E_K^-)$, using (2.1), we infer that

$$\mathcal{I}_{\lambda,s}(U) \leq \frac{1}{2}||U||_Y^2 - \frac{\mu_1}{2}\frac{1}{\lambda_{k,s}}||U||_Y^2 = \frac{1}{2}\left(1 - \frac{\mu_1}{\lambda_{k,s}}\right)||U||_Y^2 \leq 0.$$

(ii) Let $U \in \Gamma_2$, then $U = W + s\vec{e}_\varepsilon$ with $W = (w_1, w_2) \in E_k^-$ and $\vec{e} := (B(P_+ u_\varepsilon), C(P_+ u_\varepsilon)) = (Be_\varepsilon, Ce_\varepsilon)$, where the positive constants B and C are chosen as in Remark 2.1.

Hence

$$\begin{aligned}
\mathcal{I}_{\lambda,s}(U) \quad &\leq \quad \frac{1}{2}\left(1 - \frac{\mu_1}{\lambda_{k,s}}\right)||W||_Y^2 + \frac{s^2}{2}(B^2 + C^2)||e_\varepsilon||_{X_0^s}^2 \\
&- \quad \frac{1}{\alpha + \beta}\int_\Omega (w_1 + sBe_\varepsilon + u_{rt})_+^\alpha (w_2 + sCe_\varepsilon + v_{rt})_+^\beta dx \\
&- \quad \frac{\xi_1}{\alpha + \beta}\int_\Omega (w_1 + sBe_\varepsilon + u_{rt})_+^{\alpha+\beta} dx - \frac{\xi_2}{\alpha + \beta}\int_\Omega (w_2 + sCe_\varepsilon + v_{rt})_+^{\alpha+\beta} dx.
\end{aligned}$$

Consider the maximum value $\hat{\alpha}$ of the function $\alpha(\rho)$ like in (4.6), and define

$$s_0 := \frac{\sqrt{2\frac{s}{N}S^{N/2s}(1-\frac{\mu_2}{\lambda_{k+1,s}})^{\frac{N}{2s}}\frac{1}{(1+\xi)^{\frac{N-2}{2s}}}}}{\sqrt{\sup\limits_{0<\varepsilon\leq 1}||\vec{e}_\varepsilon||_Y^2}} = \frac{\sqrt{2\hat{\alpha}}}{\sqrt{\sup\limits_{0<\varepsilon\leq 1}||\vec{e}_\varepsilon||_Y^2}}. \tag{4.14}$$

In order to comply the condition $\mathcal{I}_{\lambda,s}|_{\Gamma_2} < \hat{\alpha}$, we distinguish in our analysis two cases.
First case: If $0 \leq s \leq s_0$.
The expression of $\mathcal{I}_{\lambda,s}$ provides the estimate

$$\mathcal{I}_{\lambda,s}(U) \leq \frac{s^2}{2}||\vec{e}_\varepsilon||_Y^2 \leq \frac{s_0^2}{2}\sup_{0<\varepsilon\leq 1}||\vec{e}_\varepsilon||_Y^2 = \hat{\alpha}.$$

What concludes this case.
Second case: $s > s_0$.
Define

$$K := \sup\left\{ \left\|\frac{W+(u_{r,t},v_{r,t})}{s}\right\|_{L^\infty\times L^\infty} : s_0 \leq s \leq R,\ \|W\|_Y = r,\ W \in E_k^-\right\},$$

with $K > 0$ independent of R.
Then, by (4.9) and (4.10) we have

$$\begin{aligned} e_\varepsilon(0) &= (P_+u_\varepsilon)(0) = u_\varepsilon(0) - P_-u_\varepsilon(0) \\ &\geq \frac{C_0}{\|\tilde{u}\|_{L^{2_s^*}(\mathbb{R}^N)}}\varepsilon^{-\frac{(N-2s)}{2}} - c\varepsilon^{\frac{N-2s}{2}} \to +\infty, \end{aligned}$$

as $\varepsilon \to 0$ because $N > 2s$.
By the continuity of e_ε, we have

$$\Omega_\varepsilon = \{x \in \Omega : e_\varepsilon(x) = (P_+u_\varepsilon)(x) > K\} \neq \emptyset$$

for $\varepsilon > 0$ small enough. Therefore, by Lemmas 4.4 and 4.5, for $j = 1$ and $z_{r,t} = u_{r,t}$ or $j = 2$ and $z_{r,t} = v_{r,t}$, we have

$$\begin{aligned} &\int_{\Omega_\varepsilon}\left(Be_\varepsilon + \frac{w_j+z_{r,t}}{s}\right)^{\alpha+\beta}dx \geq \int_{\Omega_\varepsilon}|Be_\varepsilon|^{2_s^*}dx + \int_{\Omega_\varepsilon}\left|\frac{w_j+z_{r,t}}{s}\right|^{\alpha+\beta}dx \\ &-C\int_{\Omega_\varepsilon}\left(|Be_\varepsilon|^{2_s^*-1}\left|\frac{w_j+z_{r,t}}{s}\right| + |Be_\varepsilon|\left|\frac{w_j+z_{r,t}}{s}\right|^{2_s^*-1}\right)dx \end{aligned} \tag{4.15}$$

and

$$\int_{\Omega_\varepsilon}\Big(Be_\varepsilon+\frac{w_1+u_{r,t}}{s}\Big)_+^\alpha\Big(Ce_\varepsilon+\frac{w_2+v_{r,t}}{s}\Big)_+^\beta dx \tag{4.16}$$
$$\geq\int_{\Omega_\varepsilon}B^\alpha C^\beta|e_\varepsilon|^{\alpha+\beta}dx+\int_{\Omega_\varepsilon}\left|\frac{w_1+u_{r,t}}{s}\right|^\alpha\left|\frac{w_2+v_{r,t}}{s}\right|^\beta dx$$
$$-K\int_{\Omega_\varepsilon}\Big(\left|\frac{w_1+u_{r,t}}{s}\right|^{\alpha-1}\left|\frac{w_2+v_{r,t}}{s}\right|^\beta|Be_\varepsilon|+\left|\frac{w_1+u_{r,t}}{s}\right|^{\alpha-1}|Ce_\varepsilon|^\beta|Be_\varepsilon|$$
$$+|Be_\varepsilon|^{\alpha-1}\left|\frac{w_2+v_{r,t}}{s}\right|^\beta\left|\frac{w_1+u_{r,t}}{s}\right|+|Be_\varepsilon|^{\alpha-1}|Ce_\varepsilon|^\beta\left|\frac{w_1+u_{r,t}}{s}\right|\Big)\,dx$$
$$-K\int_{\Omega_\varepsilon}\Big(\left|\frac{w_1+u_{r,t}}{s}\right|^{\alpha-1}\left|\frac{w_2+v_{r,t}}{s}\right||Ce_\varepsilon|^{\beta-1}|Be_\varepsilon|+|Be_\varepsilon|^\alpha\left|\frac{w_2+v_{r,t}}{s}\right|^\beta$$
$$+|Be_\varepsilon|^\alpha|Ce_\varepsilon|^{\beta-1}\left|\frac{w_2+v_{r,t}}{s}\right|\Big)\,dx$$
$$-K\int_{\Omega_\varepsilon}\Big(\left|\frac{w_1+u_{r,t}}{s}\right|^\alpha\left|\frac{w_2+v_{r,t}}{s}\right|^{\beta-1}|Ce_\varepsilon|+\left|\frac{w_1+u_{r,t}}{s}\right|^\alpha|Ce_\varepsilon|^\beta$$
$$+|Be_\varepsilon|^{\alpha-1}\left|\frac{w_2+v_{r,t}}{s}\right|^{\beta-1}\left|\frac{w_1+u_{r,t}}{s}\right||Ce_\varepsilon|\Big)\,dx$$
$$-K\int_{\Omega_\varepsilon}\Big(\left|\frac{w_2+v_{r,t}}{s}\right|^{\beta-1}|Be_\varepsilon|^\alpha|Ce_\varepsilon|+|Ce_\varepsilon|^{\beta-1}\left|\frac{w_1+u_{r,t}}{s}\right|^\alpha\left|\frac{w_2+v_{r,t}}{s}\right|\Big)\,dx.$$

