On the existence and uniqueness of weak solutions to elliptic equations with a singular drift

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Abstract

In this paper we study the Dirichlet problem for a scalar elliptic equation in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ with a singular drift of the form $b_0 = b - \alpha \frac{x'}{|x'|^2}$ where $x' = (x_1, x_2, 0)$, $\alpha \in \mathbb{R}$ is a parameter and b is a divergence free vector field having essentially the same regularity as the potential part of the drift. Such drifts naturally arise in the theory of axially symmetric solutions to the Navier-Stokes equations. For $\alpha < 0$ the divergence of such drifts is positive which potentially can ruin the uniqueness of solutions. Nevertheless, for $\alpha < 0$ we prove existence and Hölder continuity of a unique weak solution which vanishes on the axis $\Gamma := \{ \, x \in \mathbb{R}^3 : \ |x'| = 0 \, \}$.

1 Introduction and Main Results

Assume $\Omega \subset \mathbb{R}^3$ is a bounded domain with the Lipschitz boundary $\partial\Omega$. Without loss of generality we can assume Ω contains the origin. We consider the following boundary value problem:

$$\left\{ \begin{array}{rcl} -\Delta u + b_0 \cdot \nabla u &=& -\operatorname{div} f \qquad \text{in} \quad \Omega, \\ u|_{\partial\Omega} &=& 0. \end{array} \right. \tag{1.1}$$

Here $u : \Omega \to \mathbb{R}$ is unknown, $b_0 : \Omega \to \mathbb{R}^3$ and $f : \Omega \to \mathbb{R}^3$ are given functions.

In this paper we study the problem (1.1) for the drift b_0 of a special form. Our motivating example is

$$b_0(x) = -\alpha \, \frac{x'}{|x'|^2}, \qquad x' = (x_1, x_2, 0), \qquad \alpha \in \mathbb{R} \tag{1.2}$$

where $\alpha \in \mathbb{R}$ is a parameter. Clearly, in this case the drift does not belong to $L_2(\Omega)$, so instead we assume that b_0 belongs to some critical weak Morrey space

$$b_0 \in L^{2,1}_w(\Omega), \tag{1.3}$$

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where $L^{p,\lambda}_w(\Omega)$ is the weak Morrey space equipped with the quasinorm

$$\|b_0\|_{L^{p,\lambda}_w(\Omega)} \;:=\; \sup_{x_0\in\Omega}\sup_{r<1} r^{-\frac{\lambda}{p}}\,\|b_0\|_{L_{p,w}(B_r(x_0)\cap\Omega)}$$

and $L_{p,w}(\Omega)$ is the weak Lebesgue space equipped with the quasinorm

$$\|b_0\|_{L_{p,w}(\Omega)} \;:=\; \sup_{s>0} s\,|\{\, x\in\Omega:\ |b_0(x)|>s\,\}|^{\frac{1}{p}}. \tag{1.4}$$

We define the bilinear form $\mathcal{B}[u,\eta]$ by

$$\mathcal{B}[u,\eta] \;:=\; \int\limits_{\Omega} \eta\, b_0\cdot\nabla u\; dx. \tag{1.5}$$

Note that for b_0 satisfying (1.3) the bilinear form (1.5) generally speaking is not well-defined for $u\in W^1_2(\Omega)$ where we denote by $W^k_p(\Omega)$ the standard Sobolev space, see notation at the end of this section. Nevertheless, $\mathcal{B}[u,\eta]$ is well-defined at least for $u\in W^1_p(\Omega)$ with $p>2$ and $\eta\in L_{\frac{2p}{p-2}}(\Omega)$. So, instead of the standard notion of weak solutions from the energy class $W^1_2(\Omega)$ we introduce the definition of p-weak solutions to the problem (1.1), see also the related definitions in [15], [18], [23]:

Definition 1.1. *Assume $p>2$, $b_0\in L_{p'}(\Omega)$, $p'=\frac{p}{p-1}$, and $f\in L_1(\Omega)$. We say u is a p-weak solution to the problem (1.1) if $u\in \overset{\circ}{W}{}^1_p(\Omega)$ and u satisfies the identity*

$$\int\limits_{\Omega}\nabla u\cdot\nabla\eta\,dx \;+\; \mathcal{B}[u,\eta] \;=\; \int\limits_{\Omega} f\cdot\nabla\eta\; dx, \qquad \forall\,\eta\in C_0^\infty(\Omega). \tag{1.6}$$

If $b\in L_2(\Omega)$ and $u\in \overset{\circ}{W}{}^1_2(\Omega)$ satisfy (1.6) then we call u a weak solution to the problem (1.1). Obviously, in this case p-weak solutions are some subclass of weak solutions.

Note that if u is a weak or a p-weak solution to (1.1) and $f\in L_2(\Omega)$ then by density arguments we can extend the class of test functions in (1.6) from $\eta\in C_0^\infty(\Omega)$ to all functions $\eta\in \overset{\circ}{W}{}^1_2(\Omega)\cap L_\infty(\Omega)$. We can consider two special cases of the drift b_0:

$$\operatorname{div} b_0 \;\le\; 0 \quad \text{in} \quad \mathcal{D}'(\Omega), \tag{1.7}$$

or

$$\operatorname{div} b_0 \;\ge\; 0 \quad \text{in} \quad \mathcal{D}'(\Omega), \tag{1.8}$$

where we denote by $\mathcal{D}'(\Omega)$ the space of distibutions on Ω. The principal difference between the cases (1.7) and (1.8) is due to the fact that under the assumption (1.7) the quadratic form $\mathcal{B}[u,u]$ provides (at least, formally) a positive support to the quadratic form of the elliptic operator in (1.1), while in the case of (1.8) the quadratic form $\mathcal{B}[u,u]$ is non-positive and hence it "shifts" the operator to the "spectral area". For example, it is well-known that in the case (1.8) the uniqueness for the problem (1.1) can be violated even for smooth solutions. Indeed, the function $u(x)=c(1-|x|^2)$ is a solution to the problem (1.1) in the unite ball $\Omega=\{\,x\in\mathbb{R}^n:\ |x|<1\,\}$ corresponding to $b_0(x)=n\,\frac{x}{|x|^2}$ with $b_0\in L_{n,w}(\Omega)$ satisfying (1.8), see also the discussion in [6], [12].

The case (1.7) under the assumption (1.3) was studied in [7] (see also [6] for the 2D case). In this paper we focus on the case (1.8) which is much more subtle and for now we are not able to treat it in full generality. So, we restrict ourselves to the drifts with the potential part of some specific form. Namely, we assume that

$$b_0(x) \; = \; b(x) \, + \, \alpha \, \frac{x'}{|x'|^2}, \tag{1.9}$$

where $b : \Omega \to \mathbb{R}^3$ satisfies the divergence-free condition (in the sense of distributions)

$$\operatorname{div} b = 0 \qquad \text{in} \quad \mathcal{D}'(\Omega). \tag{1.10}$$

Note that from (1.3) we obtain

$$b \in L^{2,1}_w(\Omega). \tag{1.11}$$

The relation (1.9) can be viewed as the Helmogoltz decomposition of the vector field b_0

$$b_0 \; = \; b + \alpha \, \nabla h \quad \text{a.e. in} \quad \Omega, \tag{1.12}$$

where the potential part of the drift $h : \Omega \to \mathbb{R}$ is specified by

$$h(x) \; = \; \ln \frac{1}{|x'|}. \tag{1.13}$$

In this case we have

$$-\Delta h \; = \; 2\pi \, \delta_\Gamma \quad \text{in} \quad \mathcal{D}'(\Omega), \qquad \Gamma \; = \; \{ \, x \in \bar{\Omega} : \, x' = 0 \, \}, \tag{1.14}$$

where δ_Γ is the delta-function concentrated on Γ, i.e.

$$\langle \delta_\Gamma, \varphi \rangle \; := \; \int\limits_\Gamma \varphi(x) \, dl_x, \qquad \forall \, \varphi \in C_0^\infty(\mathbb{R}^3).$$

Certainly, in the case of (1.9) the identity (1.6) reduces to

$$\int\limits_\Omega \nabla u \cdot (\nabla \eta + b\eta) \, dx \, + \, 2\pi \, \alpha \int\limits_\Gamma u(x) \eta(x) \, dl_x \; = \; \int\limits_\Omega f \cdot \nabla \eta \; dx, \quad \forall \, \eta \in C_0^\infty(\Omega). \tag{1.15}$$

Note that for $u \in W^1_p(\Omega)$ with $p > 2$ the trace $u|_\Gamma$ of u on Γ satisfies

$$u|_\Gamma \in W_p^{1-\frac{2}{p}}(\Gamma), \tag{1.16}$$

where $W^s_p(\Gamma)$, $s > 0$, is the Slobodetskii-Sobolev space, see, for example, [2]. So, for p-weak solutions in the sense of Definition 1.1 the second term in the left-hand side of (1.15) is well-defined. Note also that the condition (1.8) corresponds to $\alpha \leq 0$ in (1.9) and (1.15).

The drift of type (1.9) plays an important role in the theory of axially symmetric solutions to the Navier-Stokes equations, see, for example, [19], [30], [31], [32], [33], [35], [38]. In the axially symmetric case the Navier-Stokes system can reduced to the scalar equation

$$\partial_t u - \Delta u + \Big(v - \alpha \, \frac{x'}{|x'|^2} \Big) \cdot \nabla u \; = \; 0 \qquad \text{in} \quad \mathbb{R}^3 \times (0, T), \tag{1.17}$$

where $v = v(x,t)$ is the divergence-free velocity field and $u = u(x,t)$ is some auxiliary scalar function. For example, for axially symmetric solutions without swirl (i.e. if $v(x,t) = v_r(r,z,t)e_r + v_z(r,z,t)e_z$ where e_r, e_φ, e_z is the standard cylindrical basis) the equation (1.17) is satisfied for $\alpha = 2$ and $u = \frac{\omega_\varphi}{r}$, where $\omega_\varphi := v_{r,z} - v_{z,r}$ and $r = |x'|$. In the case of general axially symmetric solutions $v(x,t) = v_r(r,z,t)e_r + v_\varphi(r,z,t)e_\varphi + v_z(r,z,t)e_z$ the equation (1.17) holds for $\alpha = -2$ and $u = rv_\varphi$.

It is well-known in the Navier-Stokes theory (see [19], [30], [33], [35], [38]) that while in the case $\alpha > 0$ some results like Liouville-type theorems assume no special conditions on the solutions u to the equation (1.17) besides a proper decay of the drift v, the analogues results in the case $\alpha < 0$ require the additional condition $u|_\Gamma = 0$. Our equation (1.1) under the assumption (1.9) can be considered as the elliptic model for the general equation (1.17). The main goals of the present paper is to investigate the equation (1.17) from the point of view of the "general theory" (i.e. without the assumption on the axial symmetry of u and other specific properties of solutions to the Navier-Stokes equations). In particular, we would like to clarify the role which the condition $u|_\Gamma = 0$ plays in the theory. On the other hand, our present contribution can be viewed as a 3D extension of the results obtained earlier in [6] in the 2D case.

