ALMOST SPECIAL REPRESENTATIONS OF WEYL GROUPS

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Introduction

0.1. Let W be an irreducible Weyl group and let $\hat W$ be the set of (isomorphism classes of) irreducible representations of W over $\mathbf{C}$. Let $c\subset\hat W$ be a family of W (see [L79],[L84]). Let Γ_c be the finite group associated in [L84] to c and let $c\to M(\Gamma_c)$ be the imbedding defined in [L84]. (For any finite group Γ, we denote by $M(\Gamma)$ the set of Γ-conjugacy of pairs (x,σ) where $x\in\Gamma$ and σ is an irreducible representation over $\mathbf{C}$ of the centralizer of x in Γ.)

In [L82] a set Con_c of (not necessarily irreducible) representations of W with all irreducible components in c was defined and it was conjectured that these are exactly the representations of W carried by the various left cells [KL] contained in the two-sided cell associated to c. (This conjecture was proved in [L86].)

We can view Con_c as a subset of the $\mathbf{C}$-vector space $\mathbf{C}[c]$ (with basis c) and hence with a subset of the $\mathbf{C}$-vector space $\mathbf{C}[M(\Gamma_c)]$ (with basis $M(\Gamma_c)$) via the imbedding $\mathbf{C}[c]\subset\mathbf{C}[M(\Gamma_c)]$ induced by $c\subset M(\Gamma_c)$. Note that $\mathrm{Ind}_1^{\Gamma_c}(1)$ can be also viewed as an element of $\mathbf{C}[M(\Gamma_c)]$ (namely $\sum_\rho\dim(\rho)(1,\rho)$ where ρ runs over the irreducible representations of Γ_c). We have $\mathrm{Ind}_1^{\Gamma_c}(1)\in Con_c$ except when

(a) $|c|$ equals $2,4,11$ or 17;

(in these cases W is of exceptional type). To remedy this, we enlarge Con_c to the subset $Con_c^+=Con_c\cup\mathrm{Ind}_1^{\Gamma_c}(1)$ of $\mathbf{C}[M(\Gamma_c)]$. (We have $Con_c^+=Con_c$ whenever c is not as in (a)).

The main result of this paper is a definition of

a subset $\mathcal{A}_{\Gamma_c}\subset c$ in canonical bijection with Con_c^+ such that each element of $\mathcal{A}_{\Gamma_c}$ appears with nonzero coefficient in the corresponding element of Con_c^+.

In [L79a] a specific representation $sp_c\in c$ in c was defined (it was later called the *special representation*); it corresponds to $(1,1)\in M(G_c)$. One of its properties is that it appears with coefficient 1 in any element of Con_c^+. We always have $sp_c\in\mathcal{A}_{\Gamma_c}$. In fact, sp_c corresponds to $\mathrm{Ind}_1^{\Gamma_c}(1)\in Con_c^+$. Thus the representations in $\mathcal{A}_{\Gamma_c}$ generalize sp_c; we call them *almost special* representations of W. (This name is justified in 2.4.)

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We will show elsewhere that (in the case where W is of simply laced type) the irreducible representation of W attached in [L15] to a stratum of G is almost special.

0.2. Our definition of $\mathcal{A}_{\Gamma_c}$ relies on the theory of new basis [L19],[L20],[L23].

Let $\mathcal{Z}_{\Gamma_c}$ be the set of pairs $(\Gamma' \subset \Gamma'')$ of subgroups of Γ_c with Γ' normal in Γ''. For $(\Gamma' \subset \Gamma'') \in \mathcal{Z}_{\Gamma_c}$ let

$$\mathbf{s}_{\Gamma',\Gamma''} : \mathbf{C}[M(\Gamma''/\Gamma')] \to \mathbf{C}[M(\Gamma_c)]$$

be the $\mathbf{C}$-linear map defined in [L20, 3.1]. In [L23, 2.3] to c we have associated a subset X_{Γ_c} of $\mathcal{Z}_{\Gamma_c}$.

Let $\bar{X}_{\Gamma_c} = X_{\Gamma_c} \cup \{(S_1, S_1)\}$. (We denote by S_n the symmetric group in n letters.) We have $(S_1, S_1) \in X_{\Gamma_c}$ if and only if c is not as in 0.1(a); in these cases we have $\bar{X}_{\Gamma_c} = X_{\Gamma_c}$. Now,

(a) $(\Gamma' \subset \Gamma'') \mapsto \mathbf{s}_{\Gamma',\Gamma''}(1,1)$ *is a bijection of* $\bar{X}_{\Gamma_c}$ *onto a subset* $S(\Gamma_c)$ *of* $\mathbf{C}[M(\Gamma_c)]$ *which is a part of a basis of* $\mathbf{C}[M(\Gamma_c)]$.

(See [L19],[L23]). We have $Con_c^+ \subset S(\Gamma_c)$. More precisely, (a) restricts to a bijection of

$$\bar{X}_{\Gamma_c,*} := \{(\Gamma', \Gamma'') \in \bar{X}_{\Gamma_c}; \Gamma' = \Gamma''\}$$

onto Con_c^+. Let

$$\underline{\bar{X}}_{\Gamma_c} = \{\Gamma'; (\Gamma', \Gamma') \in \bar{X}_{\Gamma_c,*}\}.$$

In [L20] a bijection ϵ from a certain basis of $\mathbf{C}[M(\Gamma_c)]$ (containing $S(\Gamma_c)$) to $M(\Gamma_c)$ is defined. This restricts to an injective map from $S(\Gamma_c)$) to $M(\Gamma_c)$ whose image is equal to the image of $c \subset M(\Gamma_c)$ (if c is not as in 0.1(a)) and is equal to the image of $c \subset M(\Gamma_c)$, disjoint union with a single element $(1, ?) \in M(\Gamma_c) - c$ (if c is as in 0.1(a)). From the definition of ϵ, the following holds.

(b) $\epsilon(B)$ *appears with nonzero coefficient in* B *for any* $B \in S(\Gamma_c)$.

For $\Gamma'_0 \in \underline{\bar{X}}_{\Gamma_c}$ let $\bar{X}_{\Gamma_c}^{\Gamma'_0}$ be the set of all $(\Gamma', \Gamma'') \in \bar{X}_{\Gamma_c}$ such that Γ' is conjugate to Γ'_0. The following statement will be verified in §1,§2.

(c) *For* $\Gamma'_0 \in \underline{\bar{X}}_{\Gamma_c}$, *the function* $(\Gamma', \Gamma'') \mapsto |\Gamma''|$ *on* $\bar{X}_{\Gamma_c}^{\Gamma'_0}$ *reaches its maximum at a unique* (Γ', Γ'').

For $\Gamma'_0 \in \underline{\bar{X}}_{\Gamma_c}$ (with corresponding (Γ', Γ'') defined by (c)) we set

$B_{\Gamma'_0} = \mathbf{s}_{\Gamma',\Gamma''}(1,1) \in S(\Gamma_c)$,

$E_{\Gamma'_0}$ = element of c which maps to $\epsilon(B_{\Gamma'_0})$ under $c \subset M(\Gamma_c)$. (If c is as in 0.1(a), we necessarily have $\epsilon(B_{\Gamma'_0}) \neq (1, ?)$.)

We can now define

$${}^1S(\Gamma_c) = \{B_{\Gamma'_0}; \Gamma'_0 \in \underline{\bar{X}}_{\Gamma_c}\} \subset S(\Gamma_c),$$

$$\mathcal{A}_{\Gamma_c} = \{E_{\Gamma'_0}; \Gamma'_0 \in \underline{\bar{X}}_{\Gamma_c}\} \subset c.$$

The elements of $\mathcal{A}_{\Gamma_c}$ are said to be the *almost special* representations of W in c.

We have a bijection $\mathcal{A}_{\Gamma_c} \xrightarrow{\sim} Con_c^+$ given by $E_{\Gamma'_0} \mapsto \mathbf{s}_{\Gamma'_0,\Gamma'_0}(1,1)$ for $\Gamma'_0 \in \underline{\bar{X}}_{\Gamma_c}$.