Then, using the estimates (4.15) and (4.16), we can see that, for $\varepsilon>0$ small enough,

$$\begin{aligned}\mathcal{I}_{\lambda,s}(U)\quad&\leq\quad\frac{1}{2}\Big(1-\frac{\mu_1}{\lambda_{k,s}}\Big)||W||_Y^2+\frac{s^2}{2}(B^2+C^2)||e_\varepsilon||^2_{X_0^s}\\&\quad-\frac{s^{2_s^*}}{2_s^*}(B^\alpha C^\beta+\xi_1B^{2_s^*}+\xi_2C^{2_s^*})||e_\varepsilon||^{2_s^*}_{L^{2_s^*}(\Omega_\varepsilon)}\\&\quad+K\frac{s^{2_s^*}}{2_s^*}\Big(\|e_\varepsilon\|^{2_s^*-1}_{L^{2_s^*-1}(\Omega_\varepsilon)}+\|e_\varepsilon\|_{L^1(\Omega_\varepsilon)}+\|e_\varepsilon\|^{\alpha+1}_{L^{\alpha+1}(\Omega_\varepsilon)}+\|e_\varepsilon\|^{\beta+1}_{L^{\beta+1}(\Omega_\varepsilon)}\\&\quad+\|e_\varepsilon\|^{\alpha-1}_{L^{\alpha-1}(\Omega_\varepsilon)}+\|e_\varepsilon\|^{\beta-1}_{L^{\beta-1}(\Omega_\varepsilon)}+\|e_\varepsilon\|^{\beta}_{L^{\beta}(\Omega_\varepsilon)}+\|e_\varepsilon\|^{\alpha}_{L^{\alpha}(\Omega_\varepsilon)}\Big).\end{aligned}$$

Now, for each $j\in\{\alpha,\beta,\alpha-1,\beta-1\}$, there exists $C_j>0$ such that

$$\|e_\varepsilon\|^j_{L^j(\Omega_\varepsilon)}\leq C_j\|e_\varepsilon\|^{2_s^*-1}_{L^{2_s^*-1}(\Omega_\varepsilon)}$$

and, by lemma 4.6, for each $j\in\{\alpha+1,\beta+1\}$, there exists $K_j>0$ such that

$$\|e_\varepsilon\|^j_{L^j(\Omega_\varepsilon)}\leq K_j(\|e_\varepsilon\|^{2_s^*-1}_{L^{2_s^*-1}(\Omega_\varepsilon)}+\varepsilon^p),$$

with $p>2$. Therefore, using the above estimate and the Lemmas 4.1, 4.2 and 4.3, we get

$$\mathcal{I}_{\lambda,s}(U)\leq\frac{1}{2}\Big(1-\frac{\mu_1}{\lambda_{k,s}}\Big)\|W\|_Y^2+\Phi_\varepsilon(s),$$

where

$$\Phi_\varepsilon(s):=\frac{s^2}{2}(B^2+C^2)S^{\frac{N}{2s}}-\frac{s^{2_s^*}}{2_s^*}(B^\alpha C^\beta+\xi_1B^{2_s^*}+\xi_2C^{2_s^*})S^{\frac{N}{2s}}+Ks^{2_s^*}O(\varepsilon^q)$$

with $q=\min\{\dfrac{N-2s}{2},p\}$. Then, applying Lemma 4.7, we obtain

$$\begin{aligned}
\mathcal{I}_{\lambda,s}(U) &\le \frac{1}{2}\Big(1-\frac{\mu_1}{\lambda_{k,s}}\Big)r^2+\frac{s}{N}\left(\frac{\Big[(B^2+C^2)S^{\frac{N}{2s}}\Big]^N}{\Big[(B^\alpha C^\beta+\xi_1B^{2_s^*}+\xi_2C^{2_s^*})S^{\frac{N}{2s}}\Big]^{N-2s}}\right)^{\frac{1}{2s}}+O(\varepsilon^q)\\
&= \frac{1}{2}\Big(1-\frac{\mu_1}{\lambda_{k,s}}\Big)r^2+\frac{s}{N}\left(\frac{(B^2+C^2)^{\frac{N}{2s}}}{(B^\alpha C^\beta+\xi_1B^{2_s^*}+\xi_2C^{2_s^*})^{\frac{N-2s}{2s}}}\right)S^{\frac{N}{2s}}+O(\varepsilon^q).
\end{aligned}$$

Since $\lambda_{k,s}<\mu_1$ and $\varepsilon>0$ can be made arbitrarily small, we can choose $r>0$ to be arbitrarily large in the inequality above such that $\mathcal{I}_{\lambda,s}(U)<0$. This leads to the conclusion stated in the proposition for $U\in\Gamma_2$.
(iii) Let $U\in\Gamma_3$. It can be expressed by Γ_3 definition as $U=W+R\vec{e}_\varepsilon$ with $W\in E_k^-\cap B_r(0)$. Analogously to the case (ii), we get

$$\begin{aligned}
&\mathcal{I}_{\lambda,s}(U)\le\frac{1}{2}\Big(1-\frac{\mu_1}{\lambda_{k,s}}\Big)\|W\|_Y^2+(B^2+C^2)\frac{R^2}{2}\|e_\varepsilon\|^2_{X_0^s}\\
&-\frac{B^\alpha C^\beta}{2_s^*}R^{2_s^*}\int_\Omega\Big(e_\varepsilon+\frac{w_1+u_{r,t}}{BR}\Big)_+^\alpha\Big(e_\varepsilon+\frac{w_2+v_{r,t}}{CR}\Big)_+^\beta dx\\
&-\frac{\xi_1B^{2_s^*}}{2_s^*}R^{2_s^*}\int_\Omega\Big(e_\varepsilon+\frac{w_1+u_{r,t}}{BR}\Big)_+^{2_s^*}dx-\frac{\xi_2C^{2_s^*}}{2_s^*}R^{2_s^*}\int_\Omega\Big(e_\varepsilon+\frac{w_2+v_{r,t}}{CR}\Big)_+^{2_s^*}dx.
\end{aligned}$$

Due to the boundedness of the functions $W\in E_k^-\cap B_r(0)$, $u_{r,t}$ and $v_{r,t}$, there exists $k>0$ such that $\|w_1+u_{r,t}\|_{L^\infty}\le k$ and $\|w_2+v_{r,t}\|_{L^\infty}\le k$. Again, since $e_\varepsilon(0)=P_+u_\varepsilon(0)\to\infty$ as $\varepsilon\to0$, there exists $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$, we have $e_\varepsilon(0)>2k$. Then, by the continuity of e_ε we can find $R_1=R_1(\varepsilon)>0$ and $\eta=\eta(\varepsilon)>0$ such that $|\chi|\ge\eta$ for all $R>R_1$, where

$$\chi:=\left\{x\in\Omega\ :\ e_\varepsilon(x)+\frac{w_1(x)+u_{r,t}(x)}{BR}>1\text{ and }e_\varepsilon(x)+\frac{w_2(x)+v_{r,t}(x)}{CR}>1\right\}.$$

Then, we find $\varepsilon_0,R_0>0$ such that for $0<\varepsilon<\varepsilon_0$ and $R>R_0$, we have

$$\mathcal{I}_{\lambda,s}(U)\le0,\quad\text{for all }U\in\Gamma_3.$$

Indeed, let $R_0>\max\{R_1,R_2\}$, where R_0 is such that $\alpha R_0^2-R_0^{2_s^*}<0$, with

$$\alpha=\frac{(B^2+C^2)2_s^*}{2(B^\alpha C^\beta+\xi_1B^{2_s^*}+\xi_2C^{2_s^*})}(\eta^{-1}\|e_\varepsilon\|^2_{X_0^s}).$$

