BOUNDARY UNIQUE CONTINUATION IN PLANAR DOMAINS BY CONFORMAL MAPPING

STEFANO VITA

Abstract. Let $\Omega \subset \mathbb{R}^2$ be a chord arc domain with small constant. We show that a nontrivial harmonic function which vanishes continuously on a relatively open set of the boundary cannot have the norm of the gradient which vanishes on a subset of positive surface measure (arc length). This result was previously known to be true, and conjectured in higher dimensions by Lin, in Lipschitz domains. Let now $\Omega \subset \mathbb{R}^2$ be a C^1 domain with Dini mean oscillations. We prove that a nontrivial harmonic function which vanishes continuously on a relatively open subset of the boundary $\partial\Omega \cap B_1$ has a finite number of critical points in $\overline{\Omega} \cap B_{1/2}$. The latter improves some recent results by Kenig and Zhao. Our technique involves a conformal mapping which moves the boundary where the harmonic function vanishes into an interior nodal line of a new harmonic function, after a further reflection. Then, size estimates of the critical set - up to the boundary - of the original harmonic function can be understood in terms of estimates of the *interior* critical set of the new harmonic function and of the critical set - up to the boundary - of the conformal mapping.

1. Introduction

Let Ω be a domain in the plane $\mathbb{R}^2$. In this paper we are concerned with the local behaviour of planar harmonic functions in Ω near a given open piece of the boundary where they vanish continuously; that is, we consider weak solutions to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \cap B_1, \\ u = 0 & \text{on } \partial\Omega \cap B_1. \end{cases} \tag{1.1}$$

Here we assume that $0 \in \partial\Omega$, $z = (x, y) \in \mathbb{R}^2$, $B_1 = \{|z| < 1\}$ and $\Omega \cap B_1$ is simply connected. Aim of our work is to provide a conformal mapping Θ which locally moves the above boundary problem into

$$\begin{cases} \Delta U = 0 & \text{in } \{Y > 0\} \cap B_1, \\ U = 0 & \text{on } \{Y = 0\} \cap B_1. \end{cases}$$

Here $u = U \circ \Theta$ and $Z = (X, Y) = \Theta(x, y)$. After an odd reflection of U across the line $\{Y = 0\}$, one ends up with a harmonic function in a ball. At the end of this procedure, the map Θ turned the original boundary $\partial\Omega$ into an interior nodal line of U

$$\Theta(\partial\Omega \cap B_1) \subset \{Y = 0\}.$$

Date: May 8, 2024.

2020 *Mathematics Subject Classification.* 30C20, 31A05, 35J25, 42B37.

Key words and phrases. Unique continuation, conformal mapping, harmonic measure, chord arc domain, Dini mean oscillation.

Then, the boundary behaviour of the solution u can be understood locally in terms of the *interior* behaviour of the new harmonic function U and the boundary behaviour of the conformal mapping Θ. This transformation provides solutions to classical problems in boundary unique continuation.

1.1. **Boundary unique continuation in chord arc domains with small constant.** *Boundary unique continuation* typically concerns the following question: given a nontrivial solution u of (1.1), and depending on the regularity of the domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, how big can the singular set $S(u) = \{u = |\nabla u| = 0\}$ be - whenever it make sense - restricted to the boundary? Similar questions can be raised for the full critical set $C(u) = \{|\nabla u| = 0\}$ inside the domain and up to the boundary. In this context, a famous problem was proposed by Lin [28].

Conjecture 1.1. Let us consider a harmonic function in a Lipschitz domain Ω in $\mathbb{R}^n$ vanishing continuously on a relatively open subset V of the boundary $\partial\Omega$. Suppose that the normal derivative $\partial_\nu u$ vanishes in a subset of V with positive surface measure. Then $u \equiv 0$.

The validity of the result above was first established in $C^{1,1}$ domains [28], convex Lipschitz domains [2], $C^{1,\alpha}$ and $C^{1,\mathrm{Dini}}$ domains [1, 24], C^1 domains and Lipschitz domains with small Lipschitz constant [36]. We would like to mention also [29, 14] for estimates of singular and critical sets in case of Lipschitz convex domains and Lipschitz domains with small Lipschitz constant, respectively. The conjecture in its generality is still open.

