IMMORTAL SOLUTIONS OF THE KÄHLER-RICCI FLOW

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In memory of Steve Zelditch

Abstract. We survey some recent developments on solutions of the Kähler-Ricci flow on compact Kähler manifolds which exist for all positive times.

1. Introduction

Let (X^n, ω_0) be a compact Kähler manifold, and consider the Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)), \quad \omega(0) = \omega_0. \tag{1.1}$$

Up to rescaling the time parameter by a factor of 2, this flow is precisely Hamilton's Ricci flow [28] starting at the Kähler metric ω_0. As noticed already by Hamilton [28, p.257], the evolved metrics remain Kähler (with respect to the fixed complex structure on X), and so we can identify them with their Kähler forms $\omega(t)$.

A key observation of Cao [3] is that Kähler-Ricci flow, which is an evolution equation for a metric tensor, is in fact equivalent to a parabolic PDE for a scalar function. Indeed, if we let

$$\alpha(t) := \omega_0 - t\mathrm{Ric}(\omega_0), \tag{1.2}$$

which is a family of closed real $(1,1)$-forms (positive definite for $t > 0$ sufficiently small, but not necessarily for t large), then it is easy to see (see e.g. [55, §3.2]) that a family $\omega(t)$ of Kähler metrics on X (with $t \in [0, T)$ for some $T > 0$) solves (1.1) if and only if it is of the form

$$\omega(t) = \alpha(t) + i\partial\overline{\partial}\varphi(t), \tag{1.3}$$

where the smooth functions $\varphi(t)$ (which vary smoothly in $t \in [0, T)$) satisfy

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t) = \log \frac{(\alpha(t) + i\partial\overline{\partial}\varphi(t))^n}{\omega_0^n} \\ \varphi(0) = 0 \\ \alpha(t) + i\partial\overline{\partial}\varphi(t) > 0. \end{cases} \tag{1.4}$$

This is a (strictly) parabolic complex Monge-Ampère equation, which is a fully nonlinear PDE (as long as $n \geqslant 2$). Thus, short time existence of a unique smooth solution $\varphi(t)$ is guaranteed by standard PDE theory, with

$t \in [0, T_{\max})$ where $0 < T_{\max} \leqslant +\infty$ is the maximal time of (forward) existence (see also Hamilton's paper [28] for the original approach, which works for general Ricci flows on closed manifolds).

A solution $\omega(t)$ of the Kähler-Ricci flow (1.1) is called *immortal* if $T_{\max} = +\infty$. The goal of this note is to give an overview of what is known about such immortal solutions, and formulate some well-known open problems.

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2. When do immortal solutions exist?

Suppose (X^n, ω_0) is a compact Kähler manifold and $\omega(t), t \in [0, T_{\max})$, is a solution of the Kähler-Ricci flow (1.1), with maximal existence time $0 < T_{\max} \leqslant +\infty$. The Ricci form $\mathrm{Ric}(\omega)$ of any Kähler metric ω on X is a closed real $(1,1)$-form whose cohomology class

$$[\mathrm{Ric}(\omega)] \in H^{1,1}(X, \mathbb{R}), \tag{2.5}$$

is independent of ω, and equals $2\pi c_1(X)$, where $c_1(X)$ denotes the first Chern class of the manifold X. Thus, if in the flow equation (1.1) we pass to cohomology classes, we get the ODE

$$\frac{\partial}{\partial t}[\omega(t)] = -2\pi c_1(X), \quad [\omega(0)] = [\omega_0], \tag{2.6}$$

whose solution is given by

$$[\omega(t)] = [\omega_0] - 2\pi t c_1(X). \tag{2.7}$$

In particular, for $t \in [0, T_{\max})$ the cohomology class $[\omega_0] - 2\pi t c_1(X)$ contains a Kähler metric, and thus lies in the Kähler cone

$$\mathcal{C}_X \subset H^{1,1}(X, \mathbb{R}),$$

of cohomology classes which contain a Kähler metric. The following theorem of Tian-Zhang [52], which improved earlier results of Cao [3] and Tsuji [59, 60], gives a nice cohomological characterization of $T_{\max}$ (see also [49, Theorem 3.3.1], [55, Theorem 3.1] and [62, Theorem 3.1] for detailed expositions):

Theorem 2.1. *Given (X^n, ω_0) a compact Kähler manifold, the maximal existence time $T_{\max}$ of the Kähler-Ricci flow* (1.1) *is given by*

$$T_{\max} = \sup\{t > 0 \mid [\omega_0] - 2\pi t c_1(X) \in \mathcal{C}_X\}. \tag{2.8}$$

It follows that $T_{\max} = +\infty$ is equivalent to $-c_1(X) \in \overline{\mathcal{C}_X}$.