Then, for $\varepsilon>0$ above and $R>R_0$ we find

$$\begin{aligned}
\mathcal{I}_{\lambda,s}(U) &\le \frac{1}{2}\Big(1-\frac{\mu_1}{\lambda_{k,s}}\Big)\|W\|_Y^2+(B^2+C^2)\frac{R^2}{2}\|e_\varepsilon\|^2_{X_0^s(\Omega)}\\
&\quad-\frac{R^{2_s^*}}{2_s^*}B^\alpha C^\beta|\chi|-\xi_1\frac{R^{2_s^*}}{2_s^*}B^{2_s^*}|\chi|-\xi_2\frac{R^{2_s^*}}{2_s^*}C^{2_s^*}|\chi|\\
&\le (B^2+C^2)\frac{R^2}{2}\|e_\varepsilon\|^2_{X_0^s(\Omega)}-(B^\alpha C^\beta+\xi_1B^{2_s^*}+\xi_2C^{2_s^*})\frac{R^{2_s^*}}{2_s^*}\eta<0.
\end{aligned}$$

This completes the proof. ∎

Lemma 4.9 *Let $s \in (0,1)$, $\lambda_{k,s} \leq \mu_1 \leq \mu_2 < \lambda_{k+1,s}$ and $N > 6s$. Then we have the following estimate*

$$\max_{Q} \mathcal{I}_{\lambda,s} < \frac{s}{N} S^{\frac{N}{2s}}$$

Proof: Let $\varepsilon < \varepsilon_0$ fixed that the linking theorem geometry holds. For $W + s\vec{e}_\varepsilon \in Q$, we have

$$\begin{aligned} \mathcal{I}_{\lambda,s}(W + s\vec{e}_\varepsilon) \quad &\leq \quad \frac{1}{2}\Big(1 - \frac{\mu_1}{\lambda_{k,s}}\Big)\|W\|_Y^2 + \frac{s^2}{2}||\vec{e}_\varepsilon||_Y^2 - \frac{\mu_1}{2} s^2 ||\vec{e}_\varepsilon||^2_{L^2\times L^2} \\ &\quad - \int_\Omega F(w + s\vec{e} + U_T)dx. \end{aligned}$$

Let s_0 be defined as in (4.14).

First case: If $0 < s \leq s_0$. Arguing as in the proof of Lemma 4.8 and bearing in mind (4.2), we can see that

$$\mathcal{I}_{\lambda,s}(W + s\vec{e}_\varepsilon) \leq \frac{s^2}{2}||\vec{e}_\varepsilon||_Y^2 \leq \frac{s_0^2}{2} \sup_{0<\varepsilon\leq 1} ||\vec{e}_\varepsilon||_Y^2 = \hat{\alpha} < \frac{s}{N} \frac{1}{(1+\xi)^{\frac{N-2s}{2s}}} S^{\frac{N}{2s}}. \qquad (4.17)$$

Now, by Lemma 2.1 and Remark 2.1, we get

$$\begin{aligned} S^{\frac{N}{2s}} \quad &= \quad \frac{(B^\alpha C^\beta + \xi_1 B^{2_s^*} + \xi_2 C^{2_s^*})^{\frac{N-2s}{2s}}}{(B^2 + C^2)^{\frac{N}{2s}}} S_s^{\frac{N}{2s}} \\ &\leq \quad (1+\xi)^{\frac{N-2s}{2s}} \frac{\Big[(B^2 + C^2)^{\frac{2_s^*}{2}}\Big]^{\frac{N-2s}{2s}}}{(B^2 + C^2)^{\frac{N}{2s}}} S_s^{\frac{N}{2s}} = (1+\xi)^{\frac{N-2s}{2s}} S_s^{\frac{N}{2s}}, \end{aligned}$$

and consequently by the estimate (4.17), we conclude that

$$\mathcal{I}_{\lambda,s}(W + s\vec{e}_\varepsilon) < \frac{s}{N} S_s^{\frac{N}{2s}}.$$

Second case: Let $s > s_0$. As in the proof of Lemma 4.8, from (4.15), Lemma 4.1 and Lemma 4.3, we derive

$$\mathcal{I}_{\lambda,s}(W + s\vec{e}_\varepsilon) \quad \leq \quad \frac{1}{2}s^2\left(||\vec{e}_\varepsilon||_Y^2 - \mu_1||\vec{e}_\varepsilon||^2_{L^2\times L^2}\right) - \int_\Omega F(w + s\vec{e} + U_T)dx.$$

On the other hand,

$$\begin{aligned} &F(w + s\vec{e} + U_T) = \frac{1}{2_s^*}\Big[(sB)^\alpha\Big(e_\varepsilon + \frac{w_1 + u_{r,t}}{sB}\Big)_+^\alpha (sC)^\beta\Big(e_\varepsilon + \frac{w_2 + v_{r,t}}{sC}\Big)_+^\beta \\ &+\xi_1(sB)^{2_s^*}\Big(e_\varepsilon + \frac{w_1 + u_{r,t}}{sB}\Big)_+^{2_s^*} + \xi_2(sC)^{2_s^*}\Big(e_\varepsilon + \frac{w_2 + v_{r,t}}{sC}\Big)_+^{2_s^*}\Big]. \end{aligned}$$

Using previous arguments, we get

$$\mathcal{I}_{\lambda,s}(W + s\vec{e}_\varepsilon) \leq \Phi_\varepsilon(s),$$

where

$$\begin{aligned}\Phi_\varepsilon(s) &:= \frac{1}{2}s^2\left(||\vec{e}_\varepsilon||_Y^2 - \mu_1||\vec{e}_\varepsilon||^2_{L^2\times L^2}\right)\\ &- \frac{s^{2_s^*}}{2_s^*}(B^\alpha C^\beta + \xi_1 B^{2_s^*} + \xi_2 C^{2_s^*})\|e_\varepsilon\|^{2_s^*}_{L^{2_s^*}(\Omega_\varepsilon)} + Ks^{2_s^*}O(\varepsilon^q)\end{aligned}$$

with $q := \min\left\{\frac{N-2s}{2}\ ,\ p\right\} > 2s$ (because $N > 6s$ and $p > 2$).

Applying Lemma 4.7 to the function Φ_ε, we have by Lemmas 4.1, 4.2 and 4.3 and by the choice of B and C,

$$\begin{aligned}\Phi_\varepsilon(s) &\leq \Phi_s(s_\varepsilon)\\ &\leq \frac{s}{N}\frac{\left[(B^2+C^2)S^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) - \mu_1 Ce^{2s} + O(\varepsilon^{N-2s})\right]^{\frac{N}{2s}}}{\left[(B^\alpha C^\beta + \xi_1 B^{2_s^*} + \xi_2 C^{2_s^*})S^{\frac{N}{2s}} + O(\varepsilon^N) + O(\varepsilon^{N-2s})\right]^{\frac{N-2s}{2s}}} + O(\varepsilon^q)\\ &\leq \frac{s}{N}\left[\frac{(B^2+C^2)}{(B^\alpha C^\beta + \xi_1 B^{2_s^*} + \xi_2 C^{2_s^*})^{\frac{N-2s}{N}}}S\right]^{\frac{N}{2s}} - \frac{\mu_1}{N}O(\varepsilon^{2s}) + O(\varepsilon^q).\end{aligned}$$

Since $q > 2s$, taking $\varepsilon > 0$ sufficiently small, we obtain

$$\mathcal{I}_{\lambda,s}(W + s\vec{e}_\varepsilon) \leq \frac{s}{N}S_s^{\frac{N}{2s}}.$$

■

4.2 The Palais-Smale condition for the functional $\mathcal{I}_{\lambda,s}$

In this subsection we discuss a compactness property for the functional $\mathcal{I}_{\lambda,s}$, given by the Palais-Smale condition.

Lemma 4.10 *If $k \geq 0$ and $\lambda_{k,s} < \mu_1 \leq \mu_2 < \lambda_{k+1,s}$. Then every $(PS)_c$ sequence of $\mathcal{I}_{\lambda,s}$ is bounded.*

Proof The Fréchet derivative of the functional $\mathcal{I}_{\lambda,s}$ is given by

$$\mathcal{I}'_{\lambda,s}(u,v)(\varphi,\psi) = \langle(u,v),(\varphi,\psi)\rangle_Y - \int_\Omega (A(u,v),(\varphi,\psi))_{\mathbb{R}^2}dx - \int_\Omega(\nabla F(u+u_T, v+v_T),(\varphi,\psi))_{\mathbb{R}^2}dx,$$

for every $(u,v),(\varphi,\psi) \in Y(\Omega)$.