The two dimensional case is commonly known to be true. The argument uses the subharmonicity of $\log |\nabla u|$, see [1, 37]. Among the other motivations, the paper aims to give an alternative proof of Conjecture 1.1 in the two dimensional case. The idea is the following: after composing with the conformal mapping Θ, one has

$$|\nabla u|^2 = |\mathrm{det} D\Theta| \cdot |\nabla U|^2 \circ \Theta,$$

where $D\Theta$ stands for the Jacobian of Θ. Then, the critical set of u is locally controlled in size by the critical set of U and by the critical set of the conformal mapping

$$C(\Theta) = \{|\mathrm{det} D\Theta| = 0\}.$$

Roughly speaking, the first is *small*, being an *interior* critical set - up to perform an odd reflection - of a harmonic function (it consists of a finite number of points), and the latter is concentrated along the boundary $\partial\Omega$ and has zero surface measure. The critical set of the conformal mapping we provide is in fact the critical set of an auxiliary positive harmonic function vanishing on $\partial\Omega$; that is,

$$|\mathrm{det} D\Theta| = |\nabla v|^2, \qquad \begin{cases} \Delta v = 0 & \text{in } \Omega \cap B_1, \\ v > 0 & \text{in } \Omega \cap B_1, \\ v = 0 & \text{on } \partial\Omega \cap B_1. \end{cases}$$

The *flatness* of a Lipschitz domain allows to *keep* the interior critical points of v far from the boundary, uniformly with respect to the Lipschitz constant (see for instance [7, Theorem 11.10] or [26, Theorem 2.9]). So, up to further localize and standard covering (with a finite number of balls - centered at the boundary - which depends on the Lipschitz constant), one can suppose that v has no interior critical points, which for having local invertibility of the conformal mapping. Moreover, along the boundary, $|\nabla v|$ is σ-a.e. comparable to the density of the harmonic measure with respect to the surface measure $d\sigma$. Besides, the harmonic measure is mutually absolutely continuous with respect to $d\sigma$. The latter two facts imply that the critical set of v along the boundary has zero surface measure. In the two dimensional case, by surface measure we mean the arc length, which corresponds to the one dimensional Hausdorff measure restricted to $\partial\Omega$, i.e. $\sigma = \mathcal{H}^1 \llcorner \partial\Omega$.

An interesting consequence of our conformal approach is the extension of the two dimensional result in Conjecture 1.1 to a more general class of planar domains, the chord arc domains with small constant (δ-chord arc domains for $\delta > 0$ small enough [19]). In two dimensions, a bounded chord arc domain is a Jordan domain for which the boundary is locally rectifiable and there exists a constant $\lambda > 0$ such that

$$\sigma(\gamma(z_1, z_2)) \leq \lambda |z_1 - z_2|, \qquad \forall z_1, z_2 \in \partial\Omega,$$

where $\gamma(z_1, z_2)$ is the shortest arc in the boundary connecting z_1 and z_2, and $\sigma(\gamma(z_1, z_2))$ is its length. In general, a chord arc domain in $\mathbb{R}^n$ is a non-tangentially accessible (NTA) domain [15] whose boundary is Ahlfors-David regular, i.e. the surface measure on boundary balls of radius r grows like r^{n-1}. Then, the notion of δ-chord arc domain for $\delta \leq \delta^*$ small enough (chord arc domain with small constant) requires δ-Reifenberg flatness for δ small enough plus a control in BMO sense of the oscillation of the unit normal vector to the boundary.

As we will see, the *regularity* and the *quantitative flatness* of a chord arc domain with small constant provide the conditions together for constructing the conformal mapping Θ, just as in the case of a Lipschitz domain. The non-tangential accessibility of the domain, together with the Ahlfors-David regularity, allows to extend the gradient up to the boundary and to have at σ-a.e. boundary point a well defined tangent plane as well as a normal vector. In addition, the mutual absolute continuity between the harmonic measure and the surface measure remains valid, along with the σ-a.e. comparability of $|\nabla v|$ to the density of the harmonic measure with respect to the surface measure [25]. These facts ensure that the critical set of v along the boundary still has zero surface measure. The δ-Reifenberg flatness, for $\delta \leq \delta^*$ small enough (a δ-approximation by Lipschitz graph domains would suffice [26]), ensures that there are no interior critical points of v near the boundary, uniformly with respect to δ^*. This, once again, provides uniform localization and covering. Our first result can be stated as follows

Theorem 1.2. *Let us consider a harmonic function in a chord arc domain with small constant Ω in $\mathbb{R}^2$ vanishing continuously on a relatively open subset V of the boundary $\partial\Omega$. Suppose that the norm of the gradient $|\nabla u|$ vanishes in a subset of V with positive arc length. Then $u \equiv 0$. Actually, given a chord arc domain with small constant $\Omega \subset \mathbb{R}^2$ and a nontrivial solution u to* (1.1)*, then*

$$\mathcal{H}^1(C(u) \cap \overline{\Omega} \cap B_{1/2}) = 0.$$

In light of the above result, one might wonder if Conjecture 1.1 could be formulated in chord arc domains with small constant in any dimension.