The main result the present paper are the following two theorems:

Theorem 1.1. *Assume $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain containing the origin, b_0 is given by* (1.9) *with $\alpha < 0$, b satisfies* (1.10), (1.11) *and $f \in L_q(\Omega)$ with $q > 3$. Then every p–weak solution to the problem* (1.1) *satisfying the condition*

$$u|_\Gamma \ = \ 0 \tag{1.18}$$

is Hölder continuous. Namely, there exists $\mu \in (0,1)$ depending only on q, α, $\|b\|_{L^{2,1}_w(\Omega)}$ and the Lipschitz constant of $\partial\Omega$ such that if for some $p > 2$ a function u is a p–weak solution to the problem (1.1) *corresponding to the right-hand side $f \in L_q(\Omega)$ and satisfying the condition* (1.18) *in the sense of traces then u is Hölder continuous on $\bar\Omega$ with the exponent μ and the estimate*

$$\|u\|_{C^\mu(\bar\Omega)} \ \le \ c\ \|f\|_{L_q(\Omega)}, \tag{1.19}$$

holds with the constant $c > 0$ depending only on Ω, α, q, $\|b\|_{L^{2,1}_w(\Omega)}$ and the Lipschitz constant of $\partial\Omega$.

We emphasize that the Hölder exponent μ in Theorem 1.1 does not depend on p, so in this theorem we need the assumption $p > 2$ only to have the trace in (1.18) to be well-defined.

Theorem 1.2. *Assume $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain containing the origin, b_0 is given by* (1.9) *with $\alpha < 0$, b satisfies* (1.10), (1.11) *and assume $q > 3$. Then there exists $p > 2$ depending only on α, q, $\|b\|_{L^{2,1}_w(\Omega)}$ and the Lipschitz constant of $\partial\Omega$ such that for any $f \in L_q(\Omega)$ there exists a unique p-weak solution u to the problem* (1.1) *satisfying the condition* (1.18) *Moreover, this solution satisfies the estimate*

$$\|u\|_{W^1_p(\Omega)} \ \le \ c\ \|f\|_{L_q(\Omega)}, \tag{1.20}$$

with a constant $c > 0$ depending only on Ω, α, q, $\|b\|_{L^{2,1}_w(\Omega)}$ and the Lipschitz constant of $\partial\Omega$.

Note that the condition (1.18) is essential for the uniqueness in Theorem 1.2, see the example of non-uniqueness for the Dirichlet problem in 2D case, for example, in [6]. Note also that for sufficiently regular drift b_0 the uniqueness for the Dirichlet problem (1.1) holds even without the condition (1.18) as it follows from the maximum principle for the problem (1.1), see, for example, [12] for the further discussion.

So, our main message is: if the drift b_0 in (1.1) is singular then the "bad" sign of its divergence (1.8) can ruin the uniqueness for the problem (1.1). But if the drift is singular along a particular curve Γ (and the singular part of the drift is harmonic away from this curve) then the additional condition (1.18) (compensating the singularity of the drift) provides the existence and uniqueness in the class of p-weak solutions as well as guarantees some other "good" properties of solutions such as the Hölder continuity in Theorem 1.1 (see also the Liouville property in [19] and [28]). From our point of view the most interesting result in Theorem 1.2 is the existence of solutions satisfying the condition (1.18) as their uniqueness follows directly from the energy identity.

There are many papers devoted to the investigation of the problem (1.1) in the case of divergence free drift, [1], [9], [11], [12], [13], [26], [27], [28], [34], [36], [41], [42] and references there. Papers devoted to the non-divergence free drifts are not so numerous (see [3], [4], [6], [15], [16], [17], [18], [20], [21], [22], [23], [24], [28] for the references). Our present contribution can be viewed as a 3D analogue of the results obtained earlier in [6] in the 2D case.

Our paper is organized as follows. In Section 2 we introduce some auxiliary results and derive the estimates of the bilinear form which are based on Fefferman [10] and Chiarenza and Frasca [8] inequalities. In Section 3 we prove a priori global boundedness of p–weak solutions to the problem (1.1) satisfying the condition (1.18). Note that this result holds for a supercritical drift b, but it is heavily based on the boundary conditions in (1.1) and has not its analogue in the local setting, see the discussion in [12]. In Section 4 we adopt the De Giorgi technique to investigation of the problem with the singular drift of type (1.9). The basic assumption which allows us to use the De Giorgi technique in more or less standard way is the density condition (4.2). To show the validity of this condition for the points on the singular curve Γ we follow the method developed in [5], see also [38]. In Section 5 we prove Theorem 1.1. Finally, in Section 6 we prove Theorem 1.2.

In the paper we use the following notation. For any $a,\ b \in \mathbb{R}^n$ we denote by $a \cdot b = a_k b_k$ their scalar product in $\mathbb{R}^n$. Repeated indexes assume the summation from 1 to n. An index after comma means partial derivative with respect to x_k, i.e. $f_{,k} := \frac{\partial f}{\partial x_k}$. We denote by $L_p(\Omega)$ and $W^k_p(\Omega)$ the usual Lebesgue and Sobolev spaces. We do not distinguish between functional spaces of scalar and vector functions and omit the target space in notation. $C_0^\infty(\Omega)$ is the space of smooth functions compactly supported in Ω. The space $\overset{\circ}{W}{}^1_p(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^1_p(\Omega)$ norm and $W^{-1}_p(\Omega)$ is the dual space for $\overset{\circ}{W}{}^1_{p'}(\Omega)$, $p' = \frac{p}{p-1}$. The space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$. By $C^\mu(\bar{\Omega})$, $\mu \in (0,1)$ we denote the spaces of Hölder continuous functions on $\bar{\Omega}$. The symbols $\rightharpoonup$ and $\to$ stand for the weak and strong convergence respectively. We denote by $B_R(x_0)$ the ball in $\mathbb{R}^n$ of radius R centered at x_0 and write B_R if $x_0 = 0$. We write B instead of B_1 and denote $S_1 := \partial B_1$. For a domain

$\Omega \subset \mathbb{R}^n$ we also denote $\Omega_R(x_0) := \Omega \cap B_R(x_0)$. For $u \in L_\infty(\omega)$ we denote

$$\operatorname*{osc}_{\omega} u \ := \ \operatorname*{ess\,sup}_{\omega} u - \operatorname*{ess\,inf}_{\omega} u.$$

We denote by $L^{p,\lambda}(\Omega)$ the Morrey space equipped with the norm

$$\|u\|_{L^{p,\lambda}(\Omega)} \ := \ \sup_{x_0 \in \Omega} \sup_{R < \operatorname{diam} \Omega} R^{-\frac{\lambda}{p}} \|u\|_{L_p(\Omega_R(x_0))}.$$

$(f)_\omega$ stands for the average of f over the domain $\omega \subset \mathbb{R}^n$:

$$(f)_\omega \ := \ \fint_{\omega} f \, dx \ = \ \frac{1}{|\omega|} \int_{\omega} f \, dx.$$

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2 Auxiliary results

In this section we present several auxiliary results. Within this section we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain for arbitrary $n \geq 2$. The first result shows that by relaxing an exponent of the integrability of a function we can always switch from a weak Morrey norm to a regular one.

Proposition 2.1. *For any $p \in (1, n)$ and $1 \leq q < p \leq r \leq n$ there are positive constants c_1 and c_2 depending only on n, p, q and Ω such that*

$$c_1 \|b\|_{L^{q,n-q}(\Omega)} \;\leq\; \|b\|_{L^{p,n-p}_w(\Omega)} \;\leq\; c_2 \|b\|_{L^{r,n-r}(\Omega)}.$$

Proof. The result follows from the Hölder inequality for Lorentz norms, see [14, Section 4.1]. □

The next result is the estimate of the quadratic form corresponding to the drift term satisfying (1.9) and (1.10).

Proposition 2.2. *Assume $p > 2$, $p' = \frac{p}{p-1}$ and b_0 is given by (1.9) with $b \in L_{p'}(\Omega)$ satisfying (1.10). Then for any $\alpha \in \mathbb{R}$ and any $u \in \overset{\circ}{W}{}^1_p(\Omega)$ the bilinear form $\mathcal{B}[u, \eta]$ defined in (1.5) satisfies the identity*

$$\mathcal{B}[u, u] \;=\; \pi\alpha \int\limits_{\Gamma} |u(x)|^2 \, dl_x \tag{2.1}$$

where the integral in the right-hand side is understood in the sense of traces.

Proof. For a smooth function $u \in C_0^\infty(\Omega)$ the relation (2.1) follows by integration by parts. For an arbitrary function $u \in \overset{\circ}{W}{}^1_p(\Omega)$ with $p > 2$ the corresponding relation follows from the continuity of the trace operator from $W^1_p(\Omega)$ to $L_p(\Gamma)$. □

The next proposition proved by Chiarenza and Frasca in [8] is the well-known extension of the result of C. Fefferman [10] for $p = 2$. This theorem is one of basic tools in our proofs of both Theorems 1.1 and 1.2.

Proposition 2.3. *Assume $p \in (1, n)$, $r \in (1, \frac{n}{p}]$ and $V \in L^{r,n-pr}(\Omega)$. Then*

$$\int\limits_{\Omega} |V| \, |u|^p \, dx \;\leq\; c_{n,r,p} \|V\|_{L^{r,n-rp}(\Omega)} \|\nabla u\|^p_{L_p(\Omega)}, \qquad \forall \, u \in C_0^\infty(\Omega).$$

with the constant $c_{n,r,p} > 0$ depending only on n, r and p.

Our next result is the estimate of the bilinear form corresponding to the drift term. This result is a direct consequence of Proposition 2.3.

Proposition 2.4. *Assume $r \in \left(\frac{2n}{n+2}, 2\right)$ and $b \in L^{r,n-r}(\Omega)$. Then there exists $c > 0$ depending only on n and r such that for any $u \in W^1_2(\Omega)$ and $\zeta \in C_0^\infty(\Omega)$ the following estimate holds:*

$$\int\limits_{\Omega} \zeta^3 \, |b| \, |u|^2 \, dx \;\leq\; c \, \|b\|_{L^{r,n-r}(\Omega)} \, \|\nabla(\zeta^2 u)\|_{L_2(\Omega)} \, \|\nabla(\zeta u)\|_{L_{\frac{2n}{n+2}}(\Omega)}. \tag{2.2}$$

Moreover, for any $\theta \in \left(\frac{n}{r} - \frac{n}{2}, 1\right)$ *there exists* $c > 0$ *depending only on* n, r *and* θ *such that for any* $u \in W_2^1(\Omega)$ *and any* $\zeta \in C_0^\infty(\Omega)$ *satisfying* $0 \le \zeta \le 1$ *we have*

$$\int\limits_\Omega \zeta^{1+\theta} \, |b| \, |u|^2 \, dx \ \le \ c \, \|b\|_{L^{r,n-r}(\Omega)} \|u\|^{1-\theta}_{L_2(\Omega)} \|\nabla(\zeta u)\|^{1+\theta}_{L_2(\Omega)} \, |\Omega|^{\theta/n}. \tag{2.3}$$

If we additionally assume $u \in \overset{\circ}{W}{}^1_2(\Omega)$ *then the estimates* (2.2) *and* (2.3) *remains true for an arbitrary* $\zeta \in C_0^\infty(\mathbb{R}^n)$.