We have a bijection ${}^1S(\Gamma_c) \to Con_c^+$ given by $B_{\Gamma'_0} \mapsto \mathbf{s}_{\Gamma'_0,\Gamma'_0}(1,1)$ for $\Gamma'_0 \in \underline{\bar{X}}_{\Gamma_c}$.

We have a bijection ${}^1S(\Gamma_c) \to \mathcal{A}_{\Gamma_c}$ given by $B_{\Gamma'_0} \mapsto E_{\Gamma'_0}$ for $\Gamma'_0 \in \underline{\bar{X}}_{\Gamma_c}$.

The following statement will be verified in §1,§2.

(d) *Let* $\Gamma'_0 \in \underline{\bar{X}}_{\Gamma_c}$. *Let* $(\Gamma', \Gamma'') \in \bar{X}_{\Gamma_c}^{\Gamma'_0}$. *Assume that* $(x,\sigma) \in M(\Gamma_c)$ *appears with nonzero coefficient in* $\mathbf{s}_{\Gamma',\Gamma''}(1,1)$. *Then* $(x,\sigma) \in M(\Gamma_c)$ *appears with nonzero coefficient in* $\mathbf{s}_{\Gamma'_0,\Gamma'_0}(1,1)$.

Assume now that $(x,\sigma) \in M(\Gamma_c)$ corresponds to $E_{\Gamma'_0}$ under $c \subset M(\Gamma_c)$. By (b), (x,σ) appears with nonzero coefficient in $\mathbf{s}_{\Gamma',\Gamma''}(1,1)$ where (Γ',Γ'') is defined by Γ'_0 as in (c). Using (d) we see that

(e) (x,σ) *appears with nonzero coefficient in* $\mathbf{s}_{\Gamma'_0,\Gamma'_0}(1,1)$.

0.3. Notation. Let F be the field $\mathbf{Z}/2$. For a,b in $\mathbf{Z}$ we write $a \ll b$ whenever $b-a \ge 2$. For a,b in $\mathbf{Z}$ let $[a,b] = \{z \in \mathbf{Z}; a \le z \le b\}$. For a finite set E we write $|E|$ for the cardinal of E.

1. Classical types

1.1. Let $D \in 2\mathbf{N}$. Let $\mathcal{I}_D$ be the set of all intervals $I = [a,b]$ where $1 \le a \le b \le D$. For $I = [a,b], I' = [a',b']$ in $\mathcal{I}_D$ we write $I \prec I'$ whenever $a' < a \le b < b'$; we write $I \spadesuit I'$ if $a' - b \ge 2$ or $a - b' \ge 2$. Let $\mathcal{I}_D^1$ be the set of all $I = [a,b] \in \mathcal{I}_D$ such that $a = b \mod 2$. For $I = [a,b] \in \mathcal{I}_D^1$ we define $\kappa(I) \in \{0,1\}$ by $\kappa(I) = 0$ if a,b are even, $\kappa(I) = 1$ if a,b are odd.

A sequence $I_* = (I_1, I_2, \dots, I_r)$ in $\mathcal{I}_D^1$ is said to be *admissible* if

$$I_* = ([a_1,b_1],[a_2,b_2],\dots,[a_r,b_r]), r \ge 1$$

where

$$a_1 \le b_1, a_2 \le b_2, \dots, a_r \le b_r,$$

$$a_2 - b_1 = 2, a_3 - b_2 = 2, \dots, a_r - b_{r-1} = 2.$$

For such I_* we define $\kappa(I_*) \in \{0,1\}$ by $\kappa(I_*) = 0$ if all (or some) a_i, b_i are even, $\kappa(I_*) = 1$ if all (or some) a_i, b_i are odd.

For $I = [a,b] \in \mathcal{I}_D^1$ let

$$I^{ev} = \{x \in I; x = a+1 \mod 2\} = \{x \in I; x = b+1 \mod 2\}.$$

1.2. Let R_D^1 be the set whose elements are the subsets of $\mathcal{I}_D^1$. Let $B \in R_D^1$. We consider the following properties $(P_0), (P_1)$ that B may or may not have:

(P_0) If $I \in B, I' \in B$ then either $I = I'$ or $I \spadesuit I'$ or $I \prec I'$ or $I' \prec I$.

(P_1) If $I \in B$ and $x \in I^{ev}$ then there exists $I' in B$ such that $x \in I', I' \prec I$.

Let S_D be the set of all $B \in R_D^1$ that satisfy $(P_0), (P_1)$. (In [L19], two sets S_D, S'_D are introduced and showed to be equal. What we call S_D in this paper was called S'_D in [L19].)

For $B \in S_D, I \in B$ let $m_{I,B} = |\{I' \in B; I \subset I'\}|$.

1.3. For $B \in S_D$, $I = [a,b] \in B$ we set

$$X_{I,B} = \{I' \in B; I' \prec I, m_{I',B} = m_{I,B} + 1\}.$$

Assuming that $a < b$, we show:

(a) *$X_{I,B}$ is an admissible sequence*

$$([a_1, b_1], [a_2, b_2], \dots, [a_r, b_r])$$

(see 1.1) with $a_1 = a+1, b_r = b-1$. Moreover $\kappa(X_{I,B}) = 1 - \kappa(I)$.
We have $a+1 \in I^{ev}$. By (P_1) we can find $[a_1, b_1] \in B$ such that $a < a_1 \le a+1 \le b_1 < b$; we must have $a_1 = a+1$ and we can assume that b_1 is maximum possible. Then $m_{[a_1,b_1],B} = m_{I,B} + 1$. If $b_1 = b-1$ then we stop. Assume now that $b \ge b_1 + 3$. Let $a_2 = b_1 + 2$. We have $a_2 \in I^{ev}$ hence by (P_1) we can find $[x, b_2] \in B$ such that $a < x \le a_2 \le b_2 < b$; we can assume that b_2 is maximum possible. Then $m_{[x,b_2],B} = m_{I,B} + 1$. Since $[a_1, b_1]\spadesuit[x, b_2]$, we must have $x = a_2$. If $b_2 = b-1$ then we stop. Assume now that $b \ge b_2 + 3$. Let $a_3 = b_2 + 2$. We have $a_3 \in I^{ev}$ hence by (P_1) we can find $[x, b_3] \in B$ such that $a < x \le a_3 \le b_3 < b$; we can assume that b_3 is maximum possible. Then $m_{[x,b_3],B} = m_{I,B} + 1$. Since $[a_2, b_2]\spadesuit[x, b_3]$, we must have $x = a_3$. This process continues in this way and it eventually stops. This proves (a). (The last statement of (a) is obvious.)

1.4. Let $B \in S_D$. For $I = [a,b] \in B$ we have

(a) $|\{I' \in B; I' \subset I\}| = (b-a+2)/2$.

See [L20, 1.3(d)]. We now write the various $I \in B$ such that $m_{I,B} = 1$ in a sequence $[a_1, b_1], [a_2, b_2], \dots, [a_r, b_r]$ where

$$1 \le a_1 \le b_1 \ll a_2 \le b_2 \ll \dots \ll a_r \le b_r.$$

From (a) we deduce

$$\begin{aligned}|B| &= \sum_{i \in [1,r]} (b_i - a_i + 2)/2 \\ &= -a_1/2 - ((a_2 - b_1) + (a_3 - b_2) + \dots + (a_r - b_{r-1}))/2 + b_r/2 + r \\ &\le (b_r - a_1)/2 - (r-1) + r \le (D-1)/2 + 1 = (D+1)/2.\end{aligned}$$

Since $D \in 2\mathbf{N}$ it follows that $|B| \le D/2$. We see that:

(b) *the condition that $|B| = D/2$ is that either*

-*each of $a_2 - b_1, a_3 - b_2, \dots, a_r - b_{r-1}$ equals 2 except one of them which equals 3 and $a_1 = 1, b_r = D$,* or

-*each of $a_2 - b_1, a_3 - b_2, \dots, a_r - b_{r-1}$ equals 2 and $a_1 = 1, b_r = D-1$,* or

-*each of $a_2 - b_1, a_3 - b_2, \dots, a_r - b_{r-1}$ equals 2 and $a_1 = 2, b_r = D$.*

We set $S_D^{D/2} = \{B \in S_D; |B| = D/2\}$.