1.2. $(n-2)$ **dimensional size control of singular and critical sets in $C^{1,\mathrm{DMO}}$ domains.** When the boundary is more regular, one may ask for stronger information on the size of singular and critical sets. That is, consider the following

Problem. Let u be a nontrivial solution to (1.1) in a domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$. Identify the conditions on the boundary under which one can control

$$\mathcal{H}^{n-2}(S(u) \cap \overline{\Omega} \cap B_{1/2}) \leq C, \tag{1.2}$$

or either

$$\mathcal{H}^{n-2}(C(u)\cap\overline{\Omega}\cap B_{1/2})\le C. \tag{1.3}$$

This problem was recently addressed by Kenig and Zhao in a series of papers [21, 22, 23], using also techniques developed by Naber and Valtorta [30]. The property (1.2) holds true in $C^{1,\mathrm{Dini}}$ domains [21], and counterexamples are provided below this threshold [22]. Actually, in [21] the authors obtain upper estimates for the $(n-2)$ dimensional Minkowski content, which coincides with the $(n-2)$ dimensional Hausdorff measure in the present planar case. The property (1.3), which is stronger than (1.2), holds in $C^{1,\alpha}$ domains [23]. The size bounds in [21, 23] are uniform prescribing a bound on the Almgren frequency function of the solution at a macroscopic scale as well as a control over the $C^{1,\mathrm{Dini}}$ (respectively $C^{1,\alpha}$) character of the boundary parametrization.

Our conformal approach allows us to prove the stronger property (1.3), again in two dimensions, any time the critical set of the conformal mapping has the suitable bound in measure

$$\mathcal{H}^{0}(C(\Theta)\cap\overline{\Omega}\cap B_{1/2})\le C.$$

Here $\mathcal{H}^0$ stands for the counting measure. Our focus here is not on establishing conditions on the boundary that yield the precise bound above, although many examples could be constructed. Instead, we concentrate our analysis on a set of hypothesis which implies an empty critical set, i.e.

$$C(\Theta)\cap\overline{\Omega}\cap B_{1/2}=\emptyset.$$

With respect to our conformal mapping, this happens any time the following two conditions hold:

- (P1) any solution u to (1.1) belongs to $C^1_{\mathrm{loc}}(\overline{\Omega}\cap B_1)$ (and the same holds true for solutions having homogeneous Neumann boundary condition at $\partial\Omega$);
- (P2) the Hopf lemma holds true: any solution v to (1.1) which is positive in $\Omega\cap B_1$ has $\partial_\nu v<0$ on $\partial\Omega\cap B_1$.

As we proved in [12], the properties above hold true when the domain is C^1 with Dini mean oscillations ($C^{1,\mathrm{DMO}}$), and this is true in any dimension. This class of domains was recently introduced in [12] and strictly contains the $C^{1,\mathrm{Dini}}$ class, see Section 3.1 for the precise definition. For reader's convenience, we would like to provide here a two dimensional example of a $C^{1,\mathrm{DMO}}$ local parametrization which fails to be $C^{1,\mathrm{Dini}}$: the domain is given, locally around $0\in\partial\Omega$, by

$$\Omega\cap B_{1/2}=\{y>\varphi(x)\}\cap B_{1/2},\qquad \partial\Omega\cap B_{1/2}=\{y=\varphi(x)\}\cap B_{1/2},$$

with $z=(x,y)\in\mathbb{R}^2$ and

$$\varphi(x)=\frac{x}{|\log|x||^{1/2}},\qquad |x|<1/2.$$

The following is our second result

Theorem 1.3. *Let $n=2$, Ω be a $C^{1,\mathrm{DMO}}$ domain and u be a nontrivial solution to* (1.1)*. Then,*

$$\mathcal{H}^{n-2}(C(u)\cap\overline{\Omega}\cap B_{1/2})<\infty.$$

We believe that the above result is still valid in any dimension $n\ge 2$. Let us stress the fact that Theorem 1.3 is not in contradiction with [22], since the counterexamples proposed there *do not see* our intermediate condition. We refer to Remark 3.1 for a detailed explanation of this fact. Finally, although not the focus of the present paper, in Remark 3.2 we will discuss the possibility of obtaining uniform size bounds prescribing macroscopic controls over the frequency function of the solution and on the $C^{1,\mathrm{DMO}}$ character of the boundary parametrization.

2. The construction of the conformal mapping in chord arc domains

The conformal mapping Θ we are going to construct is the same for proving both Theorem 1.2 and Theorem 1.3. However, the construction in the first case is more delicate, since in chord arc domains one has to work with generalized gradients, defined as non-tangential limits, and the invertibility of the map is more subtle. The full section should be intended as the proof of Theorem 1.2.

2.1. **Chord arc domains with small constant.** In this brief subsection we would like to introduce the chord arc domains with small constant. We refrain from providing precise definitions here, but we do indicate some references for readers who wish to go deeper into these notions.

A NTA (non-tangentially accessible) domain Ω is one which enjoys an interior Harnack Chain condition, as well as interior and exterior Corkscrew conditions. This notion was introduced in [15]. A chord arc domain is a NTA domain whose boundary is Ahlfors-David regular, i.e. the surface measure on boundary balls of radius r grows like r^{n-1}. We refer to [5] for precise definitions and nice characterizations of chord arc domains. We also would like to refer to [25] for the proof of the mutual absolute continuity between arc length and harmonic measure in chord arc domains, see also [9, 10, 34] for the result in any dimension and [8, 16] for further references. For the definition of a δ-chord arc domain, or chord arc domain with small constant, we refer to [19, Definition 1.10]: it is a chord arc domain which is δ-Reifenberg flat for $\delta \le \delta^*$ small enough [32] and there is sufficient control on the oscillation of the unit normal vector (in the BMO sense), see also [31, 17, 18, 20, 6]. Finally, we refer to [26, 27] for useful considerations on the gradient of positive harmonic functions vanishing continuously on a relatively open subset of a chord arc boundary with small constant.