Let $(U_n) \subset Y(\Omega)$ be a $(PS)_c$-sequence, i.e. satisfying $\mathcal{I}_{\lambda,s}(U_n) = c + o(1)$ and $\langle\mathcal{I}'_{\lambda,s}(U_n),\Psi\rangle = o(1)\|\Psi\|_Y,\ \forall\,\Psi = (\psi,\xi) \in Y(\Omega)$.

Therefore

$$\begin{aligned}\mathcal{I}_{\lambda,s}(U_n) - \frac{1}{2}\mathcal{I}'_{\lambda,s}(U_n)U_n &= \frac{1}{2}\int_\Omega(\nabla F(U_n+U_T),U_n)_{\mathbb{R}^2}dx - \int_\Omega F(U_n+U_T)dx\\ &\leq c + o(1) + o(1)\|U_n\|_Y.\end{aligned} \tag{4.18}$$

From (4.18), we get

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega (\nabla F(U_n+U_T),U_n)_{\mathbb{R}^2}dx - \int_\Omega F(U_n+U_T)dx\\
&= \frac{1}{2}\int_\Omega \Big(\frac{\alpha}{\alpha+\beta}(u_n+u_{r,t})_+^{\alpha-1}(v_n+v_{r,t})_+^{\beta}u_n \ +\xi_1(u_n+u_{r,t})_+^{\alpha+\beta-1}u_n\\
&+\frac{\beta}{\alpha+\beta}(u_n+u_{r,t})_+^{\alpha}(v_n+v_{r,t})_+^{\beta-1}v_n \ +\xi_2(v_n+v_{r,t})_+^{\alpha+\beta-1}v_n\Big)\,dx\\
&-\frac{1}{\alpha+\beta}\int_\Omega \Big((u_n+u_{r,t})_+^{\alpha}(v_n+v_{r,t})_+^{\beta}+\xi_1(u_n+u_{r,t})_+^{\alpha+\beta}+\xi_2(v_n+v_{r,t})_+^{\alpha+\beta}\Big)\,dx \qquad (4.19)\\
&\le c+o(1)+o(1)\|U_n\|_Y.
\end{aligned}
$$

Now note that

$$
\begin{aligned}
&\int_\Omega \Big((u_n+u_{r,t})_+^{\alpha-1}(v_n+v_{r,t})_+^{\beta}u_n\Big)\,dx\\
&=\int_\Omega \Big((u_n+u_{r,t})_+^{\alpha-1}(u_n+u_{r,t})_+(v_n+v_{r,t})_+^{\beta}\Big)\,dx \qquad (4.20)\\
&-\int_\Omega \Big((u_n+u_{r,t})_+^{\alpha-1}(v_n+v_{r,t})_+^{\beta}u_{r,t}\Big)\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega \Big((u_n+u_{r,t})_+^{\alpha+\beta-1}u_n\Big)\,dx \qquad (4.21)\\
&=\int_\Omega (u_n+u_{r,t})_+^{\alpha+\beta}dx-\int_\Omega (u_n+u_{r,t})_+^{\alpha+\beta-1}u_{r,t}dx.
\end{aligned}
$$

Substituting (4.20), (4.21) and expressions similar to these in (4.19), yields that

$$
\left.
\begin{aligned}
\int_\Omega (u_n+u_{r,t})_+^{\alpha}(v_n+v_{r,t})_+^{\beta}dx,\\
\int_\Omega (u_n+u_{r,t})_+^{\alpha+\beta}dx,\\
\int_\Omega (v_n+v_{r,t})_+^{\alpha+\beta}dx
\end{aligned}
\right\} \le c+o(1)+o(1)\|U_n\|_Y. \qquad (4.22)
$$

Now, using (2.1) and that $\Psi=U_n^+=(u_n^+,v_n^+)\in E_k^+$, we obtain

$$
\begin{aligned}
\Big(1-\frac{\mu_2}{\lambda_{k+1,s}}\Big)\|U_n^+\|_Y^2 &\le \|U_n^+\|_Y^2-\int_\Omega (AU_n^+,U_n^+)_{\mathbb{R}^2}dx\\
&= \int_\Omega (\nabla F(U_n+U_T),U_n^+)_{\mathbb{R}^2}dx-\langle \mathcal{I}'_{\lambda,s}(U_n),(U_n^+)\rangle \qquad (4.23)\\
&\le \int_\Omega F_u(U_n+U_T)|u_n^+|dx+\int_\Omega F_v(U_n+U_T)|v_n^+|dx+C\|U_n^+\|_Y.
\end{aligned}
$$

Hence, by Remark 4.1 (iii), there exists a constant $K>0$ such that

$$
\begin{aligned}
F_u(U)&\le K\Big((u)_+^{\alpha+\beta-1}+(v)_+^{\alpha+\beta-1}\Big)\\
F_v(U)&\le K\Big((u)_+^{\alpha+\beta-1}+(v)_+^{\alpha+\beta-1}\Big).
\end{aligned}
$$

Then

$$\begin{aligned}
&\int_\Omega F_u(U_n+U_T)|u_n^+|dx+\int_\Omega F_v(U_n+U_T)|v_n^+|dx\\
\leq& K\int_\Omega\Big((u_n+u_T)_+^{\alpha+\beta-1}+(v_n+v_T)_+^{\alpha+\beta-1}\Big)|u_n^+|dx\\
+&K\int_\Omega\Big((u_n+u_T)_+^{\alpha+\beta-1}+(v_n+v_T)_+^{\alpha+\beta-1}\Big)|v_n^+|dx.
\end{aligned}$$

and using Hölder's inequality with $p=\frac{\alpha+\beta}{\alpha+\beta-1}$ and $q=\alpha+\beta$, Young's inequality, we deduce that

$$\begin{aligned}
&\int_\Omega F_u(U_n+U_T)|u_n^+|dx+\int_\Omega F_v(U_n+U_T)|v_n^+|dx\\
\leq& K\Big\{\varepsilon\|u_n^+\|^2_{L^{2_s^*}}+C_\varepsilon\Big[\|(u_n+u_T)_+\|^{2(2_s^*-1)}_{L^{2_s^*}}+\|(v_n+v_T)_+\|^{2(2_s^*-1)}_{L^{2_s^*}}\Big]\Big\}\\
+&K\Big\{\varepsilon\|v_n^+\|^2_{L^{2_s^*}}+C_\varepsilon\Big[\|(u_n+u_T)_+\|^{2(2_s^*-1)}_{L^{2_s^*}}+\|(v_n+v_T)_+\|^{2(2_s^*-1)}_{L^{2_s^*}}\Big]\Big\}.
\end{aligned}$$

Using (4.22), in view of the embedding $X(\Omega)\hookrightarrow L^r(\Omega)$, $\forall\, r\leq 2_s^*$, we get

$$\begin{aligned}
&\int_\Omega F_u(U_n+U_T)|u_n^+|dx+\int_\Omega F_v(U_n+U_T)|v_n^+|dx\\
&\leq \varepsilon C_1\|U_n^+\|_Y^2+C_2C_\varepsilon+4\varepsilon_n\|U_n\|_Y^{\frac{N+2s}{N}}.
\end{aligned}$$

By (4.23), taking $\varepsilon>0$ small enough, we conclude that

$$\|U_n^+\|_Y^2\leq C_3+C_4\|U_n\|_Y^{\frac{N+2s}{N}}+C_5\|U_n^+\|_Y. \tag{4.24}$$

Analogously, the following estimate is valid

$$\|U_n^-\|_Y^2\leq C_6+C_7\|U_n\|_Y^{\frac{N+2s}{N}}+C_8\|U_n^+\|_Y. \tag{4.25}$$

Using the estimates (4.24) and (4.25), we get

$$\|U_n\|_Y^2\leq C+C\|U_n\|_Y^{\frac{N+2s}{N}}+C\|U_n\|_Y.$$

Since $\dfrac{N+2s}{N}<2$, we conclude that (U_n) is bounded in $Y(\Omega)$. ■

Lemma 4.11 *If $k\geq 0$ and $\lambda_{k,s}<\mu_1\leq\mu_2<\lambda_{k+1,s}$, then the functional $\mathcal{I}_{\lambda,s}$ satisfies the (PS) condition at level c with $c<\dfrac{s}{N}S_s^{\frac{N}{2s}}$.*