1.5. Let ${}^1S_D$ be the set of all $B \in S_D$ such that $\kappa(I) = 1$ for any $I \in B$ with $m_{I,B} = 1$.

Let $B \in S_D$. We set ${}^1B = B - \{I \in B; \kappa(I) = 0, m_{I,B} = 1\} \in R^1_D$. From the definitions it is clear that ${}^1B \in S_D$. We show that

(a) ${}^1B \in {}^1S_D$.

Let $I' \in {}^1B$ be such that $m_{I',{}^1B} = 1$. We have either $m_{I',B} = 1, \kappa(I') = 1$ or else $m_{I',B} = 2$ and $I' \prec I''$ for some $I'' \in B$ with $m_{I'',B} = 1$, $\kappa(I'') = 0$. In the second case we have $I' \in X_{I'',B}$, so that from 1.3(a) we have $\kappa(I') = 1$. This proves (a).

Thus $B \mapsto {}^1B$ is a well defined map $S_D \to {}^1S_D$.

1.6. Let $B \in {}^1S_D$. We write the various $I \in B$ such that $m_{I,B} = 1$ in a sequence

$$[a^1_1, b^1_1], [a^1_2, b^1_2], \dots, [a^1_{r_1}, b^1_{r_1}],$$

$$[a^2_1, b^2_1], [a^2_2, b^2_2], \dots, [a^2_{r_2}, b^2_{r_2}],$$

$$\dots,$$

$$[a^s_1, b^s_1], [a^s_2, b^s_2], \dots, [a^s_{r_s}, b^s_{r_s}]$$

whose first r_1 terms form an admissible sequence I_{*1}, the next r_2 terms form an admissible sequence I_{*2}, $\dots$, and the last r_s terms form an admissible sequence I_{*s}; we also assume that

$$a^2_1 \ge b^1_{r_1} + 4, a^3_1 \ge b^2_{r_2} + 4, \dots, a^s_1 \ge b^{s-1}_{r_{s-1}} + 4.$$

Here we have

$$r_1 \ge 1, r_2 \ge 1, \dots, r_s \ge 1, s \ge 0, \kappa(I_{*1}) = 1, \kappa(I_{*2}) = 1, \dots, \kappa(I_{*s}) = 1.$$

Let $Z(B)$ be the subset of $\mathcal{I}^1_D$ consisting of:
all $[a^i_1 - 1, b^i_{r_i} + 1]$ ($i \in [1, s]$) such that $a^i_1 \ge 2$ (this is automatic if $i \ge 2$);
all $[u, u]$ with u even, $b^{i-1}_{r_{i-1}} + 1 < u < a^i_1 - 1$ for some $i \in [2, s]$ (if $s > 1$);
all $[u, u]$ with u even, $1 < u < a^1_1 - 1$ (if $s > 0$);
all $[u, u]$ with u even, $b^s_{r_s} + 1 < u \le D$ (if $s > 0$);
all $[u, u]$ with u even, $1 < u \le D$ (if $s = 0$).

For any subset $U \subset Z(B)$ we set $B_U = B \sqcup U$; then $B_U \in S_D$ and $U \mapsto B_U$ defines a bijection from the set of subsets of $Z(B)$ to the fibre at B of the map $S_D \to {}^1S_D$, $B' \mapsto {}^1B'$. Note that $B_\emptyset = B$ and $B_{Z(B)} \in S^{D/2}_D$. Moreover, $B \mapsto B_{Z(B)}$ is the bijection ${}^1S_D \xrightarrow{\sim} S^{D/2}_D$ whose inverse is the restriction to $S^{D/2}_D$ of $S_D \to {}^1S_D$, $B' \mapsto {}^1B'$. (We use 1.4(b).)

1.7. A subset B of R^1_D is said to be in ${}^1\dot{S}_D$ if it satisfies (P_0) and if each $I \in B$ satisfies $\kappa(I)=1$. For $B \in {}^1S_D$ we set $\dot{B} = \{I \in B; \kappa(I)=1\}$. Then $B \mapsto \dot{B}$ is a map

(a) ${}^1S_D \to {}^1\dot{S}_D$.

We show:

(b) *The map (a) is a bijection.*

Let $C \in {}^1\dot{S}_D$. For $I \in C$ we set $\dot{m}_{I,C} = |\{I' \in C; I \subset I'\}|$.

For $k \in \{1,2,3,\dots\}$ we set $C[k] = \{I \in C; \dot{m}_{I,C} = k\}$.

Let $I=[a,b] \in C[k]$. As in 1.6 we can write the intervals $\{I' \in C[k+1]; I' \prec I\}$ in a sequence

$$[a^1_1,b^1_1],[a^1_2,b^1_2],\dots,[a^1_{r_1},b^1_{r_1}],$$

$$[a^2_1,b^2_1],[a^2_2,b^2_2],\dots,[a^2_{r_2},b^2_{r_2}],$$

$$\dots,$$

$$[a^s_1,b^s_1],[a^s_2,b^s_2],\dots,[a^s_{r_s},b^s_{r_s}]$$

whose first r_1 terms form an admissible sequence I_{*1}, the next r_2 terms form an admissible sequence I_{*2}, ..., and the last r_s terms form an admissible sequence I_{*s}; we also assume that

$$a^2_1 \ge b^1_{r_1}+4, a^3_1 \ge b^2_{r_2}+4, \dots, a^s_1 \ge b^{s-1}_{r_{s-1}}+4.$$

Here we have

$$r_1 \ge 1, r_2 \ge 1, \dots, r_s \ge 1, s \ge 0, \kappa(I_{*1})=1, \kappa(I_{*2})=1, \dots, \kappa(I_{*s})=1.$$

Moreover we have $a^i_1 \ge a+2, b^i_{r_i} \le b-2$ for all i.

Let Y_I be the subset of $\mathcal{I}^1_D$ consisting of:

all $[a^i_1-1, b^i_{r_i}+1]$ ($i \in [1,s]$);

all $[u,u]$ with u even, $b^{i-1}_{r_{i-1}}+1 < u < a^i_1-1$ for some $i \in [2,s]$ (if $s>1$);

all $[u,u]$ with u even, $a < u < a^1_1-1$ (if $s>0$);

all $[u,u]$ with u even, $b^s_{r_s}+1 < u < b$ (if $s>0$);

all $[u,u]$ with u even, $a<u<b$ (if $s=0$).

For $l \ge 1$ we set $B[2l-1]=C[l], B[2l]=\sqcup_{I \in C[l]} Y_I$. We set $B = \sqcup_{l \in \{1,2,3,\dots\}} B[l]$. From the definition we see that $B \in {}^1S_D$ and that $C \mapsto B$ is an inverse of the map ${}^1S_D \to {}^1\dot{S}_D$, $B \mapsto \dot{B}$. This proves (b).

We shall view any element $C \in {}^1\dot{S}_D$ as a tableau with columns indexed by $[1,D]$, with rows indexed by $\{1,2,3,\dots\}$ and with entries in $\cup_j [a_j,b_j]$. Any entry in the s-column is equal to s; the k-th row consists of the elements in $\cup_{I \in C[k]} I$.

1.8. Let $C \in {}^1\dot{S}_D$. It is an unordered set of intervals $[a_1, b_1], [a_2, b_2], \dots, [a_t, b_t]$. We can order them by requiring that $b_1 < b_2 < \dots < b_t$. We view C as a tableau as in 1.7. We associate to C a new tableau $\ddot{C}$ with columns indexed by $[1, D]$, with rows indexed by $\{1, 2, 3, \dots\}$ and with entries in $\cup_j [a_j, b_j]$. This is obtained by moving the entry of C in the s-column and row k to the same s-column and to row $k + j$ where $j \in [0, t-1]$ is defined by $b_j < s \le b_{j+1}$ (with the convention $b_0 = 0$); note that $s \le b_t$ whenever the s-column of C is nonempty.