2.2. **Non-tangential limits.** Given a chord arc bounded domain Ω in $\mathbb{R}^n$ and the surface measure σ on $\partial\Omega$, one has for σ-a.e. z the existence of the tangent plane to $\partial\Omega$ in z and of the outer unit normal vector $\nu(z)$. Then, given any parameter $\alpha > 0$, let us consider the non-tangential approach region to a point on the boundary $z \in \partial\Omega$

$$\Gamma_\alpha(z) = \{\xi \in \Omega \,:\, (1+\alpha)d(\xi, \partial\Omega) > |z - \xi|\},$$

and, given a measurable function w defined in Ω, the non-tangential maximal function at the boundary point $z \in \partial\Omega$

$$\mathcal{N}_\alpha w(z) = \sup_{\xi \in \Gamma_\alpha(z)} |w|(\xi).$$

The definitions above do not depend by the choice of $\alpha > 0$, so we can fix $\alpha = 1$ and simply write $\Gamma(z) = \Gamma_1(z)$ and $\mathcal{N}w(z) = \mathcal{N}_1 w(z)$. Then, we say that w converges non-tangentially to f at $z \in \partial\Omega$ if

$$\lim_{\xi \in \Gamma(z),\ \xi \to z} w(\xi) = f(z).$$

2.3. **Positive harmonic function vanishing on $\partial\Omega$ with no interior critical points.** The first step is the construction of a solution to

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \cap B_1 \\ v > 0 & \text{in } \Omega \cap B_1 \\ v = 0 & \text{on } \partial\Omega \cap B_1. \end{cases} \tag{2.1}$$

The existence of such a function is given by the solvability of the Dirichlet problem for the Laplacian on the chord arc domain $\Omega\cap B_2$, which is regular for the solvability of the Dirichlet problem. We can assume that $\Omega\cap B_2$ is still connected, so it is a bounded chord arc domain. One can prescribe as Dirichlet data on $\partial(\Omega\cap B_1)$ the Green function G^{z_0} of $\Omega\cap B_2$ with a given pole $z_0\in\Omega\cap B_2$. The function G^{z_0} vanishes on $\partial(\Omega\cap B_2)$ and is positive inside. Then, by the strong maximum principle v is positive inside $\Omega\cap B_1$. Moreover, this function can be built in such a way that $|\nabla v|>0$ in $\Omega\cap B_R$ for some $0<R\le 1$, see [26, Lemma 3.8]. Here we possibly have to restrict the ball. However, there exists δ^* small enough such that if Ω is δ-chord arc with $\delta\le\delta^*$ then for some possibly small positive $R=R(\delta^*)$ and some $C=C(\delta^*)>1$, and any $\xi\in\partial\Omega\cap B_{2/3}$

$$\frac{1}{C}\frac{v(z)}{d(z,\partial\Omega)}\le|\nabla v(z)|\le C\frac{v(z)}{d(z,\partial\Omega)}\qquad\forall z\in\Omega\cap B_R(\xi). \tag{2.2}$$

Then, without loss of generality we can take $\xi=0$ and suppose that $R=1$. The full result follows by a standard covering argument of $\Omega\cap B_{2/3}$ with a finite number of balls depending on δ^*.

2.4. **The size of the singular set of positive harmonic functions at the chord arc boundary.** Now, being v a nonnegative solution to (2.1) in a bounded chord arc domain, by [25] and [27, Theorem 1] we have that

(i) the harmonic measure ω is absolutely continuous with respect to σ on $\partial\Omega\cap B_1$ and $d\omega\in A_\infty(\partial\Omega\cap B_1,d\sigma)$;
(ii) the limit
$$\nabla v(z):=\lim_{\xi\in\Gamma(z),\ \xi\to z}\nabla v(\xi)$$
exists for σ-a.e. $z\in\partial\Omega\cap B_1$. Moreover, $\nabla v(z)=-|\nabla v(z)|\nu(z)$ where ν stands for the outer unit normal vector;
(iii) there exists $p>1$ such that $\mathcal{N}|\nabla v|\in L^p(\partial\Omega\cap B_1,d\sigma)$;
(iv) $d\omega=|\nabla v|d\sigma$ for σ-a.e. $z\in\partial\Omega\cap B_1$.

Summing up the information above, we have that $|\nabla v|$ can not vanish on a set of positive surface measure, otherwise $v\equiv 0$. Then, combining this information with the fact that v has no interior critical points, we have

$$\mathcal{H}^1(C(v)\cap\overline{\Omega}\cap B_1)=0, \tag{2.3}$$

with the critical points (which are singular points) all concentrated along the boundary.