Proof Let $(U_n)\subset Y(\Omega)$ be a sequence satisfying

$$\mathcal{I}_{\lambda,s}(U_n)\to c\quad\text{and}\quad\mathcal{I}'_{\lambda,s}(U_n)\to 0\quad\text{in the dual space } Y(\Omega)',$$

as $n\to\infty$. By Lemma 4.10 we have that (U_n) is bounded. Hence passing to a subsequence, we may suppose that

$$\begin{aligned}
&U_n\rightharpoonup U\quad\text{in } Y(\Omega),\\
&U_n\to U\quad\text{in } L^p(\Omega)\times L^p(\Omega),\ \text{for all } p\in[1,2_s^*),\\
&U_n\to U\quad\text{a.e}\quad\text{in }\mathbb{R}^N.
\end{aligned}\tag{4.26}$$

Hence, U is s weak solution to

$$\begin{cases}(-\overrightarrow{\Delta})^s U = AU + \nabla F(U+U_T) & \text{in } \Omega,\\ \qquad U = 0 & \text{in } \mathbb{R}^N \setminus \Omega,\end{cases} \tag{4.27}$$

that is, for any $\Psi \in Y(\Omega)$ it holds

$$\langle U, \Psi\rangle_Y - \int_\Omega (AU, \Psi)_{\mathbb{R}^2} = \int_\Omega (\nabla F(U+U_T), \Psi)_{\mathbb{R}^2}\, dx. \tag{4.28}$$

In particular, taking $\Psi = U$ in (4.28),we get

$$\|U\|_Y^2 - \int_\Omega (AU, U)_{\mathbb{R}^2}\, dx = \int_\Omega (\nabla F(U+U_T), U)_{\mathbb{R}^2}\, dx \tag{4.29}$$

Note that by (4.27) $\big(\langle \mathcal{I}'_{\lambda,s}(U), U\rangle = 0\big)$ and (4.29) we obtain

$$\mathcal{I}_{\lambda,s}(U) = \frac{1}{2}\int_\Omega (\nabla F(U+U_T), U)_{\mathbb{R}^2}\, dx - \int_\Omega F(U+U_T)dx \geq 0. \tag{4.30}$$

By applying the Brezis-Lieb Lemma [7], it follows that

$$\begin{aligned}\|(U_n+U_T)_+\|^{2_s^*}_{L^{2_s^*}\times L^{2_s^*}} &= \|(U_n-U)_+\|^{2_s^*}_{L^{2_s^*}\times L^{2_s^*}} + \|(U+U_T)_+\|^{2_s^*}_{L^{2_s^*}\times L^{2_s^*}} + o(1)\\ \|U_n - U\|_Y^2 &= \|U_n\|_Y^2 - \|U\|_Y^2 + o(1)\end{aligned} \tag{4.31}$$

and by applying the Brezis-Lieb Lemma for homogeneous functions [15], we conclude that

$$\int_\Omega F(U_n+U_T)dx = \int_\Omega F(U+U_T)dx + \int_\Omega F(U_n-U)dx + o(1). \tag{4.32}$$

Also, we have

$$\begin{aligned}\int_\Omega (\nabla F(U_n+U_T), U_n+U_T)_{\mathbb{R}^2}\, dx &- \int_\Omega (\nabla F(U+U_T), U+U_T)_{\mathbb{R}^2}\, dx\\ &= (\alpha+\beta)\int_\Omega F(U_n-U)dx.\end{aligned} \tag{4.33}$$

Then, by using $(4.26), (4.31)$ and (4.33), we deduce

$$\mathcal{I}_{\lambda,s}(U_n) = \frac{1}{2}\|U_n-U\|_Y^2 + \mathcal{I}_{\lambda,s}(U) - \int_\Omega F(U_n-U)dx + o(1). \tag{4.34}$$

On the other hand, by using (4.26), (4.29) and (4.31) and (4.32) we have

$$
\begin{aligned}
&\langle \mathcal{I}'_{\lambda,s}(U_n), U_n\rangle = \|U_n\|_Y^2 - \int_\Omega (AU_n, U_n)_{\mathbb{R}^2} dx - \int_\Omega (\nabla F(U_n+U_T), U_n)_{\mathbb{R}^2} dx \\
&= \left[\|U_n - U\|_Y^2 + \|U\|^2 + o(1)\right] - \left[\int_\Omega (AU, U)_{\mathbb{R}^2} dx + o(1)\right] \\
&\quad - \int_\Omega (\nabla F(U_n+U_T), U_n + U_T)_{\mathbb{R}^2} dx + \int_\Omega (\nabla F(U_n+U_T), U_T)_{\mathbb{R}^2} dx \\
&= \|U_n - U\|_Y^2 + \left[\|U\|_Y^2 - \int_\Omega (AU, U)_{\mathbb{R}^2} dx\right] - \left[(\alpha+\beta)\int_\Omega F(U_n - U) dx \right.\\
&\quad \left. + \int_\Omega (\nabla F(U+U_T), U + U_T)_{\mathbb{R}^2} dx\right] + \int_\Omega (\nabla F(U_n+U_T), U_T)_{\mathbb{R}^2} dx + o(1) \\
&= \|U_n - U\|_Y^2 + \left[\int_\Omega (\nabla F(U+U_T), U)_{\mathbb{R}^2} dx\right] - \left[(\alpha+\beta)\int_\Omega F(U_n - U) dx \right.\\
&\quad \left. + \int_\Omega (\nabla F(U+U_T), U)_{\mathbb{R}^2} dx + \int_\Omega (\nabla F(U+U_T), U_T)_{\mathbb{R}^2} dx\right] + \int_\Omega (\nabla F(U_n+U_T), U_T)_{\mathbb{R}^2} dx + o(1) \\
&= \|U_n - U\|_Y^2 - (\alpha+\beta)\int_\Omega F(U_n - U) dx + \int_\Omega (\nabla F(U+U_T), U_T)_{\mathbb{R}^2} dx \\
&\quad + \int_\Omega (\nabla F(U_n+U_T), U_T)_{\mathbb{R}^2} dx + o(1).
\end{aligned}
$$

Taking into account that $\langle \mathcal{I}'_{\lambda,s}(U_n), U_n\rangle \to 0$ and $\int_\Omega (\nabla F(U_n+U_T), U_T)_{\mathbb{R}^2} dx \to \int_\Omega (\nabla F(U+U_T), U_T)_{\mathbb{R}^2} dx$ as $n \to \infty$, we deduce

$$\|U_n - U\|_Y^2 = (\alpha+\beta)\int_\Omega F(U_n - U) dx + o(1). \tag{4.35}$$

Let

$$L := \lim_{n\to\infty} \|U_n - U\|_Y^2 \geq 0.$$

If $L = 0$, then $U_n \to U$ in $Y(\Omega)$ as $n \to \infty$.

Let $L > 0$. Then, by the definition of S_s,

$$S_s \leq \frac{\|U\|_Y^2}{\left(\int_\Omega |u|^\alpha |v|^\beta + \xi_1 |u|^{\alpha+\beta} + \xi_2 |v|^{\alpha+\beta} dx\right)^{\frac{2}{\alpha+\beta}}} \quad \text{for all } U = (u,v) \neq (0,0).$$

and (4.35), we can infer

$$
\begin{aligned}
\|U_n - U\|_Y^2 &\geq S_s \left(\int_\Omega (u_n - u)_+^\alpha (v_n - v)_+^\beta + \xi_1 (u_n - u)_+^{\alpha+\beta} + \xi_2 (v_n - v)_+^{\alpha+\beta} dx\right)^{\frac{2}{\alpha+\beta}} \\
&= S_s \left((\alpha+\beta)\int_\Omega F(U_n - U) dx\right)^{\frac{2}{\alpha+\beta}}
\end{aligned}
$$

which gives

$$L \geq S_s L^{\frac{N-2s}{N}}, \text{ i.e. } L \geq S_s^{\frac{N}{2s}}. \tag{4.36}$$