This shows that if X admits a Kähler metric ω_0 for which the Kähler-Ricci flow starting at ω_0 is immortal, the same will hold for *any* other Kähler metric on X. Now, cohomology classes in the closure $\overline{\mathcal{C}_X}$ are traditionally called *nef*, and $-2\pi c_1(X) = c_1(K_X)$ where $K_X = \Lambda^n T^{1,0}X^*$ denotes the canonical bundle of X. Thus, the condition $-c_1(X) \in \overline{\mathcal{C}_X}$ is often stated by saying that the canonical bundle K_X is nef. When X is projective algebraic, such manifolds are also known as smooth minimal models in birational geometry.

In summary, compact Kähler manifolds which support immortal solutions of the Kähler-Ricci flow are exactly those for which the canonical bundle is nef, and on such manifolds all solutions of the Kähler-Ricci flow are immortal. Examples of such manifolds include compact Riemann surfaces of genus $g \geqslant 1$, smooth hypersurfaces in $\mathbb{P}^{n+1}$ of degree $\geqslant n+2$, and products of these.

Using the Calabi-Yau Theorem [64] we can give a more geometric reformulation of the condition that K_X be nef. Indeed, after fixing an arbitrary Kähler metric ω on X, it is easy to see [55, Lemma 2.2] that $-2\pi c_1(X)$ is nef if and only if for every $\varepsilon > 0$ there is a smooth closed real $(1,1)$-form η_ε cohomologous to $-2\pi c_1(X)$ such that $\eta_\varepsilon \geqslant -\varepsilon\omega$ holds on X. By the Calabi-Yau Theorem [64] we can find a Kähler metric ω_ε such that $-\mathrm{Ric}(\omega_\varepsilon) = \eta_\varepsilon \geqslant -\varepsilon\omega$, or in other words

$$\mathrm{Ric}(\omega_\varepsilon) \leqslant \varepsilon\omega. \tag{2.9}$$

In other words, K_X is nef if and only if for every $\varepsilon > 0$ there is a Kähler metric ω_ε on X that satisfies (2.9) (i.e. ω_ε has "almost nonpositive Ricci curvature" in a certain sense). The choice of ω is of course irrelevant, as any two Kähler metrics on X are uniformly equivalent.

Combining Theorem 2.1 with Demailly-Păun's numerical characterization of $\mathcal{C}_X$ in [11], we obtain the following geometric characterization of $T_{\max}$, which was conjectured by Feldman-Ilmanen-Knopf in [14, p.204]:

Theorem 2.2. *Given (X^n, ω_0) a compact Kähler manifold, the maximal existence time $T_{\max}$ of the Kähler-Ricci flow* (1.1) *is given by*

$$T_{\max} = \inf \left\{ t > 0 \;\middle|\; \text{there exists } V \subset X \, s.t. \int_V ([\omega_0] - 2\pi t c_1(X))^{\dim V} = 0 \right\}, \tag{2.10}$$

where here V is a closed irreducible positive-dimensional analytic subvariety of X.

In other words, if the flow ceases to exist at a finite time $T_{\max}$, there has to be such a subvariety V whose volume in the induced evolving metrics

$$\mathrm{Vol}(V, \omega(t)) = \frac{1}{(\dim V)!} \int_V \omega(t)^{\dim V} = \frac{1}{(\dim V)!} \int_V ([\omega_0] - 2\pi t c_1(X))^{\dim V}$$

is converging to zero as $t \uparrow T_{\max}$. The union of all such subvarieties is itself a closed analytic subvariety, and it is in fact equal to the singularity formation set of the flow (where the curvature of $\omega(t)$ is blowing up), as proved by Collins and the author [7] solving another conjecture in [14].