2.5. **The harmonic conjugate.** Let us construct the harmonic conjugate of v by solving

$$\nabla\overline{v}=J\nabla v\quad\text{in }\Omega\cap B_1,\qquad J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \tag{2.4}$$

Here J is the clockwise rotation matrix of angle $\pi/2$, such that $J^{-1}=J^T=-J$. The condition above corresponds to the Cauchy-Riemann equations. The form is closed in a simply connected domain, so (2.4) admits a solution. In particular $\overline{v}$ has the following properties

$$\nabla v\cdot\nabla\overline{v}=0,\qquad|\nabla v|=|\nabla\overline{v}|,$$

and is harmonic in $\Omega\cap B_1$. Moreover, we can suppose that $\overline{v}(0)=0$, considering instead $\tilde{v}(z)=\overline{v}(z)-\overline{v}(0)$.

2.6. **A hodograph conformal transformation and its invertibility.** Following the idea we introduced in [35], let us define the hodograph conformal mapping involving $v, \overline{v}$

$$\Theta(x,y) = (\overline{v}(x,y), v(x,y)) = (X,Y),$$

with $\Theta(0) = 0$. We are going to prove that Θ is a homeomorphism. This transformation has good properties: it is conformal, i.e. preserves harmonicity, and flattens the boundary where v vanishes. It is clear that, up to dilations, the map send the open set $\Omega \cap B_1$ into $B_1^+ = \{Y > 0\} \cap B_1$ and $\partial\Omega \cap B_1$ into $B_1' = \{Y = 0\} \cap B_1$. Actually, the map is a deformation inside the domain, so it does not preserve balls. However, this interior deformation does not affect our analysis, so for simplicity we can suppose without loss of generality that the image domain is B_1^+. The Jacobian associated with Θ is given by

$$D\Theta = \begin{pmatrix} \partial_x \overline{v} & \partial_y \overline{v} \\ \partial_x v & \partial_y v \end{pmatrix}, \qquad \text{with} \quad |\det D\Theta| = |\nabla v|^2 = |\nabla \overline{v}|^2.$$

Hence, the fact that v has no interior critical points implies the invertibility of the map between the open sets $\Omega \cap B_1$ and B_1^+. In order to extend the invertibility up to the boundary, we need to prove that the restriction of the map along the boundary is injective there. In fact, along $\partial\Omega$, the map Θ may have many critical points, but they are a set of vanishing surface measure, as we have already remarked. The injectivity follows from the following consideration. Let us consider a path γ on $\partial\Omega$ connecting two points $z_1, z_2 \in \partial\Omega$; that is, $\gamma : [t_1, t_2] \to \mathbb{R}^2$ with supp$\gamma \subset \partial\Omega$ and $\gamma(t_1) = z_1$, $\gamma(t_2) = z_2$. The curve is rectifiable by definition of chord arc domain. Then the parametrization can be chosen as a Lipschitz function. We choose the orientation in such a way that a clockwise rotation of angle $\pi/2$ of the tangent vector to the curve $\gamma'(t)$ goes in the same direction of the outer unit normal vector $\nu(\gamma(t))$. Then

$$\overline{v}(z_2) - \overline{v}(z_1) = \int_{t_1}^{t_2} J\nabla v(\gamma(t)) \cdot \gamma'(t)\, dt > 0.$$

Here, at $\partial\Omega$, we are using the fact that γ' is defined σ-a.e. and is a bounded function (the parametrization is Lipschitz continuous). Moreover, the non-tangential gradient of v equals $-|\nabla v|\nu$ as a $L^p(d\sigma)$ function for some $p > 1$, summing (ii)-(iii), and it is nonzero σ-almost everywhere. Notice that $-J\nu(\gamma(t))$ is parallel to $\gamma'(t)$ and goes in the same direction. This means that, restricted to $\partial\Omega$, the first component $\Theta_1(\gamma(t)) = \overline{v}(\gamma(t))$ of the map $\Theta \circ \gamma$ is monotone increasing in t, and hence it is injective. This allows to extend the invertibility of the conformal mapping up to the boundary.

2.7. **Size control of the critical set.** Let us consider $U = u \circ \Theta^{-1}$, which solves

$$\begin{cases} \Delta U = 0 & \text{in } B_1^+ \\ U = 0 & \text{on } B_1'. \end{cases}$$

Hence, considering the odd reflection $U(X,Y) = -U(X,-Y)$ across $\{Y = 0\}$, one ends up with a harmonic function on the ball B_1 for which $\Theta(\partial\Omega)$ is an interior nodal line $\{Y = 0\}$. Then, since

$$|\nabla u|^2 = |\mathrm{det} D\Theta| \cdot |\nabla U|^2 \circ \Theta,$$

we have

$$\mathcal{H}^1(C(u) \cap \overline{\Omega} \cap B_{1/2}) \le \mathcal{H}^1(C(\Theta) \cap \overline{\Omega} \cap B_{1/2}) + \mathcal{H}^1(C(U) \cap \overline{B_{1/2}^+}) = 0.$$

The latter is true due to (2.3) and classic size estimates of interior critical sets of harmonic functions.