Now, from (4.30), (4.34), (4.35), (4.36) we get

$$\frac{s}{N} S_s^{\frac{N}{2s}} \leq \left(\frac{2s}{N-2s} \right) \frac{L}{2_s^*} \leq c < \frac{s}{N} S_s^{\frac{N}{2s}},$$

which contradiction. ■

Proof of Theorem 1.2. In the case where $\lambda_{k,s} < \mu_1 \leq \mu_2 < \lambda_{k+1,s}$ occurs, the Proposition 4.1 and Lemma 4.8 with $\varepsilon > 0$ small enough, ensure that the functional $\mathcal{I}_{\lambda,s}$ satisfies the geometric structure required by the Linking Theorem. Therefore, it follows from the Linking Theorem without the Palais-Smale condition, that there exists a sequence $(U_n) \subset Y(\Omega)$ satisfying $\mathcal{I}_{\lambda,s}(U_n) \to c$ and $\mathcal{I}'_{\lambda,s}(U_n) \to 0$ in $Y(\Omega)'$, and by Lemma 4.9, the critical level satisfies

$$0 < c := \inf_{\gamma \in \Gamma} \sup_{U \in Q} \mathcal{I}_{\lambda,s}(\gamma(U)) \leq \frac{s}{N} S_s^{\frac{N}{2s}},$$

where $\Gamma := \{\gamma \in C^0(Q, Y(\Omega)) : \ \gamma = Id \text{ on } \partial Q\}$. By Lemma 4.10, (U_n) is bounded in $Y(\Omega)$ and consequently the Lemma 4.11 ensures that $U_n \to \overline{U}$ in $Y(\Omega)$.
If $0 = \lambda_{0,\mu} < \mu_1 \leq \mu_2 < \lambda_{1,\mu}$, to show that the functional $\mathcal{I}_{\lambda,s}$ satisfies the geometrical conditions of the Mountain Pass Theorem, it is enough to take the finite dimensional subspace $E^- = \{(0,0)\}$ and to apply the Proposition 4.1 with $E_k^+ = Y(\Omega)$ such that $R\|\vec{e}_\varepsilon\|_Y > \rho$ with $R > 0$ sufficient large to ensure that $\mathcal{I}_{\lambda,s}(R\vec{e}_\varepsilon) < 0$. The $(PS)_c$ condition is guaranteed by making $k = 0$ in the Lemmas 4.10 and 4.11. Thus, in both cases, there exists a non-trivial solution $\overline{U}$ for the problem (4.1). By [23] Remark 4.1, it follows that $\overline{U}_+ \neq 0$ and therefore, U_T and $U_T + \overline{U}$ are distinct solutions for the problem (1.2). ■

5 The resonant case

5.1 Proof of Theorem 1.3

In this subsection we discuss a compactness property for the functional $\mathcal{I}_{\lambda,s}$, given by the Palais-Smale condition for this case.

Lemma 5.1 *If $N > 6s$ and $\lambda_{k,s} = \mu_1 \leq \mu_2 < \lambda_{k+1,s}$ for $k > 1$, the functional $\mathcal{I}_s$ satisfies the (PS) condition.*

Proof We follow the notations of the previous proof.
Let $U_n \in Y(\Omega)$ such that $\mathcal{I}_s(U_n) \to c$ and $\mathcal{I}'_s(U_n) \to 0$ in the dual space $Y(\Omega)'$. Writing $Y(\Omega) = E_{k-1}^- \oplus E_k^+ \oplus Z_k$, consequently we have

$$U_n = U_n^- + U_n^+ + \beta_n Y_n := W_n + \beta_n Y_n,$$

where $U_n^- \in E_{k-1}^-$, $U_n^+ \in E_k^+ = (E_k^-)^\perp$ and $Y_n \in Z_k = span\{(\varphi_{k,s}, 0), (0, \varphi_{k,s})\}$ with $\|Y_n\|_Y = 1$.
Using similar arguments as in (4.24) and (4.25), we obtain

$$\|W_n\|_Y^2 \leq C + C\|U_n\|_Y^\tau + C\|W_n\|_Y, \tag{5.1}$$

where $\tau = \frac{N+2s}{N}$. We can assume $\|U_n\|_Y \geq 1$ (if $\|U_n\|_Y \leq 1$, the sequence (U_n) is bounded in $Y(\Omega)$). Then, since $\|U_n\|_Y \leq \|W_n\|_Y + |\beta_n|$, from (5.1), we have

$$\|W_n\|_Y^2 \leq C_1(\|W_n\|_Y + |\beta_n|)^\tau + C\|W_n\|_Y. \tag{5.2}$$

If β_n is bounded, since $\tau < 2$, by (5.2) we conclude that (U_n) is bounded in $Y(\Omega)$. Otherwise, we may assume $\beta_n \to +\infty$, therefore, from (5.2), it follows that

$$\left\|\frac{W_n}{\beta_n}\right\|_Y^2 \leq C_1 \left\{\frac{(\|W_n\|_Y + |\beta_n|)^{\tau/2}}{|\beta_n|}\right\}^2 + C\frac{1}{\beta_n}\left\|\frac{W_n}{\beta_n}\right\|_Y$$

$$\leq C_1 \left\{\frac{1}{|\beta_n|^{1-\tau/2}}\left\|\frac{W_n}{\beta_n}\right\|_Y^{\tau/2} + \frac{1}{|\beta_n|^{1-\tau/2}}\right\}^2 + C\frac{1}{\beta_n}\left\|\frac{W_n}{\beta_n}\right\|_Y. \tag{5.3}$$

Using again the fact that $\tau/2 < 1$, the above estimate yields that

$$\left\|\frac{W_n}{\beta_n}\right\|_Y^2 \leq C_2\left\|\frac{W_n}{\beta_n}\right\|_Y^{\tau} + C_3\left\|\frac{W_n}{\beta_n}\right\|_Y + C_4$$

and consequently the sequence $\left\{\dfrac{W_n}{\beta_n}\right\}$ is bounded in $Y(\Omega)$ and by (5.3), $\left\|\dfrac{W_n}{\beta_n}\right\|_Y \to 0$. Therefore, possibly up to a subsequence, $W_n/\beta_n \to 0$ a.e. in Ω and strongly in $L^q(\Omega) \times L^q(\Omega)$, $1 \leq q < 2_s^*$; $Y_n \to Y_0 \in Z_k$ a.e. in Ω and strongly in $Y(\Omega)$ and $L^q(\Omega) \times L^q(\Omega)$, $1 \leq q < 2_s^*$.
Now, taking $\beta_n Y_n \in Z_k$ as test function, we get

$$\mathcal{I}_s'(U_n)Y_n = \beta_n\left(\|Y_n\|_Y^2 - \int_\Omega (AY_n, Y_n)_{\mathbb{R}^2}dx\right) - \int_\Omega (\nabla F(U_n + U_T), Y_n)_{\mathbb{R}^2}dx.$$

Since (U_n) is a (PS)-sequence and $\frac{1}{(\beta_n)^{\frac{4s}{N-2s}}}\left(\|Y_n\|_Y^2 - \int_\Omega (AY_n, Y_n)_{\mathbb{R}^2}dx\right) \to 0$, as $n \to \infty$, we obtain that

$$o(1) = \frac{1}{(\beta_n)^{\frac{N+2s}{N-2s}}}\mathcal{I}_s'(U_n)(Y_n) = -\frac{1}{(\beta_n)^{\frac{N+2s}{N-2s}}}\int_\Omega (\nabla F(U_n + U_T), Y_n)_{\mathbb{R}^2}dx.$$

Now, from Remark 4.1 (ii),

$$\int_\Omega (\nabla F\Big(\frac{U_n + U_T}{\beta_n}\Big), Y_n)_{\mathbb{R}^2}dx = \frac{1}{(\beta_n)^{\frac{N+2s}{N-2s}}}\int_\Omega (\nabla F(U_n + U_T), Y_n)_{\mathbb{R}^2}dx \to 0. \tag{5.4}$$

On the other hand, since $U_n = W_n + \beta_n Y_n$, we have that $\dfrac{U_n}{\beta_n} \to Y_0$ in $L^q(\Omega) \times L^q(\Omega)$ for all $1 \leq q < 2_s^*$ and a.e in Ω. So, by the Dominated Convergence Theorem and by (5.4), it follows that

$$\int_\Omega (\nabla F\Big(\frac{U_n + U_T}{\beta_n}\Big), Y_n)_{\mathbb{R}^2}dx \to \int_\Omega (\nabla F(Y_0), Y_0)_{\mathbb{R}^2}dx = 0$$

and from Remark 4.1 (i), we concluded that $\displaystyle\int_\Omega F(Y_0)dx = 0$.
Finally, using the notation $Y_0 = (y_1^0, y_2^0)$, it follows that $(y_1^0)_+ = 0 = (y_2^0)_+$, contradicting $\|Y_0\|_Y = 1$ and $Y_0 \in Z_k$ with $k > 1$, which ensures that at least one of the functions is not null and changes sign. Thus (U_n) is bounded and using the fact that $N > 6s$, as in the proof of Lemmas 4.9 and 4.11, we have that (U_n) admits a convergent subsequence. ■

5.2 Geometry in resonant case

In this subsection, we demonstrate that the functional $\mathcal{I}_{\lambda,s}$ satisfies the geometric structure required by the Linking Theorem in resonant case, that is, we obtain the following result.