3. Volume growth of immortal solutions

Given (X^n, ω_0) a compact Kähler manifold with K_X nef, we know from Theorem 2.1 that the solution $\omega(t)$ of the Kähler-Ricci flow (1.1) starting at ω_0 is immortal. The main (vague) question is then the following:

Question 3.1. *What is the behavior of* $(X, \omega(t))$ *as* $t \to +\infty$*?*

To start getting a feeling for this question, let us examine the behavior of the total volume of X with respect to the evolving metrics. Using Stokes' Theorem we have

$$\mathrm{Vol}(X, \omega(t)) = \frac{1}{n!}\int_X \omega(t)^n = \frac{1}{n!}\int_X ([\omega_0] - 2\pi t c_1(X))^n, \tag{3.11}$$

and this is a polynomial in t of degree equal to the *numerical dimension* of K_X

$$m := \max\{k \geqslant 0 \mid [-c_1(X)]^k \neq 0 \text{ in } H^{2k}(X, \mathbb{R})\}, \quad 0 \leqslant m \leqslant n, \tag{3.12}$$

since

$$\frac{1}{n!}\int_X ([\omega_0] - 2\pi t c_1(X))^n = t^m \frac{(2\pi)^m \binom{n}{m}}{n!} \underbrace{\int_X \omega_0^{n-m} \wedge (-c_1(X))^m}_{>0} + O(t^{m-1}). \tag{3.13}$$

We will now consider separately the behavior of the flow according to the value of the numerical dimension m.

4. The case $m = 0$

The first case to consider is when $m = 0$, which by definition means that $c_1(X) = 0$ in $H^2(X, \mathbb{R})$. Compact Kähler manifolds with this property are known as *Calabi-Yau*, and a fundamental theorem of Yau [64] shows that every Kähler class contains precisely one Ricci-flat Kähler metric. In this case, the behavior of the Kähler-Ricci flow is well-understood thanks to a classic result of Cao [3], see also [49, §3.4.2–3.4.3], [62, §4.2] for expositions:

Theorem 4.1. *Let* X^n *be a compact Calabi-Yau manifold, and* ω_0 *a Kähler metric on* X*. Then the solution* $\omega(t)$ *of the Kähler-Ricci flow* (1.1) *starting at* ω_0 *converges smoothly as* $t \to +\infty$ *to the unique Ricci-flat Kähler metric cohomologous to* ω_0*.*

Furthermore, this convergence is exponentially fast (in all C^k norms). A proof of this folklore statement can be found in [58, p.2941], using the method of [41].

5. The case $m = n$

Next, we assume that $m = n$. From (3.13) we see that in this case $\mathrm{Vol}(X, \omega(t))$ grows to infinity as $t \to +\infty$, and so if we hope to obtain convergence to a reasonable limit we will have to renormalize the flow. The standard renormalization is to consider

$$\tilde{\omega}(s) := e^{-s}\omega(e^s - 1), \quad s \in [0, +\infty), \tag{5.14}$$

which now satisfies the Kähler-Ricci flow

$$\frac{\partial}{\partial s}\tilde{\omega}(s) = -\mathrm{Ric}(\tilde{\omega}(s)) - \tilde{\omega}(s), \quad \tilde{\omega}(0) = \omega_0. \tag{5.15}$$

and

$$\mathrm{Vol}(X, \tilde{\omega}(s)) = \frac{(2\pi)^n}{n!}\int_X (-c_1(X))^n + O(e^{-s}). \tag{5.16}$$

The condition that $m = n$ is equivalent to $\int_X (-c_1(X))^n > 0$, which is usually stated by saying that K_X is nef and big. An important class of manifolds that satisfy this condition are those for which $-c_1(X) \in \mathcal{C}_X$, in which case we say that K_X is ample.

The following result was first obtained by Cao [3] when K_X is ample (see [49, §3.4.1], [62, §4.1] for expositions), and by Tsuji [59] and Tian-Zhang [52] in general (see [55, Theorem 5.3] for an exposition):

Theorem 5.1. *Let X^n be a compact Kähler manifold with K_X nef and big, and let*

$$Z := \bigcup_{V \subset X,\ \int_V (-c_1(X))^{\dim V} = 0} V. \tag{5.17}$$

Then Z is a proper closed analytic subvariety of X, and there is a Kähler-Einstein metric ω_∞ on $X \backslash Z$ that satisfies

$$\mathrm{Ric}(\omega_\infty) = -\omega_\infty.$$

Given any Kähler metric ω_0 on X, the solution $\tilde{\omega}(s)$ of the Kähler-Ricci flow (5.15) *starting at ω_0 converges to ω_∞ locally smoothly on $X \backslash Z$ as $s \to +\infty$.*

It was also shown by Zhang [67] that the scalar curvature of $\tilde{\omega}(s)$ is uniformly bounded independent of s. The subvariety Z is empty if and only if K_X is ample. In this case, the Kähler-Einstein metric ω_∞ was constructed by Aubin [1] and Yau [64].