3. The construction of the conformal mapping in $C^{1,\mathrm{DMO}}$ domains

The conformal mapping Θ we consider for the proof of Theorem 1.3 is the same we built in the previous section, but enjoys better properties. The full section should be intended as the proof of Theorem 1.3.

3.1. $C^{1,\mathrm{DMO}}$ **domains.** First, let us recall the definition of $C^{1,\mathrm{Dini}}$ domains in $\mathbb{R}^n$ with $n \geq 2$. In this case, the local boundary parametrization φ is a C^1 function and the modulus of continuity of its gradient is a Dini function. This means that locally

$$\Omega \cap B_1 = \{x_n > \varphi(x')\} \cap B_1, \qquad \partial\Omega \cap B_1 = \{x_n = \varphi(x')\} \cap B_1, \tag{3.1}$$

with $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $\varphi \in C^1(\overline{B_1'})$ with $B_1' = B_1 \cap \{x_n = 0\}$, $\varphi(0) = 0$, $\nabla_{x'}\varphi(0) = 0$. Then, there exist a positive constant and a modulus of continuity η such that for all $i = 1, ..., n-1$ and $x', y' \in B_1'$

$$|\partial_i\varphi(x') - \partial_i\varphi(y')| \leq C\eta(|x' - y'|)$$

with

$$\int_0^1 \frac{\eta(r)}{r} dr < \infty. \tag{3.2}$$

Let us now proceed with the definition of $C^{1,\mathrm{DMO}}$ domains. Here, the boundary of the domain is locally parametrized by a C^1 function φ whose partial derivatives $\partial_i\varphi$ are of Dini mean oscillations. In other words, the parametrization is as in (3.1) with $\varphi \in C^1(\overline{B_1'})$, $\varphi(0) = 0$, $\nabla_{x'}\varphi(0) = 0$ and

$$\eta_i(r) = \sup_{x_0 \in B_1'} \fint_{B_r(x_0) \cap B_1'} |\partial_i\varphi(x') - \langle \partial_i\varphi \rangle_{x_0,r}| dx', \qquad \text{with } \langle \partial_i\varphi \rangle_{x_0,r} = \fint_{B_r(x_0) \cap B_1'} \partial_i\varphi(x')\, dx',$$

is a Dini function for any $i = 1, ..., n-1$, i.e. satisfies (3.2).

3.2. $C^{1,\mathrm{DMO}}$ **domains enjoy properties (P1) and (P2).** As we remarked in [12], after a standard local flattening of the $C^{1,\mathrm{DMO}}$ boundary, the validity of (P1) follows by C^1 boundary regularity up to a flat boundary where homogeneous Dirichlet or Neumann boundary conditions are prescribed, for solutions of PDEs with DMO coefficients. We refer to [11, Proposition 2.7] and [13, Theorem 1.2]. Moreover, always after a standard flattening, property (P2) follows by the Hopf Lemma proved in [33] on flat boundaries and DMO coefficients.

Remark 3.1. We would like to remark here that our result in $C^{1,\mathrm{DMO}}$ domains is not in contradiction with the counterexample in [22]. In fact, the two dimensional example there is given by a local paramentrization with fails to be $C^{1,\mathrm{Dini}}$ but is convex. As we pointed out in [12, Proposition 3.1], a local $C^{1,\mathrm{DMO}}$ parametrization φ which is convex satisfies the $C^{1,\mathrm{Dini}}$-paraboloid condition [4]; that is,

$$\omega(r) = \sup_{|x| \leq r} \frac{\varphi(x)}{|x|}$$

is a Dini function. The same consideration above explains also why the validity of the Hopf lemma in $C^{1,\mathrm{DMO}}$ domains is not in contradiction with the counterexample in [3].

3.3. **Positive harmonic function vanishing on $\partial\Omega$ with no critical points.** The first step is the construction of a solution to (2.1). The existence of such a function is done as in Section 2.3. However, properties (P1)-(P2) together imply that $v \in C^1$ up to the boundary and that $\partial_\nu v < 0$ on $\partial\Omega \cap B_1$, which says, up to restrict a bit the ball $R \le 1$, that v has no critical points in $\overline{\Omega} \cap B_R$. This further localization can be done uniformly, so again we can suppose $R = 1$.

3.4. **The harmonic conjugate.** Let us construct the harmonic conjugate $\overline{v}$ of v as in Section 2.5. Let us remark that $\overline{v}$ solves

$$\begin{cases} \Delta\overline{v} = 0 & \text{in } \Omega \cap B_1 \\ \partial_\nu \overline{v} = 0 & \text{on } \partial\Omega \cap B_1. \end{cases}$$

Then, by (P1) again, one has $\overline{v} \in C^1$ up to $\partial\Omega$.