For example, ${}^1\dot{S}_4$ consists of 5 tableaux: $(\emptyset)$, (1); (3); $\begin{pmatrix}1 & & 3\end{pmatrix}$; (123). The corresponding tableaux $\ddot{C}$ are $(\emptyset)$, (1); (3); $\begin{pmatrix}1 & & \\ & & 3\end{pmatrix}$; (123). Now ${}^1\dot{S}_6$ consists of 14 tableaux:

$(\emptyset)$; (1); (3); (5); $\begin{pmatrix}1 & & 3\end{pmatrix}$; $\begin{pmatrix}3 & & 5\end{pmatrix}$; $\begin{pmatrix}1 & & & & 5\end{pmatrix}$; $\begin{pmatrix}1 & & 3 & & 5\end{pmatrix}$; (123); $\begin{pmatrix}123 & & 5\end{pmatrix}$; (345); $\begin{pmatrix}1 & & 345\end{pmatrix}$; (12345); $\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ & & 3 & & \end{pmatrix}$.

The corresponding tableaux $\ddot{C}$ are

$(\emptyset)$; (1); (3); (5); $\begin{pmatrix}1 & \\ & 3\end{pmatrix}$; $\begin{pmatrix}3 & \\ & 5\end{pmatrix}$; $\begin{pmatrix}1 & \\ & 5\end{pmatrix}$; $\begin{pmatrix}1 & & \\ & 3 & \\ & & 5\end{pmatrix}$; (123); $\begin{pmatrix}123 & \\ & 5\end{pmatrix}$; (345); $\begin{pmatrix}1 & \\ & 345\end{pmatrix}$; (12345); $\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ & & 3 & & \end{pmatrix}$.

Here are some further examples.

If $C = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ & & 3 & & \end{pmatrix}$ then $\ddot{C} = \begin{pmatrix}1 & 2 & 3 & & \\ & & 3 & 4 & 5\end{pmatrix}$.

If $C = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ & & 3 & & & & \end{pmatrix}$ then $\ddot{C} = \begin{pmatrix}1 & 2 & 3 & & & & \\ & & 3 & 4 & 5 & 6 & 7\end{pmatrix}$.

If $C = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ & & & & 5 & & \end{pmatrix}$ then $\ddot{C} = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & & \\ & & & & 5 & 6 & 7\end{pmatrix}$.

If $C = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ & & 3 & & 5 & & \end{pmatrix}$ then $\ddot{C} = \begin{pmatrix}1 & 2 & 3 & & & & \\ & & 3 & 4 & 5 & & \\ & & & & 5 & 6 & 7\end{pmatrix}$.

If $C = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ & & 3 & 4 & 5 & & \end{pmatrix}$ then $\ddot{C} = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & & \\ & & 3 & 4 & 5 & 6 & 7\end{pmatrix}$.

If $C = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ & & 3 & 4 & 5 & 6 & 7 & & \\ & & & & 5 & & & & \end{pmatrix}$ then

$\ddot{C} = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & & & & \\ & & 3 & 4 & 5 & 6 & 7 & & \\ & & & & 5 & 6 & 7 & 8 & 9\end{pmatrix}$.

We show:

(a) *Let $j \in [1, t]$. Let k be such that $[a_j, b_j] \in C[k]$. In row j of $\ddot{C}$, b_j appears and $b_j + 1$ does not appear.*

In rows $j+1, j+2, \dots, j+k-1$ of $\ddot{C}$, b_j and b_j+1 appear. In any other row of $\ddot{C}$, b_j and b_j+1 do not appear.
Assume first that $b_j < D$. Then in C, b_j appears in rows $1, 2, \dots, k$ and b_j+1 appears in rows $1, 2, \dots, k-1$. Since $b_{j-1} < b_j \le b_j$, $b_j < b_j+1 \le b_{j+1}$ we see that in $\ddot{C}$, b_j appears in rows $1+(j-1), 2+(j-1), \dots, k+(j-1)$ and b_j+1 appears in rows $1+j, 2+j, \dots, (k-1)+j$. This proves (a) in our case. Now assume that $b_j = D$ (in this case $j = t$ and $k = 1$). Then in C, b_t appears in row 1 and in no other row. We have $b_{t-1} < b_t \le b_t$. Hence in $\ddot{C}$, b_t appears in row $1+(t-1)$ and in no other row. Thus (a) again holds.

We show:

(b) *Let $i \in [1, t]$. Let k be such that $[a_i, b_i] \in C[k]$. Define $j \in [0, t-1]$ by $b_j < a_i \le b_{j+1}$. In row $k+j$ of $\ddot{C}$, a_i appears and $a_i - 1$ does not appear.*
In rows $j+1, j+2, \dots, j+k-1$ of $\ddot{C}$, a_i and $a_i - 1$ appear. In any other row of $\ddot{C}$, a_i and $a_i - 1$ do not appear.
Assume first that $a_i > 1$. Then in C, a_i appears in rows $1, 2, \dots, k$ and $a_i - 1$ appears in rows $1, 2, \dots, k-1$. Then (since b_j, a_i are odd) we have $b_j < a_i - 1 \le b_{j+1}$. Hence in $\ddot{C}$, a_i appears in rows $1+j, 2+j, \dots, k+j$ and $a_i - 1$ appears in rows $1+j, 2+j, \dots, (k-1)+j$. This proves (b) in our case. Now assume that $a_i = 1$ (in this case we have $k = 1$). Then in C, a_i appears in row 1 and in no other row. We have $b_0 < a_i \le b_1$. Hence in $\ddot{C}$, a_i appears in row 1 and in no other row. Thus (b) again holds.

Now let $h \in \cup_j [a_j, b_j]$ be such that $h \ne a_j, h \ne b_j$ for all $j \in [1, t]$. We show:

(c) *Any row of $\ddot{C}$ that contains h must also contain $h+1$.*

(d) *Any row of $\ddot{C}$ that contains h must also contain $h-1$.*

There is a well defined $j \in [0, t-1]$ such that $b_j < h < b_{j+1}$.

We prove (c). Assume first that $h+1 < b_{j+1}$. Then in C, h appears in rows $1, 2, \dots, k$ and $h+1$ appears in rows $1, 2, \dots, k$ (for some k). In $\ddot{C}$, h appears in rows $j+1, j+2, \dots, j+k$ and $h+1$ appears in rows $j+1, j+2, \dots, j+k$. Hence in this case (c) holds. Next we assume that $h+1 = b_{j+1}$. Then in C, h appears in rows $1, 2, \dots, k$ and $h+1$ appears in rows in rows $1, 2, \dots, k+1$ (for some k). In $\ddot{C}$, h appears in rows $j+1, j+2, \dots, j+k$ and $h+1$ appears in rows $j+1, j+2, \dots, j+k+1$. We see again that (c) holds.

We prove (d). Assume first that $b_j < h-1$. Then in C, h appears in rows $1, 2, \dots, k$ and $h-1$ appears in rows $1, 2, \dots, k$ (for some k). In $\ddot{C}$, h appears in rows $j+1, j+2, \dots, j+k$ and $h-1$ appears in rows $j+1, j+2, \dots, j+k$. Hence in this case (d) holds.

Next we assume that $b_j = h-1$. Then in C, h appears in rows $1, 2, \dots, k$ and $h-1$ appears in rows $1, 2, \dots, k, k+1$ (for some k). Moreover in $\ddot{C}$, h appears in rows $j+1, j+2, \dots, j+k$ and $h-1$ appears in rows $j, j+1, j+2, \dots, j+k$. We see again that (d) holds.

From (a)-(d) we deduce:

(e) *For $j \in [1, t]$, the row j of $\ddot{C}$ consists of $a_{i_j}, a_{i_j}+1, a_{i_j}+2, \dots, b_j$ for a well* *defined* $i_j \in [1,t]$ *such that* $a_{i_j} \le b_j$. *Moreover,* $j \mapsto i_j$ *is a permutation of* $[1,t]$.

We show:

(f) *For* $u \in [2,t]$ *we have* $a_{i_{u-1}} < a_{i_u}$.

We set $i = i_u$.

Assume first that $[a_i, b_i] \in C[k], k \ge 2$. By (b), one row of $\ddot{C}$ contains a_i but not $a_i - 1$ (hence it is necessarily the row u) and the row just above it (that is row $u-1$) contains a_i and $a_i - 1$. (We use that $k \ge 2$.) Now that row consists of $a_{i_{u-1}}, a_{i_{u-1}} + 1, \dots, b_{u-1} - 1, b_{u-1}$. Thus we have $a_{i_{u-1}} \le a_i - 1 < a_i \le b_{u-1}$. In particular, $a_{i_{u-1}} < a_i$.