Proposition 2.4 is proved in [7, Proposition 3.4].

3 Boundedness of weak solutions

In this section we establish global boundedness of p–weak solutions to the problem (1.1). Note that this result holds for a supercritical drift b, so we do not need the critical condition (1.11) in this section. Note also, that our result is global, i.e. it is heavily based on the homogeneous (or, more generally, regular) Dirichlet boundary conditions in (1.1) and this result is not valid in the local setting, see, for example, counterexamples and discussion in [12]. For the related results in local setting we see the recent paper [1] and reference there.

Theorem 3.1. *Assume $p > 2$, $p' = \frac{p}{p-1}$ and b_0 is given by (1.9) with $b \in L_{p'}(\Omega)$ satisfying (1.10). Assume $q > 3$ and $f \in L_q(\Omega)$. Then for any $\alpha \in \mathbb{R}$ any p-weak solution $u \in \overset{\circ}{W}{}^1_p(\Omega)$ to the problem (1.1) satisfying (1.18) is essentially bounded and satisfies the estimate*

$$\|u\|_{L_\infty(\Omega)} \ \le \ c \, \|f\|_{L_q(\Omega)}, \tag{3.1}$$

with a constant $c > 0$ depending only on Ω and q.

Proof. For $m > 0$ we define a truncation $T_m : \mathbb{R} \to \mathbb{R}$ by $T_m(s) := m$ for $s \ge m$, $T_m(s) = s$ for $s < m$. For any $s \in \mathbb{R}$ we also denote $(s)_+ := \max\{s, 0\}$. Now we fix some $m > 0$ and and denote $\bar{u} := T_m(u)$. Then for any $k \ge 0$ we have

$$(\bar{u} - k)_+ \in L_\infty(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega), \qquad \nabla(\bar{u} - k)_+ = \chi_{\Omega[k<u<m]} \nabla u$$

where $\Omega[k < u < m] = \{ \, x \in \Omega : \ k < u(x) < m \, \}$. Define $\eta := (\bar{u} - k)_+$ and note that from (1.18) for any $m \ge 0$ and any $k \ge 0$ we obtain

$$(u - m)_+|_\Gamma = 0, \qquad \eta|_\Gamma \ = \ 0 \tag{3.2}$$

in the sense of traces. Approximating η by smooth functions we can take η as a test function in (1.6). For $k \ge m$ we have $\eta \equiv 0$ and hence $\mathcal{B}[u, \eta] = 0$. For $k < m$ we obtain

$$\mathcal{B}[u, \eta] \ = \ \mathcal{B}[\eta, \eta] \ + \ (m - k) \int\limits_\Omega b_0 \cdot \nabla(u - m)_+ \, dx.$$

From Proposition 2.2 taking into account (3.2) we obtain $\mathcal{B}[\eta, \eta] = 0$. On the other hand, from (1.10) and (1.14) taking into account (3.2) we obtain

$$\int\limits_\Omega b_0 \cdot \nabla(u - m)_+ \, dx \ = \ 2\pi\alpha \int\limits_\Gamma (u - m)_+(x) \, dl_x \ = \ 0$$

and hence $\mathcal{B}[u, \eta] = 0$. So, for $A_k := \{ \, x \in \Omega : \ \bar{u}(x) > k \, \}$ from (1.6) we obtain

$$\int\limits_\Omega |\nabla(\bar{u} - k)_+|^2 \, dx \ \le \ \|f\|^2_{L_q(\Omega)} |A_k|^{1 - \frac{2}{q}}, \qquad \forall \, k \ge 0,$$

which implies (see [25, Chapter II, Lemma 5.3])

$$\operatorname*{ess\,sup}_\Omega \bar{u} \ \le \ c \, |\Omega|^\delta \, \|f\|_{L_q(\Omega)}, \qquad \delta := \tfrac{1}{3} - \tfrac{1}{q},$$

with some constant $c > 0$ depending only on q. As this estimate is uniform with respect to $m > 0$ we conclude u is essentially bounded from above and

$$\operatorname*{ess\,sup}_{\Omega} u \;\leq\; c\,|\Omega|^{\delta}\,\|f\|_{L_q(\Omega)}.$$

Applying the same procedure to $\bar{u} := T_m(-u)$ instead of $\bar{u} := T_m(u)$ we obtain $u \in L_\infty(\Omega)$ as well as (3.1). $\square$

4 De Giorgi classes

In this section we introduce the modified De Giorgi classes which are convenient for the study of solutions to the elliptic equations with coefficients from Morrey spaces. These classes were used before in [7]. In this section we use the following notation: for $u \in L_\infty(\Omega)$ and $B_\rho(x_0) \subset \Omega$ we denote

$$m(x_0,\rho) := \inf_{B_\rho(x_0)} u, \qquad M(x_0,\rho) := \sup_{B_\rho(x_0)} u, \qquad \omega(x_0,\rho) := M(x_0,\rho) - m(x_0,\rho).$$

Definition 4.1. *Assume $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $n \geq 2$. For $u \in W_2^1(\Omega)$ we define $A_k := \{\, x \in \Omega : \ u(x) > k \,\}$. We say $u \in DG(\Omega, k_0)$ if there exist constants $\gamma > 0$, $F > 0$, $\beta \geq 0$, $q > n$ such that for any $B_R(x_0) \subset \Omega$, any $0 < \rho < R$ and any $k \geq k_0$ the following inequality holds*

$$\begin{gathered} \int\limits_{A_k \cap B_\rho(x_0)} |\nabla u|^2 \, dx \ \leq \ \frac{\gamma^2}{(R-\rho)^2} \left(1 + \frac{R^\beta}{(R-\rho)^\beta} \right) \int\limits_{A_k \cap B_R(x_0)} |u-k|^2 \, dx \ + \\ + \ F^2 \, |A_k \cap B_R(x_0)|^{1-\frac{2}{q}}. \end{gathered} \tag{4.1}$$

To avoid overloaded notation, when we need to specify constants in Definition 4.1 we allow some terminological license and say that the class $D(\Omega, k_0)$ corresponds to the constants γ, F, β, q instead of including these constants in the notation of the functional class.

The main result of this section is the following weak version of the maximum principle:

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $n \geq 2$. Denote by $DG(\Omega, k_0)$ the De Giorgi class with parameters γ, β, q, F and assume $u \in DG(\Omega, k_0)$. Then u is locally essentially bounded from above in Ω and for any $\delta \in (0,1)$ there exists $\theta \in (0,1)$ depending only on δ, γ, β, q such that for any $B_{4R}(x_0) \subset \Omega$ if*

$$\big| \{\, x \in B_{2R}(x_0) : \ u(x) \leq k_0 \,\} \big| \ \geq \ \delta \, |B_{2R}| \tag{4.2}$$

then

$$\sup_{B_R(x_0)} u \ \leq \ (1-\theta) \sup_{B_{4R}(x_0)} u \ + \ \theta \, k_0 + c_1 \, F \, R^{1-\frac{n}{q}}, \tag{4.3}$$

where $c_1 > 0$ depends only on n, γ, β, q.

Proposition 4.1 is a simple combination of statements of Lemmas 4.1 — 4.3 below. These Lemmas are standard and their proofs can be found, for example, in [7], see also [25].

Lemma 4.1. *Assume $u \in DG(\Omega, k_0)$. Then u is locally essentially bounded from above in Ω and for any $B_{2R}(x_0) \subset \Omega$*

$$\sup_{B_R(x_0)} (u-k_0)_+ \ \leq \ c_* \left[\Big(\ \fint\limits_{B_{2R}(x_0)} |(u-k_0)_+|^2 \ dx \Big)^{1/2} + F \, R^{1-\frac{n}{q}} \right] \tag{4.4}$$

where $(u-k_0)_+ := \max\{u-k_0, 0\}$ and $c_* > 0$ depends only on n, γ and β.

Lemma 4.2. *Assume $u \in DG(\Omega, k_0)$. Then there exists $\delta_0 \in (0,1)$ depending on n, γ, β in Definition 4.1 of the De Giorgi class such that for any $B_{2R}(x_0) \subset \Omega$ if*

$$|B_{2R}(x_0) \cap A_{k_0}| \ \le \ \delta_0 \, |B_{2R}|$$

then either

$$\sup_{B_R(x_0)} (u-k_0)_+ \ \le \ \frac{1}{2} \sup_{B_{4R}(x_0)} (u-k_0)_+ \tag{4.5}$$

or

$$\sup_{B_{4R}(x_0)} (u-k_0)_+ \ \le \ 4c_* \, F \, (2R)^{1-\frac{n}{q}} \tag{4.6}$$

where $c_* > 0$ is a constant from (4.4).

Lemma 4.3. *Assume $u \in DG(\Omega, k_0)$. Then for any $\delta \in (0,1)$ there exists $s \in \mathbb{N}$ depending only on δ, n, γ, β such that if for some $B_{4R}(x_0) \subset \Omega$ we have*

$$|B_{2R}(x_0) \setminus A_{k_0}| \ \ge \ \delta \ |B_{2R}|$$

then either

$$|B_{2R}(x_0) \cap A_{\bar{k}}| \ \le \ \delta_0 \ |B_{2R}|, \tag{4.7}$$

or

$$\sup_{B_{4R}(x_0)} (u-k_0)_+ \ \le \ 2^s \, F \, R^{1-\frac{n}{q}}. \tag{4.8}$$

Here $\delta_0 \in (0,1)$ is the constant from Lemma 4.2 and we denote

$$\bar{k} \ = \ M(x_0, 4R) - \frac{1}{2^s}\Big(M(x_0,4R) - k_0\Big), \qquad M(x_0,4R) \ := \ \sup_{B_{4R}(x_0)} u.$$

Proposition 4.1 provides the control of the oscillation of a function belonging to the modified De Giorgi class if the assumption (4.2) is satisfied. The validity of (4.2) for the points x_0 on the singular curve Γ of the drift (1.9) follows from the following weak form of the Harnak inequality which we borrow from [38], see also [6]. From now on we restrict ourselves to the case $n = 3$ so that the trace $u|_\Gamma$ for a p-weak solution $u \in W^1_p(\Omega)$ is well-defined.