3.5. **A hodograph conformal transformation and its invertibility.** Let us define the hodograph conformal mapping involving $v, \overline{v}$ as in Section 2.6

$$\Theta(x,y) = (\overline{v}(x,y), v(x,y)) = (X,Y),$$

which is of class C^1 with $\Theta(0) = 0$. Then, since v has no critical points in $\overline{\Omega} \cap B_1$, the invertibility of the map is even more direct this time, since $|\mathrm{det} D\Theta| > 0$ in $\overline{\Omega} \cap B_1$. Hence, this time Θ is a C^1 diffeomorphism.

3.6. **Size control of the critical set.** Let us consider $U = u \circ \Theta^{-1}$, and consider again the odd reflection across $\{Y = 0\}$. One ends up again with a harmonic function on the ball B_1 for which $\Theta(\partial\Omega)$ is an interior nodal line $\{Y = 0\}$. Then, since

$$|\nabla u|^2 = |\mathrm{det} D\Theta| \cdot |\nabla U|^2 \circ \Theta,$$

we have

$$\mathcal{H}^0(C(u) \cap \overline{\Omega} \cap B_{1/2}) \le \mathcal{H}^0(C(\Theta) \cap \overline{\Omega} \cap B_{1/2}) + \mathcal{H}^0(C(U) \cap \overline{B^+_{1/2}}) = C. \tag{3.3}$$

The latter is true due to classic size estimates of interior critical sets for harmonic functions.

Remark 3.2. Similarly to [21], one could give a notion of Almgren frequency function for the solutions u to (1.1) in $C^{1,\mathrm{DMO}}$ domains

$$N(0,u,r) = \frac{r\int_{\Omega\cap B_r} |\nabla u|^2\,dx}{\int_{\Omega\cap\partial B_r} u^2\,d\sigma},$$

and quantify the bound in (3.3) in terms of a macroscopic bound on the frequency function at 0, and a bound on the $C^{1,\mathrm{DMO}}$-character of the boundary parametrization. In fact, these two bounds combined would give a macroscopic bound on the Almgren frequency for the solution U after composition with the conformal mapping. Let us remark that a control over the $C^{1,\mathrm{DMO}}$-character of the domain gives the following uniform bounds for the associated conformal mapping

$$\frac{1}{C} \le |\mathrm{det} D\Theta| \le C.$$

The bound from above is trivial, while the one from below comes from a quantified version of the Hopf lemma that can be obtained from [33].

Acknowledgement

The author would like to thank Xavier Tolsa and Matteo Levi for fruitful conversations on harmonic measure and Green functions in NTA domains. The author is research fellow of Istituto Nazionale di Alta Matematica INDAM group GNAMPA and supported by the MUR funding for Young Researchers - Seal of Excellence (ID: SOE_0000194. Acronym: ADE. Project Title: Anomalous diffusion equations: regularity and geometric properties of solutions and free boundaries).