In general, a compact Kähler manifold with K_X nef and big must be projective (by a classical result of Moishezon, see e.g. [39, Theorem 2.2.26]), and by a result of Kawamata-Shokurov [36, Theorem 2.6] there is a birational morphism $f : X \to Y$ onto a normal projective variety, with exceptional locus $\mathrm{Exc}(f) = Z$. It is then shown by Wang [61] (see also [27, 53] for earlier results, and upcoming work of Guo-Song for a new proof) that $(X, \tilde{\omega}(s))$ converges as $s \to +\infty$ in the Gromov-Hausdorff topology to the metric completion of $(X \backslash Z, \omega_\infty)$, which is a compact metric space homeomorphic to Y by [43].

6. The case $0 < m < n$: Abundance conjecture

In order to answer Question 3.1, it thus remains to address the case when $0 < m < n$, which is usually referred to as "intermediate Kodaira dimension". This is by far the hardest case, and it is open in general. However, decisive progress has been made under the additional assumption that K_X be semiample, which conjecturally always holds, as we now explain.

More precisely, we say that K_X is semiample if there is some $\ell \geqslant 1$ such that K_X^ℓ is globally generated, which means that given any $x \in X$ there is a global section $s \in H^0(X, K_X^\ell)$ with $s(x) \neq 0$. We refer the reader to [37, §2.1.B] for a thorough discussion of semiample line bundles, including the basic fact (which follows from [37, Theorem 2.1.27]) that semiample line bundles are always nef.

The fundamental *Abundance conjecture* in algebraic geometry (and its natural extension to Kähler manifolds) predicts that for the canonical bundle the converse holds as well:

Conjecture 6.1. *Let X be a compact Kähler manifold with K_X nef. Then K_X is semiample.*

The Abundance conjecture is known when $n \leqslant 3$ by [2, 8, 9].

Motivated by the Abundance conjecture, we will assume in the following section that K_X is not just nef but actually semiample, and discuss recent progress which gives a rather satisfactory picture of the behavior of $(X, \omega(t))$. We will then return to the general case of K_X nef in section 8, to discuss what is known and what is expected in that case.

7. The case $0 < m < n$: semiample canonical bundle

In this section we assume that X^n is a compact Kähler manifold with K_X semiample (hence in particular nef). As explained in [37, Theorem 2.1.27], global sections of K_X^ℓ for ℓ sufficiently divisible give a holomorphic map $f : X \to Y$, with connected fibers onto an irreducible normal projective variety Y with $\dim Y = m$, and $-c_1(X) = f^*[\omega_Y]$ for some Kähler metric ω_Y on Y (understood in the sense of analytic spaces [40] if Y is not smooth). In particular,

$$\chi := f^*\omega_Y, \tag{7.18}$$

is a smooth closed real $(1,1)$-form on X which is semipositive definite and cohomologous to $-2\pi c_1(X)$ (smoothness follows from standard properties of differential forms on analytic spaces).

There is a proper closed analytic subvariety $D \subset Y$ such that $Y^\circ := Y \backslash D$ is smooth and does not contain any critical values of f. This means that if we define $X^\circ := X \backslash f^{-1}(D)$, then $f : X^\circ \to Y^\circ$ is a proper holomorphic submersion (hence in particular a C^∞ fiber bundle, by Ehresmann's Lemma). The fibers $X_y = f^{-1}(y), y \in Y^\circ$ are automatically Calabi-Yau $(n-m)$-folds (by adjunction) which are pairwise diffeomorphic, but may not be pairwise biholomorphic in general. The fibers of f contained in $f^{-1}(D)$ on the other hand are referred to as singular fibers.

The simplest example of such fiber spaces are products $X = Y \times F$ where Y is a compact Kähler manifold with K_Y ample, and F is Calabi-Yau, with the map f being the projection onto the Y factor.