Proposition 4.2. *Assume $B_R := \{x \in \mathbb{R}^3 : \, |x| < R\}$. Assume $p > 2$, $b \in L_{p'}(B_2)$, $p' = \frac{p}{p-1}$ satisfies $\operatorname{div} b = 0$ in $\mathcal{D}'(B_2)$, and $g \in L_p(B_2)$. Assume $v \in W^1_p(B_2)$ satisfies the equation (in the sense of distributions)*

$$-\Delta v + b_0 \cdot \nabla v \ = \ -\operatorname{div} g \quad \text{in} \quad B_2, \qquad b_0 := b - \alpha \, \frac{x'}{|x'|^2}. \tag{4.9}$$

Assume $\alpha \neq 0$ and

$$0 \le v \le 2 \quad \text{in} \quad B_2, \qquad v|_\Gamma \ge 1, \qquad \Gamma := \{\, x \in \mathbb{R}^3 : \ x_1 = x_2 = 0 \,\}. \tag{4.10}$$

Then there exists constants $\delta_1 \in (0,1)$ and $\lambda_1 \in (0,1)$ depending only on $\|b\|_{L_{p'}(B_2)}$ in the explicit way specified below such that if

$$\|g\|_{L_2(B_2)} \ \le \ c_\star \, |\alpha|, \qquad \text{where} \qquad c_\star := \frac{\pi}{4|B_2|^{1/2}}, \tag{4.11}$$

then

$$\big|\{\, x \in B_2 : \ v(x) \ge \lambda_1 \,\}\big| \ \ge \ \delta_1.$$

Proof. Assume there exist $g \in L_p(B_2)$ and $v \in W^1_p(B_2)$ satisfying (4.9), (4.10), (4.11) such that

$$|\{ x \in B_2 : v(x) > \lambda_1 \}| \le \delta_1. \tag{4.12}$$

Multiplying (4.9) by an arbitrary $\eta \in C_0^\infty(B_2)$ and integrating by parts we obtain

$$2\pi\alpha \int\limits_{\Gamma \cap B_2} v(x)\eta(x)\, dl_x = \int\limits_{B_2} v\Big(\Delta\eta + b \cdot \nabla\eta\Big)\, dx + \int\limits_{B_2} g \cdot \nabla\eta\, dx. \tag{4.13}$$

Note that $v|_\Gamma \ge 1$. Choose $\eta \in C_0^\infty(B_2)$ so that

$$\eta = 1 \quad \text{on} \quad B, \qquad \|\nabla\eta\|_{L_2(B_2)} \le 4\,|B_2|^{1/2}, \qquad \|\eta\|_{C^2(\bar B_2)} \le c_{\star\star},$$

where $c_{\star\star} > 0$ is some sufficiently large absolute constant. Then from (4.11) and the Hölder inequality we obtain

$$\Big| \int\limits_{B_2} g \cdot \nabla\eta\, dx \Big| \le \pi|\alpha|.$$

Hence from (4.13) we obtain

$$\pi|\alpha| \le \Big| \int\limits_{B_2} v\Big(\Delta\eta + b \cdot \nabla\eta\Big)\, dx \Big|. \tag{4.14}$$

Denote

$$B_2[v > \lambda_1] := \{\, x \in B_2 : v(x) > \lambda_1 \,\}, \qquad B_2[v \le \lambda_1] := \{\, x \in B_2 : v(x) \le \lambda_1 \,\}.$$

From the Hölder inequality we obtain

$$\int\limits_{B_2[v>\lambda_1]} v\Big(\Delta\eta + b^{(\alpha)} \cdot \nabla\eta\Big)\, dx \le \|v\|_{L_\infty(B_2)}\, \|\eta\|_{C^2(\bar B_2)}\Big(1 + \|b\|_{L_{p'}(B_2)}\Big)|B_2[v > \lambda_1]|^{\frac1p}.$$

Taking into account (4.10), (4.12) and $\|\eta\|_{C^2(B_2)} \le c_{\star\star}$ we conclude

$$\Big| \int\limits_{B_2[v>\lambda_1]} v\Big(\Delta\eta + b \cdot \nabla\eta\Big)\, dx \Big| \le 2c_{\star\star}\Big(1 + \|b\|_{L_{p'}(B_2)}\Big)\delta_1^{\frac1p}.$$

On the other hand

$$\Big| \int\limits_{B_2[v\le\lambda_1]} v\Big(\Delta\eta + b \cdot \nabla\eta\Big)\, dx \Big| \le c_{\star\star}\Big(1 + \|b\|_{L_1(B_2)}\Big)\lambda_1.$$

Finally, we obtain

$$|\alpha| \le 2c_{\star\star}\Big(1 + \|b\|_{L_{p'}(B_2)}\Big)\delta_1^{\frac1p} + c_{\star\star}\Big(1 + \|b\|_{L_1(B_2)}\Big)\lambda_1.$$

This inequality leads to the contradiction if we fix values of λ_1, $\delta_1 \in (0,1)$ so that

$$2c_{\star\star}\Big(1 + \|b\|_{L_{p'}(B_2)}\Big)\delta_1^{\frac1p} + c_{\star\star}\Big(1 + \|b\|_{L_1(B_2)}\Big)\lambda_1 < \pi|\alpha|.$$

$\square$

Proposition 4.3. *Assume $\Omega \subset \mathbb{R}^3$ is a bounded domain which contains the origin. Denote by $\delta_1 \in (0,1)$ and $\lambda_1 \in (0,1)$ the constants from Proposition 4.2. Assume $b \in L^{2,1}_w(\Omega)$ satisfies $\operatorname{div} b = 0$ in $\mathcal{D}'(\Omega)$. Assume $f \in L_p(\Omega)$ with $p > 2$ and let u be a p-weak solution to the problem (1.1) such that*

$$u|_\Gamma \;=\; 0.$$

Assume $\alpha \neq 0$, $x_0 \in \Gamma$ and $B_{4R}(x_0) \subset \Omega$. Denote

$$k_0 \;:=\; \frac{1}{2}\Big(M(x_0,4R) + m(x_0,4R)\Big), \tag{4.15}$$

and

$$k_1 \;:=\; M(x_0,4R) \;-\; \frac{\lambda_1}{2}\,\omega(x_0,4R). \tag{4.16}$$

If $k_0 \;\geq\; 0$ then either

$$\big|\{\, x \in B_{2R}(x_0) : \;u(x) \leq k_1 \,\}\big| \;\geq\; \delta_1\,|B_{2R}| \tag{4.17}$$

or

$$\omega(x_0,4R) \;\leq\; \frac{2R^{-1/2}}{c_\star|\alpha|}\,\|f\|_{L_2(B_{4R}(x_0))} \tag{4.18}$$

where $c_\star$ is defined in (4.11).

Proof. Assume (4.18) does not hold, i.e.

$$R^{-1/2}\,\|f\|_{L_2(B_{4R}(x_0))} \;\leq\; \frac{c_\star|\alpha|}{2}\,\omega(x_0,4R). \tag{4.19}$$

For $x \in B_4$ we denote

$$u^R(x) = u(x_0 + Rx), \qquad b_0^R(x) = R\,b_0(x_0+Rx), \qquad f^R(x) = Rf(x_0+Rx).$$

Then u^R is a solution to

$$-\Delta u^R + b_0^R \cdot \nabla u^R \;=\; -\operatorname{div} f^R \qquad \text{in} \quad B_2$$

and, moreover,

$$\|b_0^R\|_{L_{p'}(B_2)} \;\leq\; c\,\|b_0\|_{L^{2,1}_w(\Omega)}$$

with some $c > 0$ independent of R. Define v and g so that

$$v(x) \;:=\; 2\,\frac{M(x_0,4R) - u^R(x)}{\omega(x_0,4R)}, \qquad g(x) \;:=\; -2\,\frac{f^R(x)}{\omega(x_0,4R)}.$$

From (4.19) we conclude that g satisfies (4.11). Moreover, v is a p-weak solution to the equation (4.9) which satisfies

$$0 \leq v \leq 2, \qquad v|_\Gamma \;\geq\; 1.$$

Hence

$$\big|\{\, x \in B_2 : \;v(x) \geq \lambda_1 \,\}\big| \;\geq\; \delta_1$$

which gives (4.17). $\square$

Now we can prove the estimates of oscillation for functions belonging to various modified De Giorgi classes. We will distinguish between the following three cases which are motivated by the properties of a p-weak solution u to the problem (1.1) (we will proof this properties in Section 5):

- Assume x_0 is an internal point away from the singular line Γ. In this case we assume $B_R(x_0) \subset \Omega \setminus \Gamma$. In this case we will show that $\pm u \in DG(B_R(x_0); k_0)$ for any $k_0 \in \mathbb{R}$, i.e. away from the singular line Γ a p-weak solution to the problem (1.1) belongs to the De Giorgi class with arbitrary starting level k_0. In this case the estimate of the oscillation of u in the ball $B_R(x_0)$ is standard.
- Assume x_0 is an internal point which belongs to the singular line Γ. In this case we assume $B_R(x_0) \subset \Omega$ and $x_0 \in \Gamma$. Due to the assumption (1.18) this case would correspond to the condition $\pm u \in DG(\Omega; 0)$ (i.e. the De Griorgi class $DG(\Omega; k_0)$ with the fixed starting level $k_0 = 0$). In this case the estimate of the oscillation of u in the ball $B_R(x_0)$ is based on Proposition 4.3.
- Finally, consider a boundary point $x_0 \in \partial\Omega$ (including the case when $x_0 \in \partial\Omega \cap \Gamma$). In this case we take arbitrary Ω_0 such that $\Omega \Subset \Omega_0$ and denote by $\bar{u}$ the zero extension of u onto $\Omega_0 \setminus \Omega$. So, this case also corresponds to the condition $\pm\bar{u} \in DG(\Omega_0; 0)$ and due to the Dirichlet condition in (1.1) and the Lipschitz continuity of $\partial\Omega$ we may assume that $\tilde{u}$ vanishes on a fixed portion of $B_R(x_0) \subset \Omega_0$ (so the density condition (4.2) in this case is satisfied for free because of the Dirichlet condition in (1.1)).

We start from the oscillation estimate for the non-singular internal points. The proof of this estimate is standard and we present it only for readers' convenience.

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $n \geq 2$, and $\pm u \in DG(\Omega, k_0)$ for any $k_0 \in \mathbb{R}$, where $DG(\Omega; k_0)$ is the De Giorgi class with the parameters γ, F, β, q (see Definition 4.1). There exists a constant $\sigma \in (0,1)$ depending only on γ, β, q, such that for any $B_{4R}(x_0) \subset \Omega$*

$$\operatorname*{osc}_{B_R(x_0)} u \ \leq\ \sigma \operatorname*{osc}_{B_{4R}(x_0)} u + c_2\, F\, R^{1-\frac{n}{q}} \tag{4.20}$$

where $c_2 > 0$ is a constant from (4.3).