Next we assume that $[a_i, b_i] \in C[1]$. Now $[a_i, b_u]$ is contained in the union of all $I \in C[1]$ and consists of consecutive numbers. Hence it is contained in one such I which is necessarily $[a_i, b_i]$. Thus we have $[a_i, b_u] \subset [a_i, b_i]$.

Assume that $a_i \le b_{u-1}$. In row 1 of C we have $a_i \le b_{u-1} < b_u$. In row 2 of C, a_i is missing. Since $a_i \le b_{u-1}$ the unique entry a_i in $\ddot{C}$ appears in a row $\le u-1$. In particular the row u of $\ddot{C}$ does not contain a_i; but it contains b_u. This contradicts $a_i = a_{i_u}$. We see that we must have $b_{u-1} < a_i$. But $a_{i_{u-1}} \le b_{u-1}$ hence $a_{i_{u-1}} < a_i$. This proves (f).

1.9. Let ${}^1\ddot{S}_D$ be the set of tableaux with columns indexed by $[1,D]$, with rows indexed by $\{1,2,3,\dots\}$ and with entries in $[1,D]$ such that any entry in the s-column is equal to s; for any $k \in [1,t]$ (some t), the row k consists of the elements in some $I_k = [c_k, d_k] \in \mathcal{I}_D^1$ with $\kappa(I_k) = 1$; for $k > t$ the row k contains no entries; we have $c_1 < c_2 < \dots < c_t, d_1 < d_2 < \dots < d_t$.

For $X, c_1 < c_2 < \dots < c_t, d_1 < d_2 < \dots < d_t$ as above we define a tableau $\dot{X}$ with columns indexed by $[1,D]$, with rows indexed by $\{1,2,3,\dots\}$ and with entries in $\cup_j [c_j, d_j]$. This is obtained by moving the entry of X in the s-column and row k to the same s-column and to row $k - j$ where $j \in [0, t-1]$ is defined by $d_j < s \le d_{j+1}$ (with the convention $d_0 = 0$); note that we necessarily have $k > j$. (Indeed, we have $s \le d_k$; if $k \le j$ then $d_k \le d_j$, hence $s \le d_j$, contradicting $d_j < s$.)

From the definitions we see that $\dot{X} \in {}^1\dot{S}_D$ and that $X \mapsto \dot{X}$ is a bijection ${}^1\ddot{S}_D \to {}^1\dot{S}_D$ inverse to $C \mapsto \ddot{C}$, ${}^1\dot{S}_D \to {}^1\ddot{S}_D$.

1.10. Let U_D be the set of all tableaux

$$\begin{pmatrix} c_1 & c_2 & \dots & c_t \\ d'_1 & d'_2 & \dots & d'_t \end{pmatrix}$$

where $c_1 < c_2 < \dots < c_t$ are odd integers in $[1,D]$, $d'_1 < d'_2 < \dots < d'_t$ are even integers in $[1,D]$ and $c_1 < d'_1, c_2 < d'_2, \dots, c_t < d'_t$.

We have an obvious bijection ${}^1\ddot{S}_D \xrightarrow{\sim} U_D$,

$$(X, c_1 < c_2 < \dots < c_t, d_1 < d_2 < \dots < d_t) \mapsto$$
$$\begin{pmatrix} c_1 & c_2 & \dots & c_t \\ d_1 + 1 & d_2 + 1 & \dots & d_t + 1 \end{pmatrix}.$$

1.11. Let $\ddot{\Sigma}_D$ be the set of all symbols

$$\Lambda = \begin{pmatrix} i_1 & i_2 & \dots & i_{(D+2)/2} \\ j_1 & j_2 & \dots & j_{(D+2)/2} \end{pmatrix}$$

where

$$\{i_1, i_2, \dots, i_{(D+2)/2}\} \sqcup \{j_1, j_2, \dots, j_{(D+2)/2}\} = [0, D+1],$$

$$i_1 < i_2 < \dots < i_{(D+2)/2}, j_1 < j_2 < \dots < j_{(D+2)/2},$$

$$i_1 < j_1, i_2 < j_2, \dots, i_{(D+2)/2} < j_{(D+2)/2}.$$

We then have $i_1 = 0, j_{(D+2)/2} = D+1$.

For Λ as above let $c_1 < c_2 < \dots < c_t$ be the odd numbers in $\{i_1, i_2, \dots, i_{(D+2)/2}\}$ (in increasing order) and let $d'_1 < d'_2 < \dots < d'_{t'}$ be the even numbers in $\{j_1, j_2, \dots, j_{(D+2)/2}\}$ (in increasing order). We have necessarily $t = t'$. We show:

(a) $$c_1 < d'_1, c_2 < d'_2, \dots, c_t < d'_t.$$

Assume now that for some $s \in [0, t], s < t$ we already know that $c_1 < d'_1, c_2 < d'_2, \dots, c_s < d'_s$. We show that $c_{s+1} < d'_{s+1}$.

Assume that $d'_{s+1} \le c_{s+1}$. Let $Z = \{i_k; k \in [1, (D+2)/2]; i_k < d'_{s+1}\}$. Then

$$Z = \{0, 2, 4, \dots, d'_{s+1} - 2\} \sqcup \{c_1, c_2, \dots, c_s\} - \{d'_1, d'_2, \dots, d'_s\}.$$

(We use that $c_1 < d'_1, c_2 < d'_2, \dots, c_s < d'_s$. We also use that $d'_{s+1} \le c_{s+1}$.) We have $|Z| = |(0, 2, 4, \dots, d'_{s+1} - 2)|$. We have $d'_{s+1} = j_m$ for some $m \in [1, (D+2)/2]$ and $i_m < d'_{s+1}$ that is $i_m \in Z$. It follows that $\{i_1, i_2, \dots, i_m\} \subset Z$ so that $m \le |Z|$. Let

$$Z' = \{j_k; k \in [1, (D+2)/2]; j_k \le d'_{s+1}\}.$$

We have $|Z'| = m$. Now

$$Z' = \{1, 3, 5, \dots, d'_{s+1} - 1 \sqcup \{d'_1, d'_2, \dots, d'_s, d'_{s+1}\} - \{c_1, s_2, \dots, c_s\}$$

so that $|Z'| = |(1, 3, 5, \dots, d'_{s+1} - 1) + 1)|$. Since $|Z'| = m$ and $m \le Z$ we have $|Z'| \le |Z|$ so that

$$|(1, 3, 5, \dots, d'_{s+1} - 1) + 1)| \le |(0, 2, 4, \dots, d'_{s+1} - 2)|.$$

This is obviously not true. This proves (a).

From (a) we see that

$$\Lambda \mapsto \begin{pmatrix} c_1 & c_2 & \dots & c_t \\ d'_1 & d'_2 & \dots & d'_t \end{pmatrix}$$

(as described above) defines a map

(b) $$\ddot{\Sigma}_D \to U_D.$$

We show:

(c) *The map (b) is injective.*

To any

$$\mu = \begin{pmatrix} c_1 & c_2 & \dots & c_t \\ d'_1 & d'_2 & \dots & d'_t \end{pmatrix} \in U_D$$

we associate a sequence

$$\mu' = (i_1, i_2, \dots, i_{(D+2)/2})$$

and a sequence

$$\mu'' = (j_1, j_2, \dots, j_{(D+2)/2})$$

as follows.

μ' consists of the elements in $\{c_1, c_2, \dots, c_t\}$ and those in $\{0, 2, 4, \dots, D\} - \{d'_1, d'_2, \dots, d'_t\}$ (in increasing order).

μ'' consists of the elements in $\{d'_1, d'_2, \dots, d'_t\}$ and those in $\{1, 3, 5, \dots, D+1\} - \{c_1, c_2, \dots, c_t\}$ (in increasing order).

From the definition we see that if $\Lambda \in \ddot{\Sigma}_D$ has image $\mu \in U_D$ under (b) then $\Lambda = \begin{pmatrix} \mu' \\ \mu'' \end{pmatrix}$. From this it is clear that the map (b) is injective. This proves (c).