Proposition 5.1 *Suppose Ω is a smooth bounded domain of $\mathbb{R}^N$, $\alpha+\beta=2_s^*$ and $\lambda_{k,s}=\mu_1\le\mu_2<\lambda_{k+1,s}$ for some $k>1$. Then*

i) there exist $\sigma,\rho>0$ such that $\mathcal{I}_s(U)\ge\sigma$ for all $U\in E_k^+$ with $\|U\|_Y=\rho$,

ii) there exists $E\in E_k^+$ and $R>0$ such that $R\|E\|_Y>\rho$ and $\mathcal{I}_s(U)\le 0$, for all $U\in\partial Q$, where $Q=(\overline{B}_R\cap E_k^-)\oplus[0,R]E$.

Proof i) Let $U=(u,v)\in E_k^+$, using the fact that $u_T,v_T<0$, estimate $|u|^\alpha|v|^\beta\le|u|^{\alpha+\beta}+|v|^{\alpha+\beta}$ and the fractional imbedding $X\hookrightarrow L^{\alpha+\beta}$, by (2.1), we have

$$\begin{aligned}\mathcal{I}_s(U) &\ge \frac{1}{2}\|U\|_Y^2-\frac{\mu_2}{2}\|U\|^2_{(L^2)^2}-C\int_\Omega(|u|^{\alpha+\beta}+|v|^{\alpha+\beta})dx\\ &\ge \frac{1}{2}\Big(1-\frac{\mu_2}{\lambda_{k+1,s}}\Big)\|U\|_Y^2-C\|U\|_Y^{\alpha+\beta},\end{aligned}$$

where $C>0$ is a constant. Since $\mu_2<\lambda_{k+1,s}$ and $\alpha+\beta>2$, for $\|U\|_Y=\rho$ small enough, we get $\mathcal{I}_s(U)\ge\sigma$.

ii) Now consider the following decomposition $\partial Q=\Gamma_1\cup\Gamma_2\cup\Gamma_3$, where
$\Gamma_1=\{U\in Y(\Omega);\ U=U_1+rE,\ \text{with}\ U_1\in E_k^-,\ \|U_1\|_Y=R,\ 0\le r\le R\}$,
$\Gamma_2=\{U\in Y(\Omega);\ U=U_1+RE,\ \text{with}\ U_1\in E_k^-,\ \|U_1\|_Y\le R\}$,
$\Gamma_3=\overline{B_R(0)}\cap E_k^-$.

Let us show that on each set Γ_i we have $\mathcal{I}_s\,|_{\Gamma_i}\le 0,\ i=1,2,3.$

Choose E as follows:

Fixed $R_0>\rho$, take $E=(e_1,e_2)\in E_k^+=(E_k^-)^\perp$ (with $e_i\ge0,\ i=1,2$) satisfying

(I) $\|E\|_Y^2<\Big(\dfrac{\mu_1}{\lambda_{k-1,s}}-1\Big)\delta^2$, where $\delta>0$ is a constant to be obtained forward.

(II) $e_1\ge 2\Big(K+\frac{\|u_T\|_{C^0}}{R_0}\Big)$ and $e_2\ge 2\Big(K+\frac{\|v_T\|_{C^0}}{R_0}\Big)$ a.e. in some $\mathcal{C}\subset\Omega$ with $|\mathcal{C}|>0$, where $K>0$ satisfies $\|V\|_{(C^0)^2}\le K\|V\|_Y$, for all $V\in E_k^-$.

Note that this choice is possible because $(E_k^-)^\perp$ has unbounded functions; E_k^- has finite dimension and $K=\sup_{\substack{\|V\|_Y=1\\ V\in E_k^-}}\|V\|_{(C^0)^2}$.

Estimates on Γ_1: For $U=U_1+rE\in\Gamma_1$, we consider $U_1=R\widehat{U}_1\in E_k^-$ with $\|\widehat{U}_1\|_E=1$ and we set $\widehat{U}_1=c_1Y+c_2E_k$, where $E_k\in Z_k=\ \text{span}\{(\varphi_{k,s},0),(0,\varphi_{k,s})\}$ and $Y\in E_{k-1}^-$ with $\|Y\|_Y=1$. Then,

$$\begin{aligned}\mathcal{I}_s(U) &\le \frac{1}{2}\|U_1\|_Y^2+\frac{r^2}{2}\|E\|_Y^2-\frac{\mu_1}{2}\|U_1\|^2_{(L^2)^2}-\int_\Omega F(U+U_T)dx\\ &\le \frac{R^2}{2}\|\widehat{U}_1\|_Y^2+\frac{R^2}{2}\|E\|_Y^2-\frac{\mu_1R^2}{2}\|\widehat{U}_1\|^2_{(L^2)^2}-\int_\Omega F(U+U_T)dx\\ &= \frac{R^2}{2}\|c_1Y+c_2E_k\|_Y^2+\frac{R^2}{2}\|E\|_Y^2-\frac{\mu_1R^2}{2}\|c_1Y+c_2E_k\|^2_{(L^2)^2}\\ &- \int_\Omega F(U+U_T)dx\\ &= \frac{R^2}{2}c_1^2(\|Y\|_Y^2-\mu_1\|Y\|^2_{(L^2)^2})+\frac{R^2}{2}c_2^2(\|E_k\|_Y^2-\mu_1\|E_k\|^2_{(L^2)^2})\\ &+ \frac{R^2}{2}\|E\|_Y^2-\int_\Omega F(U+U_T)dx.\end{aligned}$$

Consequently

$$\mathcal{I}_s(U) \leq \frac{R^2}{2} c_1^2 \Big(1 - \frac{\mu_1}{\lambda_{k-1,s}}\Big) \|Y\|_Y^2 + \frac{R^2}{2} \|E\|_Y^2 - \int_\Omega F(U + U_T) dx. \tag{5.5}$$

Now using the notation $\widehat{U}_1 = (\widehat{u}_1, \widehat{v}_1) = (c_1 y_1 + c_2 e_1^k, c_1 y_2 + c_2 e_2^k)$, where $Y = (y_1, y_2) \in E_{k-1}^- \cap B_1$ and $E_k = (e_1^k, e_2^k) \in Z_k \cap B_1$, we will prove that there exist $\delta > 0$ and $\eta > 0$ such that

$$\max_{i=1,2} \Big\{ \max_{\overline{\Omega}} \{c_1 y_i + c_2 e_i^k; \;\; |c_1| \leq \delta\} \Big\} \geq \eta > 0.$$

Indeed, by contradiction, assume that there exist sequences $(c_1^n), (c_2^n) \subset \mathbb{R}$ and $Y_n = (y_1^n, y_2^n) \subset Y(\Omega)$ with $\|Y_n\|_Y = 1$ such that $c_1^n \to 0$, $|c_2^n| = \sqrt{1 - (c_1^n)^2} \to 1$ and

$$\max_{i=1,2} \Big\{ \max_{\overline{\Omega}} \{c_1^n y_i^n + c_2^n e_i^k\} \Big\} \to 0, \text{ as } n \to \infty.$$