Proof. Define $k_0 \in \mathbb{R}$ by (4.15) and consider the case:

$$\big|\{\, x \in B_{2R}(x_0) :\ u(x) \leq k_0 \,\}\big| \ \geq\ \tfrac{1}{2}\, |B_{2R}|.$$

Then from (4.3) we obtain

$$M(x_0, R) \ \leq\ (1-\theta)\, M(x_0, 4R) + \theta\, k_0 + c_2\, F\, R^{1-\frac{n}{q}}.$$

Subtracting from both sides $m(x_0, 4R)$ we arrive at

$$\omega(x_0, R) \ \leq\ (1 - \tfrac{\theta}{2})\, \omega(x_0, 4R) \ +\ c_2\, F\, R^{1-\frac{n}{q}}. \tag{4.21}$$

In the second case

$$\big|\{\, x \in B_{2R}(x_0) :\ u(x) \geq k_0 \,\}\big| \ \geq\ \tfrac{1}{2}\, |B_{2R}|$$

we denote $v := -u$, $l_0 := -k_0$ and obtain $v \in DG(\Omega, l_0)$ and

$$\big|\{\, x \in B_{2R}(x_0) :\ v(x) \leq l_0 \,\}\big| \ \geq\ \tfrac{1}{2}\, |B_{2R}|.$$

Then from (4.3) we again arrive at (4.21). ∎

Now we present the oscillation estimate for points on the singular curve Γ:

Lemma 4.5. *Assume $\Omega \subset \mathbb{R}^3$ is a bounded domain which contains the origin. Denote by $\delta_1 \in (0,1)$ and $\lambda_1 \in (0,1)$ the constants from Proposition 4.2. Assume $b \in L^{2,1}_w(\Omega)$ satisfies $\operatorname{div} b = 0$ in $\mathcal{D}'(\Omega)$. Assume $f \in L_p(\Omega)$ with $p > 2$ and let u be a p-weak solution to the problem* (1.1) *such that*

$$u|_\Gamma \;=\; 0.$$

Assume $\alpha \neq 0$, $x_0 \in \Gamma$ and $B_{4R}(x_0) \subset \Omega$. There is a constant $\sigma \in (0,1)$ depending only on $\|b\|_{L^{2,1}_w(\Omega)}$ and α such that for any $x_0 \in \Gamma$ and any $B_{4R}(x_0) \subset \Omega$ we have

$$\operatorname*{osc}_{B_R(x_0)} u \;\le\; \sigma \operatorname*{osc}_{B_{4R}(x_0)} u \,+\, c_2\, F\, R^{1-\frac{3}{q}} \tag{4.22}$$

where $c_1 > 0$ depends only on n, γ, β, q and α.

Proof. Assume u is a p-weak solution to (1.1) and $u|_\Gamma = 0$. In the next section (see Proposition 5.2) we will show that in this case $\pm u \in DG(B_{4R}(x_0); 0)$. Define $k_0 \in \mathbb{R}$ and $k_1 \in \mathbb{R}$ by (4.15) and (4.16) respectively and assume $k_0 \ge 0$. By Proposition 4.3 either (4.17) of (4.18) hold. In the case of (4.18) we obtain by the Hölder inequality for $q > 3$

$$\omega(x_0, 4R) \;\le\; \frac{2}{c_\star|\alpha|}\, \|f\|_{L_2(B_{4R}(x_0))}\, R^{-\frac12} \;\le\; \frac{c}{c_\star|\alpha|}\, \|f\|_{L_q(B_{4R}(x_0))}\, R^{1-\frac{3}{q}}$$

and hence (4.22) follows. Assume now (4.17) holds. Note that $k_1 \ge k_0 \ge 0$ and hence $u \in DG(\Omega, k_1)$. Hence from Proposition 4.1 we conclude

$$M(x_0, R) \;\le\; (1-\theta)\, M(x_0, 4R) + \theta\, k_1 + c_1\, F\, R^{1-\frac{3}{q}}$$

and taking into account (4.16) we arrive at

$$\omega(x_0, R) \;\le\; (1 - \tfrac{\lambda_1 \theta}{2})\, \omega(x_0, 4R) \,+\, c_2\, F\, R^{1-\frac{3}{q}}. \tag{4.23}$$

Now consider the case $k_0 \le 0$. Denote $v := -u$, $l_0 := -k_0$

$$l_1 \;:=\; -m(x_0, 4R) \,-\, \frac{\lambda_0}{2}\, \omega(x_0, 4R). \tag{4.24}$$

Note that $v \in DG(\Omega; 0)$ and $l_0 \ge 0$. Applying Proposition 4.3 for v we obtain either (4.18) or

$$\big|\{\, x \in B_{2R}(x_0) : \; v(x) \le l_1 \,\}\big| \;\ge\; \delta_1\, |B_{2R}| \tag{4.25}$$

hold. In the case of (4.18) we obtain (4.22) immediately. In the case of (4.25) we apply Proposition 4.1 for v and conclude

$$-m(x_0, R) \;\le\; -(1-\theta)\, m(x_0, 4R) + \theta\, l_1 + c_1\, F\, R^{1-\frac{3}{q}}$$

and taking into account (4.24) we arrive at (4.23) again. ∎

Finally we present the oscillation estimate near the boundary:

Lemma 4.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $n \geq 2$, and $\pm u \in DG(\Omega, 0)$ where $DG(\Omega; 0)$ is the De Giorgi class with the parameters γ, F, β, q and the initial level $k_0 = 0$ (see Definition 4.1). For any $\delta > 0$ there exists a constant $\sigma \in (0,1)$ depending only on δ, γ, β, q, such that if for some $B_{4R}(x_0) \subset \Omega$ the estimate*

$$\big|\{\, x \in B_{2R}(x_0) : \; u(x) = 0 \,\}\big| \;\geq\; \delta \, |B_{2R}| \tag{4.26}$$

is valid then (4.20) *holds.*

Proof. Define $k_0 \in \mathbb{R}$ by (4.15) and consider the case $k_0 \geq 0$. Then

$$\{\, x \in B_{2R}(x_0) : \; u(x) = 0 \,\} \subset \{\, x \in B_{2R}(x_0) : \; u(x) \leq k_0 \,\}$$

and we obtain hence (4.2) holds. As $u \in DG(\Omega, k_0)$ we obtain

$$M(x_0, R) \;\leq\; (1-\theta)\, M(x_0, 4R) + \theta\, k_0 + c_1\, F\, R^{1-\frac{n}{q}}$$

and hence

$$\omega(x_0, R) \;\leq\; (1 - \tfrac{\theta}{2})\, \omega(x_0, 4R) \;+\; c_1\, F\, R^{1-\frac{n}{q}}. \tag{4.27}$$

In the case $k_0 \leq 0$ we denote $v := -u$, $l_0 := -k_0$. Then $l_0 \geq 0$ and $v \in DG(\Omega, l_0)$. As

$$\{\, x \in B_{2R}(x_0) : \; u(x) = 0 \,\} \subset \{\, x \in B_{2R}(x_0) : \; v(x) \leq l_0 \,\}$$

we obtain

$$\big|\{\, x \in B_{2R}(x_0) : \; v(x) \leq l_0 \,\}\big| \;\geq\; \delta\; |B_{2R}|$$

and hence

$$-m(x_0, R) \;\leq\; -(1-\theta)\, m(x_0, 4R) + \theta\, l_0 + c_1\, F\, R^{1-\frac{n}{q}}$$

which again leads to (4.27). $\square$

5 Hölder continuity of weak solutions

In this section we show that under assumption on the drift term (1.9), (1.10), (1.11) any p-weak solution u to the problem (1.1) belongs to De Giorgi classes from Section 4. As a consequence we obtain the proof of Theorem 1.1. First we consider an internal point x_0 away from the singular curve Γ.

Proposition 5.1. *Let all assumptions of Theorem 1.1 hold and assume $B_{4R}(x_0) \subset \Omega \setminus \Gamma$. Then for any $k_0 \in \mathbb{R}$ we have $\pm u \in DG(B_{4R}(x_0); k_0)$ where $DG(\Omega; k_0)$ is the De Giorgi class in Definition 4.1 with $n = 3$, $F = \|f\|_{L_q(\Omega)}$ and some $\gamma > 0$, $\beta > 0$ which depend only on $\|b\|_{L^{2,1}_w(\Omega)}$ and α.*

In the case $B_{4R}(x_0) \subset \Omega \setminus \Gamma$ the drift b_0 is divergence free in $B_{4R}(x_0)$. Hence Proposition 5.1 follows from [7, Section 5]. Here we outline the proof for reader's convenience.

Proof. From Theorem 3.1 we conclude $u \in L_\infty(\Omega)$. Let us fix some $r \in \left(\frac{6}{5}, 2\right)$ and $\theta \in \left(\frac{3}{r} - \frac{3}{2}, 1\right)$. Then from (1.3) and Proposition 2.1 we obtain

$$b_0 \in L^{r,3-r}(\Omega), \qquad \|b_0\|_{L^{r,3-r}(\Omega)} \le c\,\|b_0\|_{L^{2,1}_w(\Omega)}.$$

Take some radius $\rho < R$ and a cut-off function $\zeta \in C_0^\infty(B_R(x_0))$ such that

$$0 \le \zeta \le 1, \qquad \zeta \equiv 1 \quad \text{on} \quad B_\rho(x_0), \qquad |\nabla\zeta| \le \frac{c}{R-\rho}. \tag{5.1}$$

Assume $k \in \mathbb{R}$ is arbitrary and denote

$$\tilde u := (u-k)_+ \equiv \max\{u-k, 0\}, \qquad \tilde u \in L_\infty(\Omega) \cap W^1_p(\Omega). \tag{5.2}$$

Fix $m := \frac{1}{1-\theta}$ and note that $2m - 1 = m(1+\theta)$. Take $\eta = \zeta^{2m}\tilde u$ in (1.6). Taking into account

$$\operatorname{div} b_0 = 0 \quad \text{in} \quad \mathcal{D}'(B_{4R}(x_0))$$

with the help of integration by parts we obtain

$$\mathcal{B}[u,\eta] \;=\; -\,m \int\limits_\Omega \zeta^{2m-1}\, b_0 \cdot \nabla\zeta\, |\tilde u|^2\, dx. \tag{5.3}$$

As $2m - 1 = m(1+\theta)$ we obtain

$$\begin{gathered}\|\zeta^m \nabla\tilde u\|^2_{L_2(B_R(x_0))} \;\le\; c\,\|\tilde u\nabla\zeta\|^2_{L_2(B_R(x_0))} + m \int\limits_\Omega \zeta^{m(1+\theta)}\, b_0\cdot\nabla\zeta\,|\tilde u|^2\,dx \,+ \\ +\; \|f\|^2_{L_q(\Omega)}\,|A_k \cap B_R(x_0)|^{1-\frac{2}{q}}\end{gathered} \tag{5.4}$$

where $A_k := \{\, x \in \Omega : u(x) > k \,\}$. Taking into account (5.1) and applying the estimate (2.3) we obtain

$$\int\limits_\Omega \zeta^{m(1+\theta)}\, b\cdot\nabla\zeta\,|\tilde u|^2\,dx \;\le\; \frac{c\,R^\theta}{R-\rho}\,\|b_0\|_{L^{r,3-r}(\Omega)}\;\|\nabla(\zeta^m\tilde u)\|^{1+\theta}_{L_2(B_R(x_0))}\,\|\tilde u\|^{1-\theta}_{L_2(B_R(x_0))}.$$