1.12. We show:

(a) *The injective map $\ddot{\Sigma}_D \to U_D$ in 1.11(b) is a bijection.*

Note that $\ddot{\Sigma}_D$ can be viewed as the set of standard Young tableaux attached to a partition with two equal parts of size $(D+2)/2$. The number of such standard tableaux can be computed from the hook length formula so that it is equal to $(D+2)!/((D+2)/2)!(D+4)/2)!)$ that is to the Catalan number $Cat_{(D+2)/2}$. (This interpretation of Catalan numbers in terms of standard Young tableaux has been known before.)

From the bijections

$$U_D \leftarrow {}^1\ddot{S}_D \to {}^1\dot{S}_D \leftarrow {}^1S_D \to S_D^{D/2}$$

(see 1.10,1.9,1.7,1.6) we see that $|U_D| = |S_D^{D/2}|$. By [LS], $|S_D^{D/2}|$ is equal to the Catalan number $Cat_{(D+2)/2}$. We see that the map in (a) satisfies $|\ddot{\Sigma}_D| = |U_D| = Cat_{(D+2)/2}$. It follows that this map is a bijection.

(It is likely that (a) has a more direct proof which does not rely on [LS].)

We show:

(b) *If $\mu \in U_D$ and if μ', μ'' are as in the proof of 1.11(c), then* $\begin{pmatrix} \mu' \\ \mu'' \end{pmatrix} \in \ddot{\Sigma}_D$. *Moreover* $\mu \mapsto \begin{pmatrix} \mu' \\ \mu'' \end{pmatrix}$ *is the inverse of the map* $\ddot{\Sigma}_D \to U_D$ *in 1.11.*

If $\mu \in U_D$, then by (a) we can find $\Lambda \in \ddot{\Sigma}_D$ whose image under the map 1.11(b) is μ. By the proof of 1.11(c) we have $\Lambda = \begin{pmatrix} \mu' \\ \mu'' \end{pmatrix}$. It follows that $\begin{pmatrix} \mu' \\ \mu'' \end{pmatrix} \in \ddot{\Sigma}_D$.

1.13. Let V_D be the F-vector space with basis $e_1, e_2, \dots, e_D$ and with the symplectic form $(,) : V \times V \to F$ given by $(e_i, e_j) = 1$ if $i - j = \pm 1$, $(e_i, e_j) = 0$, otherwise. For any subset J of $[1, D]$ we set $e_J = \sum_{j \in J} e_j \in V_D$.

For $B \in S_D$ let $< B >$ be the subspace of V_D spanned by $\{e_I; I \in B\}$. (This is actually a basis of $< B >$, see [L19].)

For $j \in [1, D]$ and $B \in S_D$ we set $B_j = \{I \in B; j \in I\}$ and

$$\epsilon_j(B) = |B_j|(|B_j| + 1)/2 \in F$$

. For $B \in S_D$ we set

$$\epsilon(B) = \sum_{j \in [1,D]} \epsilon_j(B) e_j \in V_D.$$

We show:

(a) $$\epsilon(B) = \sum_{I \in B; m_{I,B} \in 2\mathbf{N}+1} e_I.$$

An equivalent statement is:

If $j \in [1, D]$ then $|\{I \in B_j, m_{I,B} \in 2\mathbf{N} + 1\}|$ is even if $|B_j| \in (4\mathbf{Z}) \cup (4\mathbf{Z} + 3)$ and is odd if $|B_j| \in (4\mathbf{Z} + 1) \cup (4\mathbf{Z} + 2)$.

This follows immediately from the following statement (which holds by the definition of S_D):

B_j consists of intervals $I_k \prec I_{k-1} \prec \dots \prec I_1$ in $\mathcal{I}_D^1$ such that $m_{I_k,B} = k, m_{I_{k-1},B} = k - 1, \dots, m_{I_1,B} = 1$.

For $C \in {}^1\dot{S}_D$ we define

(b) $$\dot{\epsilon}(C) = \sum_{I \in C} e_I.$$

For $\ddot{C} \in {}^1\ddot{S}_D$ we define

(c) $$\ddot{\epsilon}(\ddot{C}) = \sum_k e_{[c_k, d_k]} \in V_D$$

where $c_k, c_k + 1, c_k + 2, \dots, d_k$ are the entries in the k-th row of $\ddot{C}$. From the definitions we have

(d) $$\dot{\epsilon}(C) = \ddot{\epsilon}(\ddot{C})$$

if $C, \ddot{C}$ correspond to each other under the bijection in 1.9.

1.14. By [L19, 1.16], $B \mapsto \epsilon(B)$ is an injective map $\epsilon : S_D \xrightarrow{\sim} V_D$. By 1.13(a), for $B \in {}^1S_D$ we have $\epsilon(B) = \dot\epsilon(\dot B)$ ($\dot\epsilon$ as in 1.13(b)). Hence the restriction of ϵ to ${}^1S_D$ can be identified with $\dot\epsilon : {}^1\dot S_D \to V_D$ via the bijection 1.7(b). In particular, $\dot\epsilon$ is injective. Using 1.13(d) we see that via the bijection in 1.9, $\dot\epsilon : {}^1\dot S_D \to V_D$ becomes $\ddot\epsilon : {}^1\ddot S_D \to V_D$. In particular, $\ddot\epsilon$ is injective.

1.15. Let Σ_D be the set of all unordered pairs $\begin{pmatrix} A \\ B \end{pmatrix}$ of subsets of $[0, D+1]$ such that $[0, D+1] = A \sqcup B$, $|A| = |B| \mod 4$.

There is a unique bijection $f : V_D \to \Sigma_D$ such that

$$f(0) = \begin{pmatrix} 0 & 2 & 4 & \dots & D \\ 1 & 3 & 5 & \dots & D+1 \end{pmatrix}$$

and such that if $x \in V_D$, $f(x) = \begin{pmatrix} A \\ B \end{pmatrix}$ and $i \in [1, D]$ then

$$f(x + e_i) = \begin{pmatrix} A\sharp\{i, i+1\} \\ B\sharp\{i, i+1\} \end{pmatrix}$$

where $\sharp$ is the symmetric difference; it follows that for $1 \le i < j \le D$ we have

$$f(x + e_i + e_{i+1} + \dots + e_j) = \begin{pmatrix} A\sharp\{i, j+1\} \\ B\sharp\{i, j+1\} \end{pmatrix}.$$

1.16. We can regard $\ddot\Sigma_D$ as a subset of Σ_D. If $\ddot C \in {}^1\ddot S_D$ corresponds to $\mu \in U_D$ under 1.10 then from the definitions we have $f(\ddot\epsilon(\ddot C)) = \begin{pmatrix} \mu' \\ \mu'' \end{pmatrix}$ (notation of 1.12(b)). In particular we have

(a) $f(\ddot\epsilon(\ddot C)) \in \ddot\Sigma_D$

and (using 1.12(b)) we see that

(b) $\ddot C \mapsto f(\ddot\epsilon(\ddot C))$ *is a bijection* ${}^1\ddot S_D \xrightarrow{\sim} \ddot\Sigma_D$.

1.17. We have $V_D = V_D^0 \oplus V_D^1$ where V_D^0 is the subspace spanned by $\{e_2, e_4, \dots, e_D\}$ and V_D^1 is the subspace spanned by $\{e_1, e_3, \dots, e_{D-1}\}$. For $I \in \mathcal{I}_D^1$ we have $I = I^0 \sqcup I^1$ where $I^0 = I \cap \{2, 4, \dots, D\}$, $I^1 = I \cap \{1, 3, \dots, D-1\}$. As shown in [L19], for $B \in S_D$ we have $< B > = < B >_0 \oplus < B >_1$ where

$$< B >_0 = < B > \cap V_D^0, < B >_1 = < B > \cap V_D^1;$$

moreover,

(a) $< B >_1$ *has basis* $\{e_{I^1}; I \in B, \kappa(I) = 1\}$,

$< B >_0$ *has basis* $\{e_{I^0}; I \in B, \kappa(I) = 0\}$.