References

[1] V. Adolfsson and L. Escauriaza. *$C^{1,\alpha}$ domains and unique continuation at the boundary*. Comm. Pure Appl. Math. 50-10 (1997), 935-969.
[2] V. Adolfsson, L. Escauriaza, and C. Kenig. *Convex domains and unique continuation at the boundary*. Rev. Mat. Iberoam. 11 (1995), 519-525.
[3] D. Apushkinskaya and A. Nazarov. *A counterexample to the Hopf-Oleinik lemma (elliptic case)*. Anal. PDE 9-2 (2016), 439-458.
[4] D. Apushkinskaya and A. Nazarov. *On the boundary point principle for divergence-type equations*. Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 30-4 (2019), 677-699.
[5] J. Azzam, S. Hofmann, J.M. Martell, K. Nyström, and T. Toro. *A new characterization of chord-arc domains*. J. Eur. Math. Soc. (JEMS) 19-4 (2017), 967-981.
[6] S. Bortz and M. Engelstein. *Reifenberg flatness and oscillation of the unit normal vector*. (2017) arXiv:1708.05331.
[7] L. Caffarelli and S. Salsa. *A Geometric Approach to Free Boundary Problems*. Grad. Stud. Math., vol. 68. Am. Math. Soc., Providence, RI (2005).
[8] L. Capogna, C. Kenig, and L. Lanzani. *Harmonic measure: geometric and analytic points of view*. University Lecture Series 35, American Mathematical Society (2005).
[9] B. Dahlberg. *On the absolute continuity of elliptic measure*. Amer. J. Math. 108 (1986), 1119-1138.
[10] G. David and D. Jerison. *Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals*. Indiana Univ. Math. J. 39 (1990), 831-845.
[11] H. Dong, L. Escauriaza, and S. Kim. *On C^1, C^2, and weak type-$(1,1)$ estimates for linear elliptic operators: part II*. Math. Ann. 370 (2018), 447-489.
[12] H. Dong, S. Jeon, and S. Vita. *Schauder type estimates for degenerate or singular elliptic equations with Dini mean oscillation coefficients with application*. (2023), arXiv:2311.06846.
[13] H. Dong, J. Lee, and S. Kim. *On conormal and oblique derivative problem for elliptic equations with Dini mean oscillation coefficients*. Indiana Univ. Math. J. 69-6 (2020), 1815-1853.
[14] J. M. Gallegos. *Size of the zero set of solutions of elliptic PDEs near the boundary of Lipschitz domains with small Lipschitz constant*. Calc. Var. Partial Differential Equations 62-4 (2023), 1-52.
[15] D. Jerison and C. Kenig. *Boundary behavior of harmonic functions in non-tangentially accessible domains*. Adv. Math. 46-1 (1982), 80-147.
[16] D. Jerison and C. Kenig. *Hardy spaces, A_∞, and singular integrals on chord-arc domains*. Math. Scand. 50-2 (1982), 221-247.
[17] C. Kenig and T. Toro. *Harmonic measure on locally flat domains*. Duke Math. J. 87 (1997), 501-551.
[18] C. Kenig and T. Toro. *Free boundary regularity for harmonic measure and Poisson kernels*. Ann. of Math. (2) 150 (1999), 369-454.
[19] C. Kenig and T. Toro. *Poisson kernel characterization of Reifenberg flat chord arc domains*. Ann. Sci. Éc. Norm. Supér. (4) 36-3 (2003), 323-401.
[20] C. Kenig and T. Toro. *Free boundary regularity below the continuous threshold: 2-phase problems*. J. Reine Angew. Math. 596 (2006), 1-44.
[21] C. Kenig and Z. Zhao. *Boundary unique continuation on C^1-Dini domains and the size of the singular set*. Arch. Ration. Mech. Anal. 245 (2022), 1-88.
[22] C. Kenig and Z. Zhao. *Examples of non-Dini domains with large singular sets*. Adv. Nonlinear Stud. (Special Issue: In honor of David Jerison) 23-1 (2023), 1-33.

[23] C. Kenig and Z. Zhao. *A note on the critical set of harmonic functions near the boundary.* (2024), arXiv:2402.08881.

[24] I. Kukavika and K. Nyström. *Unique continuation on the boundary for Dini domains.* Proc. Amer. Math. Soc. 126 (1998), 441-446.

[25] M. Lavrentiev. *Boundary problems in the theory of univalent functions.* Mat. Sb. 43 (1936), 815-846 (in Russian), AMS Transl. Ser. 32 (1963), 1-35 (English translation).

[26] J. Lewis and K. Nyström. *Boundary behaviour of p-harmonic functions in domains beyond Lipschitz domains.* Adv. Calc. Var. 1 (2008), 133-177.

[27] J. Lewis and K. Nyström. *Regularity and free boundary regularity for the p-Laplace operator in Reifenberg flat and Ahlfors regular domains.* J. Amer. Math. Soc. 25-3 (2012), 827-862.

[28] F. Lin. *Nodal sets of solutions of elliptic and parabolic equations.* Comm. Pure Appl. Math. 45 (1991), 287-308.

[29] S. McCurdy. *Unique continuation on convex domains.* Rev. Mat. Iberoam. 39-1 (2023), 1-28.

[30] A. Naber and D. Valtorta. *Rectifiable-Reifenberg and the regularity of stationary and minimizing harmonic maps.* Ann. of Math. (2) 185 (2017), 131-227.

[31] C. Pommerenke. *On univalent functions, Bloch functions and VMOA.* Math. Ann. 236 (1978), 199-208.

[32] E. Reifenberg. *Solution of the Plateau Problem for m-dimensional surfaces of varying topological type.* Acta Math. 104 (1960), 1-92.

[33] F. Rendón, B. Sirakov, and M. Soares. *Boundary weak Harnack estimates and regularity for elliptic PDE in divergence form.* Nonlinear Anal. 235 (2023), 1-13.

[34] S. Semmes. *Analysis vs. geometry on a class of rectifiable hypersurfaces in $\mathbb{R}^n$.* Indiana Univ. Math. J. 39 (1990), 1005-1035.

[35] S. Terracini, G. Tortone, and S. Vita. *A priori regularity estimates for equations degenerating on nodal sets.* (2024), arXiv:2404.06980.

[36] X. Tolsa. *Unique continuation at the boundary for harmonic functions in C^1 domains and Lipschitz domains with small constant.* Comm. Pure Appl. Math. 76-2 (2023), 305-336.

[37] T. Wolff. *Counterexamples with harmonic gradients in $\mathbb{R}^3$.* Essays on Fourier analysis in honor of Elias M. Stein, Princeton Math. Ser. 42 (1991), 321-384.

Stefano Vita
Dipartimento di Matematica "Giuseppe Peano"
Università degli Studi di Torino
Via Carlo Alberto 10, 10124, Torino, Italy
Email address: stefano.vita@unito.it