Taking arbitrary $\varepsilon > 0$ and applying the Young inequality we obtain

$$\int\limits_\Omega \zeta^{m(1+\theta)} b\cdot\nabla\zeta\,|\tilde u|^2\,dx \;\le\; \varepsilon\,\|\nabla(\zeta^m\tilde u)\|^2_{L_2(B_R(x_0))} \;+$$
$$+\;\frac{c_\varepsilon}{(R-\rho)^2}\Big(\frac{R}{R-\rho}\Big)^{\frac{2\theta}{1-\theta}}\|b_0\|^{\frac{2}{1-\theta}}_{L^{r,3-r}(B_R(x_0))}\|\tilde u\|^2_{L_2(B_R(x_0))}.$$

So, if we fix sufficiently small $\varepsilon > 0$ from (5.4) for any $k\in\mathbb{R}$ and $0<\rho<R$ we obtain

$$\begin{gathered}\tfrac12\,\|\nabla(u-k)_+\|^2_{L_2(B_\rho(x_0))} \;\le\\ \le\;\frac{c}{(R-\rho)^2}\,\bigg(1+\Big(\frac{R}{R-\rho}\Big)^{\frac{2\theta}{1-\theta}}\|b_0\|^{\frac{2}{1-\theta}}_{L^{r,3-r}(\Omega)}\bigg)\,\|(u-k)_+\|^2_{L_2(B_R(x_0))}+\\ +\;\|f\|^2_{L_q(\Omega)}\,|A_k\cap B_R(x_0)|^{1-\frac2q}.\end{gathered}\tag{5.5}$$

Hence we obtain that $u\in DG(\Omega)$. Applying the same arguments to $-u$ instead of u we also obtain $-u\in DG(\Omega)$. $\square$

Now we consider an internal point x_0 laying on the singular curve Γ.

Proposition 5.2. *Let all assumptions of Theorem 1.1 hold and assume $x_0\in\Gamma$ and $B_{4R}(x_0)\subset\Omega$. Then for any $k_0\ge 0$ we have $\pm u\in DG(B_{4R}(x_0);0)$ where $DG(\Omega;0)$ is the De Giorgi class in Definition 4.1 with $n=3$, $k_0=0$, $F=\|f\|_{L_q(\Omega)}$ and some $\gamma>0$, $\beta>0$ which depend only on $\|b\|_{L^{2,1}_w(\Omega)}$ and α.*

Proof. We take arbitrary $k\ge 0$ and proceed as in the proof of Proposition 5.1. Define $\tilde u$ by (5.2). As $u|_\Gamma=0$ and $k\ge 0$ we conclude $\tilde u|_\Gamma=0$ in the sense of traces and from Proposition 2.2 we conclude

$$\mathcal{B}[\zeta^m\tilde u,\zeta^m\tilde u] \;=\; 0. \tag{5.6}$$

Hence we again arrive at (5.3) and proceed in the same way as in Proposition 5.1. $\square$

Finally we consider a point x_0 laying on the boundary $\partial\Omega$. Note that as Ω is a bounded Lipschitz domain there exist $R_*>0$ and δ_* such that for any $x_0\in\partial\Omega$ and any $R<R_*$

$$|B_R(x_0)\setminus\Omega| \;\ge\; \delta_*\,|B_R|. \tag{5.7}$$

Proposition 5.3. *Let all assumptions of Theorem 1.1 hold and denote by $\bar u$ the zero extension of u outside Ω. Assume $x_0\in\partial\Omega$ and $4R\le R_*$. Then for any $k_0\ge 0$ we have $\pm\bar u\in DG(B_{4R}(x_0);0)$ where $DG(\Omega;0)$ is the De Giorgi class in Definition 4.1 with $n=3$, $k_0=0$, $F=\|f\|_{L_q(\Omega)}$ and some $\gamma>0$, $\beta>0$ which depend only on $\|b\|_{L^{2,1}_w(\Omega)}$ and α.*

Proof. Denote $\Omega_R(x_0):=\Omega\cap B_R(x_0)$. Take a cut-off function $\zeta\in C_0^\infty(B_R(x_0))$ satisfying (5.1). Then for any $k\ge 0$ the function $\eta=\zeta^{2m}(u-k)_+$ vanishes on $\partial\Omega$ hence it is admissible for the identify (1.6). From $(u-k)_+|_\Gamma=0$ we conclude (5.6) holds. Proceeding as in the proofs of Propositions 5.1, 5.2 we arrive at

$$\begin{gathered}\tfrac12\,\|\nabla(u-k)_+\|^2_{L_2(\Omega_\rho(x_0))} \;\le\\ \le\;\frac{c}{(R-\rho)^2}\,\bigg(1+\Big(\frac{R}{R-\rho}\Big)^{\frac{2\theta}{1-\theta}}\|b\|^{\frac{2}{1-\theta}}_{L^{r,3-r}(\Omega)}\bigg)\,\|(u-k)_+\|^2_{L_2(\Omega_R(x_0))}+\\ +\;\|f\|^2_{L_q(\Omega)}\,|A_k\cap\Omega_R(x_0)|^{1-\frac2q},\end{gathered}$$

which gives (5.5) with the function $\bar{u}$ instead of u. Hence we obtain $\bar{u} \in DG(B_{4R}(x_0))$. Similarly we obtain $-\bar{u} \in DG(B_{4R}(x_0))$. □

Now we can prove Theorem 1.1. Taking into account (5.7) we can iterate estimates in Proportions 4.4, 4.5, 4.6 and obtain the oscillation estimate

$$\forall\, \rho < R \qquad \operatorname*{osc}_{B_\rho(x_0)\cap\Omega} u \;\le\; c_2 \left(\left(\frac{\rho}{R}\right)^{\mu} \|u\|_{L_\infty(\Omega)} + F\,\rho^{\mu} \right) \tag{5.8}$$

with some $\mu \in (0,1)$ depending only on $\sigma \in (0,1)$ and $q > 3$ in one of the following three cases:

(a) $B_{4R}(x_0) \subset \Omega \setminus \Gamma$,

(b) $x_0 \in \Gamma$, $B_{4R}(x_0) \subset \Omega$,

(c) $x_0 \in \partial\Omega$, $R < \frac{1}{4}R_*$.

Then the inequality (5.8) for an arbitrary $x_0 \in \bar{\Omega}$ and $R < \frac{1}{4}R_*$ can be obtained by a standard combination of inequalities (a), (b), (c). From this inequality and (3.1) the estimate (1.19) follows immediately. Theorem 1.1 is proved.

6 Existence and uniqueness of p-weak solutions

In this section we prove Theorem 1.2. First we establish the higher integrability of weak solutions to the problem (1.1).

Proposition 6.1. *Assume $b \in L_w^{2,1}(\Omega)$ satisfies (1.9), (1.10) and assume $f \in L_q(\Omega)$ with $q > 3$. Then there exists $p > 2$ depending only on q, α, $\|b\|_{L_w^{2,1}(\Omega)}$ and the Lipschitz constant of $\partial\Omega$ such that for any p-weak solution u of the problem (1.17) satisfying additional assumption (1.18) the estimate (1.20) holds with some constant $c > 0$ depending only on q, α and $\|b\|_{L_w^{2,1}(\Omega)}$ and the Lipschitz constant of $\partial\Omega$.*

Proof. Assume $p > 2$ and let u be a p-weak solution to (1.17). Then we can interpret u as a p-weak solution to the problem

$$\left\{ \begin{array}{rcl} -\Delta u + b \cdot \nabla u & = & \operatorname{div} g \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} & = & 0 \end{array} \right. \tag{6.1}$$

with the right hand side

$$g \, = \, f \, + \, \alpha \, \frac{x'}{|x'|^2} \, u.$$

and hence for any $p > 2$ we have

$$\|g\|_{L_p(\Omega)} \, \le \, \|f\|_{L_p(\Omega)} \, + \, c \, \left\| \tfrac{u}{|x'|} \right\|_{L_p(\Omega)}$$

From Theorem 1.1 we obtain $u \in C^\mu(\bar\Omega)$ with some $\mu \in (0,1)$ depending only on Ω, α, q and $\|b\|_{L_w^{2,1}(\Omega)}$. Fix some $p_0 \in \left(2, \frac{2}{1-\mu}\right)$ and denote $q_0 = \min\{q, p_0\}$. Taking into account (1.18) we obtain

$$\left\| \tfrac{u}{|x'|} \right\|_{L_{q_0}(\Omega)} \, \le \, c \, \|u\|_{C^\mu(\bar\Omega)}$$

and hence from Theorem 1.1 we arrive at

$$\|g\|_{L_{q_0}(\Omega)} \, \le \, \|f\|_{L_q(\Omega)} \, + \, c \, \|u\|_{C^\mu(\bar\Omega)} \, \le \, c \, \|f\|_{L_q(\Omega)}$$

Since $\operatorname{div} b = 0$, we conclude u is a p-weak solution of the problem (6.1) with a divergence-free drift b and the right-hand side $q \in L_{q_0}(\Omega)$ with some $q_0 > 2$. Hence from [7] we obtain there exists $p \in (2, q_0)$ such that

$$\|u\|_{W_p^1(\Omega)} \, \le \, c \, \|g\|_{L_p(\Omega)}$$

from which we obtain (1.20). $\square$

Now we turn to the proof of Theorem 1.2. We follow the method we used in [6] in the 2D case. As a first step we construct a solution of an auxiliary problem satisfying the "non-spectral" condition (1.7).

Proposition 6.2. *Assume $\alpha < 0$. Then there exists $p_1 > 2$ depending only on Ω and α such that for any $f \in C_0^\infty(\Omega \setminus \Gamma)$ there exists a unique p_1–weak solution v to the problem*

$$\left\{ \begin{array}{rcl} -\Delta v \,-\, |\alpha|\, \dfrac{x'}{|x'|^2} \cdot \nabla v & = & -|x'|^\alpha \,\mathrm{div}\, f \quad in \quad \Omega, \\ v|_{\partial\Omega} & = & 0. \end{array} \right. \tag{6.2}$$

Moreover, v is Hölder continuous in $\bar{\Omega}$.

Proposition 6.2 is proved in [7].

Now we apply the so-called "Darboux transform" to the function v to construct a solution of an auxiliary problem with $\alpha < 0$ (which corresponds to the "spectral" case (1.8)) and vanishing on Γ.