Therefore, $c_1^n y_i^n \to 0$ and $c_2^n e_i^k \to e_i^k$ and consequently

$$\max_{i=1,2} \Big\{ \max_{\overline{\Omega}} e_i^k(x) \Big\} = 0.$$

Hence, we conclude that $e_1^k \leq 0$ and $e_2^k \leq 0$ in Ω, which is a contradiction, because $k > 1$, $E_k = (e_1^k, e_2^k) \in Z_k$ and $\|E_k\|_Y = 1$ imply that at least one of the coordinate functions must change sign.
So, we conclude that there exist $\delta > 0$, $\eta > 0$ such that

$$\max \Big\{ \max_{\overline{\Omega}} \widehat{u}_1; \; \max_{\overline{\Omega}} \widehat{v}_1 \;\; : \; |c_1| \leq \delta \Big\} \geq \eta > 0 \, , \, \forall \, \widehat{U}_1 = c_1 Y + c_2 E_k \in E_k^- \text{ with } \|\widehat{U}_1\|_Y = 1.$$

Denoting $\Omega_+ = \Big\{x \in \overline{\Omega} : (\widehat{u}_1)(x) \geq \eta/2 \text{ and } (\widehat{v}_1)(x) \geq \eta/2\Big\}$. By equicontinuity of the functions $\widehat{U}_1$, we have that $|\Omega_+| \geq \nu > 0$, $\forall \widehat{U}_1 \in E_k^- \cap B_1$ and $|c_1| \leq \delta$.
Moreover

$$\frac{u_T(x)}{R} \geq -\frac{\|u_T\|_{C^0}}{R} > -\frac{\eta}{4} \text{ and } \frac{v_T(x)}{R} \geq -\frac{\|v_T\|_{C^0}}{R} > -\frac{\eta}{4}, \; \forall \, R \geq R_0 \text{ sufficiently large.}$$

Then, since $e_1, e_2 \geq 0$ in Ω,

$$\begin{aligned}
\int_\Omega F(U + U_T) dx \;\; &\geq \;\; \frac{\xi_1}{\alpha + \beta} R^{\alpha+\beta} \int_\Omega \Big(\widehat{u}_1 + \frac{u_T}{R}\Big)_+^{\alpha+\beta} dx \\
&+ \;\; \frac{\xi_2}{\alpha + \beta} R^{\alpha+\beta} \int_\Omega \Big(\widehat{v}_1 + \frac{v_T}{R}\Big)_+^{\alpha+\beta} dx \\
&\geq \;\; C R^{\alpha+\beta} \Big[\int_{\Omega_+} \Big(\widehat{u}_1 - \frac{\eta}{4}\Big)_+^{\alpha+\beta} dx + \int_{\Omega_+} \Big(\widehat{v}_1 - \frac{\eta}{4}\Big)_+^{\alpha+\beta} dx \Big] \\
&\geq \;\; C R^{\alpha+\beta} \Big[\int_{\Omega_+} \Big(\frac{\eta}{4}\Big)^{\alpha+\beta} dx + \int_{\Omega_+} \Big(\frac{\eta}{4}\Big)^{\alpha+\beta} dx \Big] \\
&\geq \;\; C R^{\alpha+\beta} \Big(\frac{\eta}{4}\Big)^{\alpha+\beta} |\Omega_+| = \widetilde{C} R^{\alpha+\beta},
\end{aligned}$$

for all R sufficiently large. Thus, from (5.5) we can conclude that there exists $R_1 > 0$ such that

$$\mathcal{I}_s(U) \leq \frac{R^2}{2} \delta^2 \Big(1 - \frac{\mu_1}{\lambda_{k-1,s}}\Big) + \frac{R^2}{2} \|E\|_Y^2 - \widetilde{C} R^{\alpha+\beta} < 0,$$

for all $R \geq R_1$.

On the other hand, if $|c_1| \geq \delta > 0$, by the choose of E, we get

$$\begin{aligned}\mathcal{I}_s(U) &\leq -\frac{R^2}{2}c_1^2\Big(\frac{\mu_1}{\lambda_{k-1,s}} - 1\Big) + \frac{R^2}{2}\|E\|_Y^2 \\ &\leq -\frac{R^2}{2}\Big[\delta^2\Big(\frac{\mu_1}{\lambda_{k-1,s}} - 1\Big) - \|E\|_Y^2\Big] < 0.\end{aligned}$$

Estimates on Γ_2: For $U = U_1 + RE \in \Gamma_2$, we have

$$\mathcal{I}_s(U_1 + RE) \leq \frac{1}{2}\|U_1\|_Y^2\Big(1 - \frac{\mu_1}{\lambda_{k,s}}\Big) + \frac{R^2}{2}\|E\|_Y^2 - \int_\Omega F(U_1 + RE + U_T)dx.$$

Since $\lambda_{k,s} = \mu_1$,

$$\mathcal{I}_s(U_1 + RE) \leq \frac{R^2}{2}\|E\|_Y^2 - \int_\Omega F(U_1 + RE + U_T)dx. \tag{5.6}$$

Now, to estimate the last integral, note that, if $U_1 = (u_1, u_2)$,

$$\begin{aligned}&\int_\Omega F(U_1 + RE + U_T)dx \\ &\geq \frac{1}{\alpha+\beta}\Big[\xi_1 R^{\alpha+\beta}\int_\Omega \Big(e_1 + \frac{u_1 + u_T}{R}\Big)_+^{\alpha+\beta} dx + \xi_2 R^{\alpha+\beta}\int_\Omega \Big(e_2 + \frac{u_2 + v_T}{R}\Big)_+^{\alpha+\beta} dx\Big]\end{aligned}$$

for $R \geq R_0$, and by (II) each integral on the right can be estimated as follows

$$\begin{aligned}\int_\Omega \Big(e_i + \frac{u_i + w_T}{R}\Big)_+^{\alpha+\beta} dx &\geq \int_\Omega \Big(e_i - \frac{\|u_i\|_{C^0} + \|w_T\|_{C^0}}{R}\Big)_+^{\alpha+\beta} dx \\ &\geq \int_\Omega \Big(e_i - \Big(K + \frac{\|w_T\|_{C^0}}{R_0}\Big)\Big)_+^{\alpha+\beta} dx \\ &\geq \int_{\mathcal{C}} \Big(K + \frac{\|w_T\|_{C^0}}{R_0}\Big)^{\alpha+\beta} dx = \Big(K + \frac{\|w_T\|_{C^0}}{R_0}\Big)^{\alpha+\beta}|\mathcal{C}|,\end{aligned}$$

for $i = 1, 2$ and $w_T \in \{u_T\ ,\ v_T\}$.

Therefore, by (5.6) and by above estimates,

$$\begin{aligned}\mathcal{I}_s(U_1 + RE) &\leq \frac{R^2}{2}\|E\|_Y^2 - c_1 R^{\alpha+\beta}\int_\Omega \Big(e_1 + \frac{u_1 + u_T}{R}\Big)_+^{\alpha+\beta} dx \\ &- c_2 R^{\alpha+\beta}\int_\Omega \Big(e_2 + \frac{u_2 + v_T}{R}\Big)_+^{\alpha+\beta} dx \leq \frac{R^2}{2}\|E\|_Y^2 - CR^{\alpha+\beta}.\end{aligned}$$

Since $\alpha + \beta > 2$, for $R \geq R_0$ we have $\mathcal{I}_s(U) < 0$, for all $U \in \Gamma_2$.

Estimates on Γ_3: For $U \in \Gamma_3$, it follows the estimate

$$\mathcal{I}_s(U) \leq \frac{1}{2}\|U\|_Y^2 - \frac{\mu_1}{2}\|U\|_{(L^2)^2}^2 \leq \frac{1}{2}\Big(1 - \frac{\mu_1}{\lambda_{k,s}}\Big)\|U\|_Y^2 = 0.$$

Therefore, for all $R \geq R_0 > 0$, follows that $\mathcal{I}_s(U) \leq 0$ for all $U \in \partial Q$, concluding the desired result. ■

Proof of Theorem 1.3.

With the previous results, we conclude the proof of Theorem 1.3 by a direct application of the Linking Theorem and arguing as in proof of Theorem 1.2 to obtain two distinct solutions for the problem (1.2). ■

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