1.18. If $\mathcal{L}$ is a subspace of V_D^δ ($\delta \in \{0,1\}$) we set

$$\mathcal{L}^! = \{x \in V^{1-\delta}; (x, \mathcal{L}) = 0\}.$$

Let $\mathcal{C}(V_D^\delta)$ be the set of subspaces of V_D^δ of the form $< B >_\delta$ for some $B \in S_D$. If $\mathcal{L} \in \mathcal{C}(V_D^\delta)$ we have $\mathcal{L}^! \in \mathcal{C}(V_D^{1-\delta})$; see [L19, §2]. Let $\mathcal{A}(V_D^1)$ be the set of all $(\mathcal{L}, \mathcal{L}') \in \mathcal{C}(V_D^1) \times \mathcal{C}(V_D^1)$ such that $\mathcal{L} \subset \mathcal{L}'$ and $\mathcal{L} \oplus \mathcal{L}'^! =< B >$ for some $B \in S_D$.

(a) *If $B \in S_D$ then $B \mapsto (< B >_1, < B >_0^!)$ is a bijection $\Phi : S_D \to \mathcal{A}(V_D^1)$;* see [L19, §2]. Let $\mathcal{A}_*(V_D^1)$ be the set of all $(\mathcal{L}, \mathcal{L}') \in \mathcal{C}(V_D^1) \times \mathcal{C}(V_D^1)$ such that $\mathcal{L} = \mathcal{L}'$. In [L19,§2] it is shown that $\mathcal{A}_*(V_D^1) \subset \mathcal{A}(V_D^1)$; more precisely if $\mathcal{L} \in \mathcal{C}(V_D^1)$ then $\mathcal{L} \oplus \mathcal{L}^! =< B >$ for a well defined $B \in S_D^{D/2}$. Moreover $B \mapsto (< B >_1, < B >_0^!)$ is a bijection $S_D^{D/2} \to \mathcal{A}_*(V_D^1)$ and $(\mathcal{L}, \mathcal{L}') \mapsto \mathcal{L} = \mathcal{L}'$ is a bijection $\mathcal{A}_*(V_D^1) \to \mathcal{C}(V_D^1)$. The composition of these bijections is a bijection $B \mapsto < B >_1$,

(b) $S_D^{D/2} \xrightarrow{\sim} \mathcal{C}(V_D^1)$.

Next we note that $B \mapsto < B >_1$ is also a bijection

(c) ${}^1S_D \xrightarrow{\sim} \mathcal{C}(V_D^1)$.

This follows from (b) since the bijection (b) is a composition

$$S_D^{D/2} \xrightarrow{\sim} {}^1S_D \to \mathcal{C}(V_D^1)$$

where the fist map is the bijection $B \mapsto {}^1B$ and the second map is the map in (c).

Here we use that

(d) $< B >_1 =< {}^1B >_1$ for any $B \in S_D$,

which follows from definitions.

We show:

(e) *For any $\mathcal{L} \in \mathcal{C}(V_D^1)$, the set*

$$\{\mathcal{L}' \in \mathcal{C}(V_D^1); (\mathcal{L}, \mathcal{L}') \in \mathcal{A}(V_D^1)\}$$

contains a unique $\mathcal{L}'$ with $|\mathcal{L}'|$ maximal.

An equivalent statement is:

(f) *For any $\mathcal{L} \in \mathcal{C}(V_D^1)$ the set $\{B' \in S_D; < B' >_1 = \mathcal{L}\}$ contains a unique B' with $| < B' >_0^! |$ maximal (that is $\dim(< B' >_0)|$ minimal).*

By (b) we have $\mathcal{L} =< B >_1$ for a well defined $B \in S_D^{D/2}$. The condition that $< B' >_1 =< B >_1$ is equivalent to $< {}^1B' >_1 =< {}^1B >_1$ (see (d)) and this is equivalent to ${}^1B' = {}^1B$ (see (c)). Hence the set in (f) is equal to

$\{B' \in S_D; {}^1B' = {}^1B\}$.

By the results in 1.6 this is the same as $\{({}^1B)_U; U \in Z({}^1B)\}$. By 1.17(a), for $U \in Z({}^1B)$ we have

$$\dim(({}^1B)_U)_0 = \dim({}^1B)_0 + |U|.$$

This is $\geq \dim({}^1B)_0$ with equality if and only if $U = \emptyset$. This proves (f) and hence (e).

For $\mathcal{L} \in \mathcal{C}(V_D^1)$ we denote by $\mathcal{L}^{max}$ the element $\mathcal{L}'$ in (e) with $|\mathcal{L}'|$ maximal. Let $\mathcal{A}^*(V_D^1)$ be the set of all $(\mathcal{L}, \mathcal{L}') \in \mathcal{C}(V_D^1) \times \mathcal{C}(V_D^1)$ such that $\mathcal{L}' = \mathcal{L}^{max}$. We have $\mathcal{A}^*(V_D^1) \subset \mathcal{A}(V_D^1)$ and

(g) $(\mathcal{L}, \mathcal{L}') \mapsto \mathcal{L}$ *is a bijection* $\mathcal{A}^*(V_D^1) \to \mathcal{C}(V_D^1)$.

From (c),(g) we see (using the proof of (f)) that

(h) $B \mapsto (< B >_1, < B >_0^!)$ *is a bijection* ${}^1S_D \xrightarrow{\sim} \mathcal{A}^*(V_D^1)$.

1.19. In this subsection we assume that $D \in \{2, 4, 6\}$. In each case we give a table with rows of the form $\alpha....\beta....\gamma$ where $\alpha \in S_D^{D/2}, \beta \in {}^1S_D$ corresponds to α and $\gamma \in \ddot{\Sigma}_D$. We write an interval $[a, b]$ as $a, a+1, a+2, \dots, b$ (without commas).

$D = 2$

$\{2\}......\{\emptyset\}........\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$

$\{1\}....\{1\}........\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$

$D = 4$

$\{2,4\}......\{\emptyset\}......\begin{pmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{pmatrix}$

$\{1,4\}.....\{1\}.......\begin{pmatrix} 0 & 1 & 4 \\ 2 & 3 & 5 \end{pmatrix}$

$\{3,234\}...\{3\}......\begin{pmatrix} 0 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix}$

$\{1,3\}....\{1,3\}......\begin{pmatrix} 0 & 1 & 3 \\ 2 & 4 & 5 \end{pmatrix}$

$\{2,123\}....\{2,123\}......\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}$

$D = 6$

$\{2,4,6\}......\{\emptyset\}.....\begin{pmatrix} 0 & 2 & 4 & 6 \\ 1 & 3 & 5 & 7 \end{pmatrix}$

$\{1,4,6\}.....\{1\}......\begin{pmatrix} 0 & 1 & 4 & 6 \\ 2 & 3 & 5 & 7 \end{pmatrix}$

$\{3,234,6\}...\{3\}.....\begin{pmatrix} 0 & 2 & 3 & 6 \\ 1 & 4 & 5 & 7 \end{pmatrix}$

$\{2,5,456\}...\{5\}.....\begin{pmatrix} 0 & 2 & 4 & 5 \\ 1 & 3 & 6 & 7 \end{pmatrix}$

$\{1,3,6\}....\{1,3\}.....\begin{pmatrix} 0 & 1 & 3 & 6 \\ 2 & 4 & 5 & 7 \end{pmatrix}$

$\{1,5,456\}...\{1,5\}......\begin{pmatrix} 0 & 1 & 4 & 5 \\ 2 & 3 & 6 & 7 \end{pmatrix}$

$\{3,5,23456\}...\{3,5\}......\begin{pmatrix} 0 & 2 & 3 & 5 \\ 1 & 4 & 6 & 7 \end{pmatrix}$

$\{1,3,5\}....\{1,3,5\}....\begin{pmatrix}0&1&3&5\\2&4&6&7\end{pmatrix}$

$\{2,123,6\}....\{2,123\}...\begin{pmatrix}0&1&2&6\\3&4&5&7\end{pmatrix}$

$\{4,345,23456\}....\{4,345\}.....\begin{pmatrix}0&2&3&4\\1&5&6&7\end{pmatrix}$

$\{2,4,12345\}....\{2,4,12345\}....\begin{pmatrix}0&1&2&4\\3&5&6&7\end{pmatrix}$

$\{1,4,345\}....\{1,4,345\}......\begin{pmatrix}0&1&3&4\\2&5&6&7\end{pmatrix}$

$\{2,123,5\}....\{2,123,5\}......\begin{pmatrix}0&1&2&5\\3&4&6&7\end{pmatrix}$

$\{3,234,12345\}....\{3,234,12345\}......\begin{pmatrix}0&1&2&3\\4&5&6&7\end{pmatrix}$

2. Exceptional types

2.1. Let W, c, Γ_c be as in 0.1. We must show that 0.2(c),0.2(d) hold.

If W is of type $A_n, n \geq 1$ we have $|c| = 1, \Gamma_c = S_1$. In this case, 0.2(c), 0.2(d) are trivial.