Proposition 6.3. *Assume $\alpha < 0$ and $q > 3$. Then there exists $p > 2$ depending only on q, α and the Lipschitz constant of $\partial\Omega$ such that for any $f \in C_0^\infty(\Omega \setminus \Gamma)$ there exists a unique p–weak solution u to the problem*

$$\left\{ \begin{array}{rcl} -\Delta u - \alpha\, \dfrac{x'}{|x'|^2} \cdot \nabla u & = & -\,\mathrm{div}\, f \qquad in \quad \Omega, \\ u|_{\partial\Omega} & = & 0, \end{array} \right. \tag{6.3}$$

which satisfies the condition (1.18). *Moreover, u is Hölder continuous in $\bar{\Omega}$ and satisfies estimates* (1.19) *and* (1.20).

Proof. Let v be a p_1-weak solution of (6.2). Denote

$$u(x) \;=\; |x'|^{|\alpha|}\, v(x)$$

If $|\alpha| \geq 1$ then the function $|x'|^{|\alpha|}$ is Lipschitz continuous. Hence we obtain $u \in W^1_{p_1}(\Omega)$. In the case $0 < |\alpha| < 1$ we have $u \in W^1_p(\Omega)$ for any p satisfying

$$2 < p \;<\; \min\left\{ p_1,\; \frac{2}{1 - |\alpha|} \right\}. \tag{6.4}$$

In any case we obtain $u \in \overset{\circ}{W}{}^1_p(\Omega)$ for some $p > 2$, u is Hölder continuous on $\bar{\Omega}$ and satisfies the condition (1.18). Let us verify u is a p-weak solution to (6.3). Indeed, taking arbitrary $\eta \in C_0^\infty(\Omega)$ and testing (6.2) by $|x'|^{|\alpha|}\eta$ with the help of identities

$$\nabla\big(|x'|^{|\alpha|}\eta\big) \;=\; |x'|^{|\alpha|}\Big(\nabla\eta + |\alpha|\,\frac{x'}{|x'|^2}\,\eta\Big), \qquad |x'|^{|\alpha|}\nabla v \;=\; \nabla u + \alpha u \frac{x'}{|x'|^2}$$

we arrive at the identity

$$\int\limits_\Omega \Big(\nabla u + \alpha\, u \frac{x'}{|x'|^2}\Big) \cdot \nabla\eta\, dx \;=\; \int\limits_\Omega f \cdot \nabla\eta\, dx, \qquad \forall\, \eta \in C_0^\infty(\Omega).$$

Taking into account $u|_\Gamma = 0$ and using integration by parts we obtain

$$\alpha \int\limits_\Omega \frac{x'}{|x|^2} \cdot \nabla u\, \eta\, dx \;=\; -\alpha \int\limits_\Omega \frac{x'}{|x|^2} \cdot \nabla \eta\, u\, dx$$

which gives

$$-\Delta u - \alpha\frac{x'}{|x'|^2}\cdot\nabla u \;=\; -\operatorname{div} f \qquad \text{in} \quad \mathcal{D}'(\Omega).$$

Now we can fix $\mu\in(0,1)$ and $p>2$ as in Theorem 1.1 and Proposition 6.1. Without loss of generality we can assume (6.4) is satisfied. Then from Theorem 1.1 and in Proposition 6.1 we obtain inequalities (1.19) and (1.20). Note that $\mu\in(0,1)$ and $p>2$ depend only on Ω, q and α. $\square$

Now we can relax the assumption on the smoothness of the right hand side f.

Proposition 6.4. *Assume $\alpha<0$, $q>3$ and assume $\mu\in(0,1)$ is defined in Theorem 1.1 and $p>2$ is defined in Proposition 6.3. Then for any $f\in L_q(\Omega)$ there exists a unique p-weak solution u of the system* (6.3) *which satisfies* (1.18). *Moreover, $u\in C^\mu(\bar\Omega)$ and the estimates* (1.19), (1.20) *hold.*

Proof. Assume $f\in L_q(\Omega)$ and take $f_\varepsilon\in C_0^\infty(\Omega\setminus\Gamma)$ so that $\|f_\varepsilon - f\|_{L_q(\Omega)}\to 0$ and $\varepsilon\to 0$. From Proposition 6.3 we obtain the existence of p-weak solutions $\{u_\varepsilon\}$ to the problem (6.3) with right hand side f_ε Moreover, using estimates (1.19) and (1.20) we obtain inequality

$$\|u_\varepsilon\|_{W^1_p(\Omega)} \,+\, \|u_\varepsilon\|_{C^\mu(\bar\Omega)} \;\le\; c\,\|f_\varepsilon\|_{L_q(\Omega)}.$$

Hence we can extract a subsequence such that $u_\varepsilon \rightharpoonup u$ in $W^1_q(\Omega)$ and $u_\varepsilon\to u$ uniformly in $\bar\Omega$. It is easy to check that u will satisfy (6.3) with right hand side f and (1.18) holds. $\square$

Now we can prove the existence of p-weak solutions to the problem (1.1) in the case of a smooth divergence free part of the drift.

Proposition 6.5. *Assume $\alpha<0$, $q>3$ and $b\in C^\infty(\Omega)$ satisfies $\operatorname{div} b=0$. Then there exist $\mu\in(0,1)$ and $p>2$ depending only on q, α, $\|b\|_{L^{2,1}_w(\Omega)}$ and the Lipschitz constant of $\partial\Omega$ such that for any $f\in L_q(\Omega)$ there exists a unique p-weak solution u to the problem* (1.1) *which satisfies the condition* (1.18). *Moreover, $u\in C^\mu(\Omega)$ and the estimates* (1.19) *and* (1.20) *hold.*

Proof. Take any $v\in L_q(\Omega)$. From Proposition 6.4 we obtain the existence the unique p-weak solution u^v to the problem

$$\left\{\begin{aligned} -\Delta u^v - \alpha\,\frac{x'}{|x'|^2}\cdot\nabla u^v \;&=\; \operatorname{div}(f-bv) \qquad \text{in} \quad \Omega,\\ u^v|_{\partial\Omega} \;&=\; 0. \end{aligned}\right. \tag{6.5}$$

such that

$$u^v|_\Gamma = 0.$$

Moreover, $u^v\in C^\mu(\bar\Omega)$ and the estimate

$$\|u^v\|_{W^1_p(\Omega)} \,+\, \|u^v\|_{C^\mu(\bar\Omega)} \;\le\; c\Big(\|f\|_{L_q(\Omega)} \,+\, \|b\|_{L_\infty(\Omega)}\|v\|_{L_q(\Omega)}\Big) \tag{6.6}$$

holds. Define the operator $A: L_q(\Omega)\to L_q(\Omega)$, $A(v):=u^v$. Applying Theorem 1.1 and Proposition 6.1 for any v_1, $v_2\in L_q(\Omega)$ we obtain the inequality

$$\|A(v_1)-A(v_2)\|_{W^1_p(\Omega)} \,+\, \|A(v_1)-A(v_2)\|_{C^\mu(\bar\Omega)} \;\le\; c\,\|b\|_{L_\infty(\Omega)}\|v_1-v_2\|_{L_q(\Omega)}$$

which implies the operator $A : L_q(\Omega) \to L_q(\Omega)$ is continuous. Moreover, from (6.6) taking into account compactness of the imbedding $W^1_p(\Omega) \hookrightarrow L_p(\Omega)$ it is easy to see that the operator $A : L_q(\Omega) \to L_q(\Omega)$ is compact. Hence we can apply the Leray-Shauder fixed point theorem. Assume $\lambda \in [0, 1]$ and $v \in L_q(\Omega)$ satisfies $v = \lambda A(v)$. Denote $u := A(v)$. Then u is a unique p-weak solution to the problem

$$\left\{ \begin{aligned} -\Delta u + \Big(\lambda b - \alpha \, \frac{x}{|x|^2}\Big) \cdot \nabla u \ &= \ - \operatorname{div} f \qquad \text{in} \quad \Omega, \\ u|_{\partial\Omega} \ &= \ 0, \end{aligned} \right.$$

which satisfies the condition (1.18). Hence from Theorem 1.1 and Proposition 6.1 we obtain the estimate

$$\|u\|_{W^1_p(\Omega)} \ + \ \|u\|_{C^\mu(\bar\Omega)} \ \le \ c\, \|f\|_{L_q(\Omega)}$$

with some constant c depending only on Ω, q, α and $\|b\|_{L_{2,w}(\Omega)}$ and independent of $\lambda \in [0, 1]$. Hence there exists $u \in L_q(\Omega)$ satisfying $u = A(u)$. $\square$

Now we can prove Theorem 1.2.

Proof. We follow the method similar to [23, Theorem 2.1], and [39],[40]. Let us fix some $q > 3$ and let $\mu \in (0, 1)$ and $p > 2$ be the constants determined in Theorem 1.1 and Proposition 1.2 respectively. Denote $p' = \frac{p}{p-1}$. As Ω is Lipschitz we can find the sequence of C^2-smooth domains $\{\Omega_k\}_{k=1}^\infty$ such that

$$\Omega_{k+1} \Subset \Omega_k, \qquad \bigcup_k \Omega_k = \Omega.$$

Moreover, it is possible to construct domains Ω_k so that the Lipschitz constants of $\partial\Omega_k$ are controlled uniformly by the Lipschitz constant of $\partial\Omega$. In particular, we can assume there are exist positive constants $\hat\delta_0$ and $\hat R_0$ independent on $k \in \mathbb{N}$ such that

$$\forall\, k \in \mathbb{N}, \qquad \forall\, R < \hat R_0, \qquad \forall\, x_0 \in \partial\Omega_k \qquad |B_R(x_0) \setminus \Omega_k| \ \ge \ \hat\delta_0\, |B_R|. \tag{6.7}$$

For a canonical domain given as a subgraph of a Lipschitz function existence of such approximation can be obtained by mollification and shift of the graph, and for general bounded Lipschitz domain the standard localization works.