If W is of type B_n or C_n, $n \geq 2$ or $D_n, n \geq 4$, we can identify $\Gamma_c = V_D^1$ for some $D \in 2\mathbf{N}$. We can identify $M(\Gamma_c) = V_D$ as in [L19, 2.8(i)]. In these cases, 0.2(c), 0.2(d) follow from 1.18(e) and the proof of 1.18(f). Now $\mathcal{A}_c$ is the same as $\ddot{\Sigma}_D$ (see 1.11) in the symbol notation [L84] for objects of $\hat{W}$ (assuming that W is of type D and c is a cuspidal family).

2.2. In the remainder of this section we assume that W is of exceptional type. Then we are in one of the following cases.

$|c| = 1, \Gamma_c = S_1$;
$|c| = 2, \Gamma_c = S'_2$;
$|c| = 3, \Gamma_c = S_2$;
$|c| = 4, \Gamma_c = S'_3$;
$|c| = 5, \Gamma_c = S_3$;
$|c| = 11, \Gamma_c = S_4$;
$|c| = 17, \Gamma_c = S_5$.

Here S'_2 (resp. S'_3) is another copy of S_2 (resp. S_3).

For the elements $(\Gamma', \Gamma'') \in \bar{X}_{\Gamma_c}$ we use the notation of [L23]. Following [L23] we give for each $\Gamma'_0 \in \underline{\bar{X}}_{\Gamma_c}$ the list

$L(\Gamma'_0) = \{\Gamma''; (\Gamma', \Gamma'') \in \bar{X}_{\Gamma_c}^{\Gamma'_0}\}$.

Assume that $|c| = 1$. Then $L(S_1) = \{S_1\}$.

Assume that $|c| \in \{2, 3\}$. Then
$L(S_1) = \{S_2, S_1\}$, $L(S_2) = \{S_2\}$.

Assume that $|c| \in \{4, 5\}$. Then
$L(S_1) = \{S_3, S_2, S_1\}$, $L(S_2) = \{S_2\}$, $L(S_3) = \{S_3\}$.

Assume that $|c|=11$. Then
$L(S_1)=\{S_4,S_3,S_2S_2,S_2,S_1\}$, $L(S_2)=\{S_2S_2,S_2\}$,
$L(S_2S_2)=\{\Delta_8,S_2S_2\}$, $L(S_3)=\{S_3\}$, $L(\Delta_8)=\{\Delta_8\}$, $L(S_4)=\{S_4\}$.
Assume that $|c|=17$. Then
$L(S_1)=\{S_5,S_4,S_3S_2,S_3,S_2S_2,S_2,S_1\}$, $L(S_2)=\{S_3S_2,S_2S_2,S_2\}$,
$L(S_2S_2)=\{\Delta_8,S_2S_2\}$, $L(S_3)=\{S_3S_2,S_3\}$, $L(\Delta_8)=\{\Delta_8\}$, $L(S_3S_2)=\{S_3S_2\}$,
$L(S_4)=\{S_4\}$, $L(S_5)=\{S_5\}$.
In each case we see that $L(\Gamma'_0)$ contains a unique term with $||$ maximum. (It is the first term of $L(\Gamma'_0)$.) We see that 0.2(c) holds in each case. Now 0.2(d) can be easily verified using the tables in [L20,§3].

2.3. Applying ϵ to $\mathbf{s}(\Gamma',\Gamma'')$ for each Γ'' in the list $L(\Gamma'_0)$ (recall that $(\Gamma',\Gamma'')\in\bar{X}_{\Gamma_c}^{\Gamma'_0}$) we obtain a list $L'(\Gamma'_0)$ of elements in $M(\Gamma_c)$; we write in the same order as the elements of $L(\Gamma'_0)$. (The notation for elements in $M(\Gamma_c)$ is taken from [L84].)
Assume that $|c|=1$. Then $L'(S_1)=\{(1,1)\}$.
Assume that $|c|\in\{2,3\}$. Then $L'(S_1)=\{(1,1),(1,\epsilon)\}$, $L'(S_2)=\{(g_2,1)\}$.
Assume that $|c|\in\{4,5\}$. Then
$L'(S_1)=\{(1,1),(1,r),(1,\epsilon)\}$, $L'(S_2)=\{(g_2,1)\}$, $L'(S_3)=\{(g_3,1)\}$.
Assume that $|c|=11$. Then
$L'(S_1)=\{(1,1),(1,\lambda^1),(1,\sigma),(1,\lambda^2),(1,\lambda^3)\}$, $L'(S_2)=\{(g_2,1),(g_2,\epsilon'')\}$,
$L'(S_2S_2)=\{(g'_2,1),(g'_2,\epsilon'')\}$, $L'(S_3)=\{(g_3,1)\}$, $L'(\Delta_8)=\{(g'_2,\epsilon')\}$, $L'(S_4)=\{(g_4,1)\}$.
Assume that $|c|=17$. Then
$L'(S_1)=\{(1,1),(1,\lambda^1),(1,\nu),(1,\lambda^2),(1,\nu'),(1,\lambda^3),(1,\lambda^4)\}$,
$L'(S_2)=\{(g_2,1),(g_2,r),(g_2,\epsilon)\}$, $L'(S_2S_2)=\{(g'_2,1),(g'_2,\epsilon'')\}$,
$L'(S_3)=\{(g_3,1),(g_3,\epsilon)\}$, $L'(\Delta_8)=\{(g'_2,\epsilon')\}$, $L'(S_3S_2)=\{(g_6,1)\}$,
$L'(S_4)=\{(g_4,1)\}$, $L'(S_5)=\{(g_5,1)\}$.
The almost special representations in c are represented by the first term in each list. They are as follows.
If $|c|=1$ we have $\mathcal{A}_{\Gamma_c}=\{(1,1)\}$.
If $|c|\in\{2,3\}$ we have $\mathcal{A}_{\Gamma_c}=\{(1,1),(g_2,1)\}$.
If $|c|\in\{4,5\}$ we have $\mathcal{A}_{\Gamma_c}=\{(1,1),(g_2,1),(g_3,1)\}$.
If $|c|=11$ we have $\mathcal{A}_{\Gamma_c}=\{(1,1),(g_2,1),(g'_2,1),(g_3,1),(g'_2,\epsilon'),(g_4,1)\}$.
If $|c|=17$ we have
$\mathcal{A}_{\Gamma_c}=\{(1,1),(g_2,1),(g'_2,1),(g_3,1),(g'_2,\epsilon'),(g_6,1),(g_4,1),(g_5,1)\}$.

2.4. In the case where $|c|=17$ we have that W must of type E_8. An element of each list $L'(\Gamma'_0)$ can be identified with an element of c (under the imbedding $c\subset M(\Gamma_c)$) represented by its dimension (with the single exception of $(1,\lambda^4)$). Then the lists $L'(\Gamma'_0)$ become:
$L'(S_1)=\{4480,5670,4536,1680,1400,70,?\}$,
$L'(S_2)=\{7168,5600,448\}$,
$L'(S_2S_2)=\{4200,2688\}$,
$L'(S_3)=\{3150,1134\}$,

$L'(\Delta_8)=\{168\}$, $L'(S_3S_2)=\{2016\}$,
$L'(S_4)=\{1344\}$, $L'(S_5)=\{420\}$.

Note that the first representation in a given list $L'(\Gamma'_0)$ has the b-invariant (see [L84, (4,1,2)]) strictly smaller than the b-invariant of any subsequent representation in the list. (We expect that this property holds for any c.) This property is similar to the defining property of special representations [L79a] and justifies the name of "almost special" representations.

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