Now we take a sequence of positive numbers $\varepsilon_k \to 0$ such that $\varepsilon_k < \operatorname{dist}\{\bar\Omega_k, \partial\Omega\}$, and define the mollification of the drift b:

$$b_k(x) \ := \int\limits_\Omega \omega_{\varepsilon_k}(x-y) b(y)\, dy, \qquad x \in \Omega,$$

where $\omega_\varepsilon(x) := \varepsilon^{-n}\omega(x/\varepsilon)$ and $\omega \in C_0^\infty(\mathbb{R}^n)$ is the standard Sobolev kernel, i.e.

$$\omega \ \ge \ 0, \qquad \operatorname{supp} \omega \in \bar B, \qquad \int\limits_{\mathbb{R}^n} \omega(x)\, dx \ = \ 1, \qquad \omega(x) = \omega_0(|x|). \tag{6.8}$$

Then $b_k \in C^\infty(\bar\Omega)$ and as $\varepsilon_k < \operatorname{dist}\{\bar\Omega_k, \partial\Omega\}$ from $\operatorname{div} b = 0$ in $\mathcal{D}'(\Omega)$ for any k we obtain

$$\operatorname{div} b_k \ = \ 0 \qquad \text{in} \quad \Omega_k. \tag{6.9}$$

Moreover, from [7, Proposition 6.1] we obtain there is a constant $c > 0$ independent on k such that

$$\|b_k\|_{L^{2,1}_w(\Omega_k)} \le c\,\|b\|_{L^{2,1}_w(\Omega)}, \qquad \|b_k - b\|_{L_{p'}(\Omega)} \to 0 \quad \text{as} \quad k \to 0. \tag{6.10}$$

For $f \in L_q(\Omega)$ we can find $f_k \in C_0^\infty(\Omega_k)$ such that $\|f_k - f\|_{L_q(\Omega)} \to 0$. From Proposition 6.5 we conclude that for any $k \in \mathbb{N}$ there exists a unique p-weak solution $u_k \in W^1_p(\Omega) \cap C^\mu(\bar{\Omega})$ to the problem

$$\left\{ \begin{array}{rcll} -\Delta u_k + \Big(b_k - \alpha\,\dfrac{x'}{|x'|^2}\Big)\cdot \nabla u_k &=& -\operatorname{div} f_k & \text{in} \quad \Omega_k, \\ u_k|_{\partial\Omega_k} &=& 0, & \end{array} \right. \tag{6.11}$$

which satisfies the condition

$$u_k|_\Gamma = 0$$

in the sense of traces. Extend functions u_k by zero from Ω_k onto Ω. From Proposition 6.1 we obtain the estimate

$$\|u_k\|_{W^1_p(\Omega)} \le c\,\|f_k\|_{L_p(\Omega)}$$

with a constant $c > 0$ depending only on q, $\|b\|_{L^{2,1}_w(\Omega)}$ and the constant $\hat{\delta}_0$ in (6.7) which is independent on k. Hence we can take a subsequence u_k such that

$$u_k \rightharpoonup u \quad \text{in} \quad W^1_p(\Omega).$$

As for $p > 2$ the trace operator is compact from $W^1_p(\Omega)$ into $L_p(\Gamma)$ we obtain

$$u|_\Gamma = 0$$

in the sense of traces. Take any $\eta \in C_0^\infty(\Omega)$, due to our construction of Ω_k for sufficiently large k we have $\operatorname{supp}\eta \subset \Omega_k$ and hence η is a suitable test function in (6.11). As $b_k \to b$ in $L_{p'}(\Omega)$ we can pass to the limit in the identity

$$\int\limits_\Omega \nabla u_k \cdot \Big(\nabla\eta + \Big(b_k - \alpha\,\frac{x'}{|x'|^2}\Big)\eta\Big)\,dx = \int\limits_\Omega f_k \cdot \nabla\eta\,dx,$$

and obtain (1.6). Hence $u \in \overset{\circ}{W}{}^1_p(\Omega)$ is a p-weak solution to the problem (1.1) satisfying (1.18) and (1.20). From Theorem 1.1 we obtain $u \in C^\mu(\bar{\Omega})$ and the estimate (1.19). The uniqueness of u follows from the estimate (1.20). $\square$

References

[1] D. Albritton, H. Dong, *Regularity properties of passive scalars with rough divergence-free drifts*, Arch. Ration. Mech. Anal. 247 (2023), no. 5, Paper No. 75.

[2] O.V. Besov, V.P. Ilin, S.M. Nikolskii, Integral Representations of Functions and Imbedding Theorems, Wiley & Sons 1979.

[3] L. Boccardo, L. Orsina, *Very singular solutions for linear Dirichlet problems with singular convection terms*. Nonlinear Anal. 194 (2020), 111437.

[4] L. Boccardo, *Weak maximum principle for Dirichlet problems with convection or drift terms*. Math. Eng. 3 (2021), no. 3, Paper No. 026.

[5] Ch.-Ch. Chen, R. Strain, T.-P. Tsai, H.-T. Yau, *Lower bounds on the blow-up rate of the axisymmetric Navier-Stokes equations. II.* Comm. Partial Differential Equations 34 (2009), no.1-3, 203–232.

[6] M. Chernobai, T. Shilkin, *Scalar elliptic equations with a singular drift*, Complex Variables and Elliptic Equations, **67** (2022), no.1, 9-33.

[7] M. Chernobai, T. Shilkin, *Elliptic equations with a singular drift from a weak Morrey space*, https://arxiv.org/abs/2208.10909

[8] F. Chiarenza, M. Frasca, *A remark on a paper by C. Fefferman*, Proc. Amer. Math. Soc., Vol. 108 (1990), no.2, 407–409.

[9] T. Cieślak, W. Ozański, *A partial uniqueness result and an asymptotically sharp nonuniqueness result for the Zhikov problem on the torus*, Calculus of Variations and Partial Differential Equations, 61 (2022), no.3.

[10] C. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. 9 (1983), 129-206.

[11] N. Filonov, *On the regularity of solutions to the equation* $-\Delta u + b\cdot\nabla u = 0$, Zap. Nauchn. Sem. of Steklov Inst. 410 (2013), 168-186; reprinted in J. Math. Sci. (N.Y.) 195 (2013), no. 1, 98–108.

[12] N. Filonov, T. Shilkin, *On some properties of weak solutions to elliptic equations with divergence-free drifts*. Mathematical analysis in fluid mechanics-selected recent results, 105-120, Contemp. Math., 710, Amer. Math. Soc., Providence, RI, 2018.

[13] S. Friedlander, V. Vicol, *Global well-posedness for an advection-diffusion equation arising in magneto-geostrophic dynamics*, Ann. Inst. H. Poincare Anal. Non Lineaire, 28 (2011), no. 2, 283–301.

[14] L. Grafakos, Classical Fourier analysis, Springer, New York, 2009.

[15] B. Kang, H. Kim, $W^{1,p}$*-estimates for elliptic equations with lower order terms*. Commun. Pure Appl. Anal. 16 (2017), 799–821.

[16] H. Kim, Y. Kim, *On weak solutions of elliptic equations with singular drifts*. SIAM J. Math. Anal. 47 (2015), no. 2, 1271–1290.

[17] H. Kim, H. Kwon, *Dirichlet and Neumann problems for elliptic equations with singular drifts on Lipschitz domains*, Transactions of the AMS 375 (2022), 6537-6574.

[18] H. Kim, T.-P. Tsai, *Existence, uniqueness, and regularity results for elliptic equations with drift terms in critical weak spaces*, SIAM J. Math. Anal. 52 (2020), no. 2, 1146–1191.

[19] G. Koch, N. Nadirashvili, G. Seregin, V. Sverak, *Liouville theorems for the Navier-Stokes equations and applications.* Acta Math. 203 (2009), no. 1, 83–105.

[20] N.V. Krylov, *Elliptic equations with VMO a, $b \in L_d$, and $c \in L_{d/2}$*. Trans. Amer. Math. Soc. 374 (2021), 2805-2822.

[21] N.V. Krylov, *Linear and fully nonlinear elliptic equations with Morrey drift.* J. Math. Sci. (N.Y.) 268 (2022), no. 4, 403–421.

[22] N.V. Krylov, *Elliptic equations in Sobolev spaces with Morrey drift and the zeroth-order coefficients*, Trans. Amer. Math. Soc. 376 (2023), no. 10, 7329–7351.

[23] H. Kwon, *Existence and uniqueness of weak solution in $W^{1,2+\varepsilon}$ for elliptic equations with drifts in weak-L^n spaces*, J. Math. Anal. Appl. 500 (2021), no. 1, 125-165.

[24] H. Kwon, *Elliptic equations in divergence form with drifts in L^2*. Proc. Amer. Math. Soc. 150 (2022), 3415-3429.

[25] O.A. Ladyzhenskaya, N.N. Uraltseva, *Linear and quasilinear equations of elliptic type*, Academic Press, 1968.

[26] V. Liskevich, Q.S. Zhang, *Extra regularity for parabolic equations with drift terms*, Manuscripta Math. 113 (2004), no. 2, 191–209.

[27] V.G. Maz'ja, I.E. Verbitskiy, *Form boundedness of the general second-order differential operator*, Comm. Pure Appl. Math. 59 (2006), 1286–1329.

[28] A.I. Nazarov, N.N. Uraltseva, *The Harnack inequality and related properties of solutions of elliptic and parabolic equations with divergence-free lower-order coefficients*, St. Petersburg Math. J. 23 (2012), no. 1, 93–115.

[29] Y. Sawano, G. Di Fazio, D.I. Hakim, Morrey Spaces: Introduction and Applications to Integral Operators and PDE's, Volumes I,II, Chapman and Hall/CRC, 2020.

[30] G. Seregin, Lecture notes on regularity theory for the Navier-Stokes equations. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.

[31] G. Seregin, *A slightly supercritical condition of regularity of axisymmetric solutions to the Navier-Stokes equations*, J. Math. Fluid Mech. 24 (2022), no. 1, paper 18.

[32] G. Seregin, *A note on local regularity of axisymmetric solutions to the Navier-Stokes equations*, J. Math. Fluid Mech. 24 (2022), no. 1, paper 27.

[33] G. Seregin, T. Shilkin, *Liouville-type theorems for the Navier–Stokes equations*, Russian Math. Surveys 73:4 (2018), 661–724.

[34] G. Seregin, L. Silvestre, V. Sverak, A. Zlatos, *On divergence-free drifts*. J. Differential Equations 252 (2012), no. 1, 505–540.

[35] G. Seregin, V. Sverak, *On type I singularities of the local axi-symmetric solutions of the Navier–Stokes equations*, Comm. Partial Differential Equations 34 (2009), no. 1-3, 171–201.

[36] L. Silvestre, V. Vicol, *Hölder continuity for a drift-diffusion equation with pressure*, Ann. Inst. H. Poincare Anal. Non Lineaire 29 (2012), no. 4, 637–652.

[37] N. Trudinger, *On imbeddings into Orlicz spaces and some applications*, Indiana Univ. Math. J. 17 No. 5 (1968), 473-483.

[38] T.-P. Tsai, *Lectures on Navier–Stokes equations*, Graduate Studies in Mathematics, AMS, 2018.

[39] G. Verchota, *Layer potentials and boundary value problems for Laplace's equation on Lipschitz domains*, ProQuest LLC, Ann Arbor, MI, thesis (Ph.D.)–University of Minnesota (1982).

[40] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal. 59 (3) (1984) 572–611.

[41] Q.S. Zhang, *A strong regularity result for parabolic equations*, Commun. Math. Phys. 244 (2004), no. 2, pp. 245–260.

[42] V.V. Zhikov, *Remarks on the uniqueness of the solution of the Dirichlet problem for a second-order elliptic equation with lower order terms*, Funct. Anal. Appl. 38 (2004), no. 3, 173–183.