As before, it is most convenient to study the Kähler-Ricci flow on X with the normalization as above (now switching notation though)

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) - \omega(t), \quad \omega(0) = \omega_0. \tag{7.19}$$

The solution $\omega(t)$ exists for all $t \geqslant 0$, and is cohomologous to

$$\hat{\omega}(t) := e^{-t}\omega_0 + (1 - e^{-t})\chi, \tag{7.20}$$

which are Kähler metrics on X for all $t \geqslant 0$.

The study of the Kähler-Ricci flow in this setting was initiated by Song-Tian in [44] when $n = 2, m = 1$, and in [45] in general. Before we can describe the picture in more detail, we need some definitions.

7.1. **The Weil-Petersson form.** First, on Y° there is a closed real $(1,1)$-form ω_{WP} (the *Weil-Petersson form*), which is semipositive definite and encodes the variation of the complex structure of the Calabi-Yau fibers X_y. In particular, ω_{WP} is identically zero if and only if all fibers $X_y, y \in Y^\circ$ are biholomorphic to each other. By the Fischer-Grauert theorem [15], this happens if and only if $f : X^\circ \to Y^\circ$ is a holomorphic fiber bundle, and in this case we say that $f : X \to Y$ is *isotrivial*. We refer the reader to [45] or [55, §5.6] for the precise construction of ω_{WP}, which originates in [19].

7.2. **The twisted Kähler-Einstein metric.** Next, by solving an elliptic complex Monge-Ampère equation on Y, Song-Tian [45] show that there exists a Kähler metric ω_{can} on Y°, of the form $\omega_{\mathrm{can}} = \omega_Y + i\partial\overline{\partial}u$, that satisfies the twisted Kähler-Einstein equation

$$\mathrm{Ric}(\omega_{\mathrm{can}}) = -\omega_{\mathrm{can}} + \omega_{\mathrm{WP}}. \tag{7.21}$$

For example, in the above-mentioned product setting where $X = Y \times F$, we have of course $Y^\circ = Y$ and $\omega_{\mathrm{WP}} = 0$ (since all the fibers are biholomorphic to F), so in this case ω_{can} is simply the Kähler-Einstein metric on Y constructed by Aubin and Yau.

7.3. **The semi-Ricci-flat form.** The last object that we will need is the semi-Ricci-flat form ω_{SRF} on X°. To define this, for each given $y \in Y^\circ$ we apply the Calabi-Yau theorem to $(X_y, \omega_0|_{X_y})$, which gives us a unique Ricci-flat Kähler metric on X_y of the form

$$\omega_0|_{X_y} + i\partial\overline{\partial}\rho_y > 0, \quad \int_{X_y} \rho_y \omega_0^{n-m} = 0, \tag{7.22}$$

and the smooth function $\rho_y : X_y \to \mathbb{R}$ is uniquely determined thanks to its integral normalization. From Yau's a priori estimates [64], it follows that ρ_y varies smoothly in $y \in Y^\circ$, so letting y vary these define a smooth function

$\rho : X^\circ \to \mathbb{R}$, so that $\rho|_{X_y} = \rho_y$. We then define a closed real $(1,1)$-form ω_{SRF} on X° by

$$\omega_{\mathrm{SRF}} := \omega_0 + i\partial\overline{\partial}\rho. \tag{7.23}$$

It is called semi-Ricci-flat because its fiberwise restrictions $\omega_{\mathrm{SRF}}|_{X_y}$ are Ricci-flat Kähler metrics. However, it is important to note that ω_{SRF} may fail to be semipositive definite on X°, see [4] for a counterexample.

7.4. **Collapsing of the Kähler-Ricci flow.** With these notations set up, we can now discuss what is known about the behavior of the evolving metrics $\omega(t)$ solving the normalized flow (7.19) on X. The first result, proved by Song-Tian [45] is the following:

Theorem 7.1. *Let (X^n, ω_0) be a compact Kähler manifold with K_X semi-ample and numerical dimension $0 < m < n$, and let $\omega(t)$ be the solution of* (7.19). *Then, as $t \to +\infty$, the metrics $\omega(t)$ converge to $f^*\omega_{\mathrm{can}}$ in the weak topology of currents on X.*

They also posed the following in [44, p.612], [45, p.306], [50, Conjecture 4.5.7], [51, p.258 and Conjecture 4.7]:

Conjecture 7.2. *In the same setting as Theorem 7.1, we have*

(a) *$\omega(t) \to f^*\omega_{\mathrm{can}}$ in the locally smooth topology on X°*
(b) *Given $K \Subset X^\circ$ there is $C > 0$ such that for all $t \geqslant 0$ we have*

$$\sup_K |\mathrm{Ric}(\omega(t))|_{\omega(t)} \leqslant C. \tag{7.24}$$

To clarify, item (a) means that given any $K \Subset X^\circ$ and $k \geqslant 0$, we have

$$\|\omega(t) - f^*\omega_{\mathrm{can}}\|_{C^k(K,\omega_0)} \to 0, \quad \text{as } t \to +\infty. \tag{7.25}$$

while item (b) simply means that the Ricci curvature of $\omega(t)$ remains uniformly bounded on K, independent of t.

The first progress towards establishing (a) was done by Fong-Zhang [17], adapting a method by the author [54] in a related elliptic PDE, who showed that the Kähler potentials of $\omega(t)$ (with respect to $\hat{\omega}(t)$) converge to f^*u (the potential of $f^*\omega_{\mathrm{can}}$) in $C^{1,\alpha}(K,\omega_0)$ for all $\alpha < 1$. This however falls short of proving (7.25) for $k = 0$, and this was only achieved later by Weinkove, Yang and the author in [57] with substantial more work (see [55, §5.5–5.13] for a detailed exposition). Using this result, Zhang and the author [58] identified the next order behavior of $\omega(t)$ when restricted to a smooth fiber, which is

$$\|(\omega(t) - e^{-t}\omega_{\mathrm{SRF}})|_{X_y}\|_{C^k(X_y,\omega_0|_{X_y})} = o(e^{-t}), \tag{7.26}$$

for any $y \in Y^\circ$ and $k \geqslant 0$. Next, it was observed in [17, 31, 58] (see also [55, §5.14] for a unified exposition, and [20] for the case of products) that Conjecture 7.2 (a) holds in the case when the smooth fibers X_y are tori or finite quotients of tori. This used crucially ideas of Gross, Zhang and the author [21] in the elliptic setting. Further progress was made by Fong-Lee [16] who proved Conjecture 7.2 (a) when f is isotrivial, and by Chu-Lee [5] who proved (7.25) for the Hölder norm C^α, $0<\alpha<1$. Both of these works are based on ideas of Hein and the author [32] in the elliptic case.

About Conjecture 7.2 (b), Song-Tian [46] showed that the scalar curvature of $\omega(t)$ is uniformly bounded on all of X, while Zhang and the author [58] showed that the full Riemann curvature tensor of $\omega(t)$ remains uniformly bounded on compact subsets $K \Subset X^\circ$ if and only if the smooth fibers X_y are tori or finite quotients of tori (see also [55, p.364]). Thus, the conjectured Ricci bound (7.24) is in some sense optimal. The bound (7.24) was established in [16] when f is isotrivial.

Very recently, Hein, Lee and the author [30] were able to confirm Conjecture 7.2 in general:

Theorem 7.3. *Conjecture 7.2 holds.*

This result can be thought of as a parabolic version of the work [33] by Hein and the author, in the elliptic setting. To prove (7.25), the authors prove a much more precise asymptotic expansion for $\omega(t)$, as follows. Given $k \geqslant 0$, and given a sufficiently small coordinate ball $B \subset Y^\circ$ (over which f is smoothly trivial, so $f^{-1}(B)$ is diffeomorphic to $B \times F$ for some Calabi-Yau manifold F), they show that, up to shrinking B, we can write on $B \times F$ for all $t \geqslant 0$,

$$\omega(t) = f^*\omega_{\mathrm{can}} + e^{-t}\omega_{\mathrm{SRF}} + \gamma_0(t) + \sum_{j=1}^{k} \gamma_{j,k}(t) + \eta_k(t), \tag{7.27}$$

where $\gamma_0(t)$ is pulled back from B and goes to zero in $C^k(B)$, the $\gamma_{j,k}(t), 1 \leqslant j \leqslant k$ go to zero in $C^k(B\times F, \omega_0)$ (at different speeds that depend on j), and the remainder $\eta_k(t)$ goes to zero in an even stronger t-dependent “shrinking” C^k norm on $B \times F$. This description clearly implies Conjecture 7.2 (a), and then plugging in this expansion with $k=2$ into the definition of $\mathrm{Ric}(\omega(t))$ and employing explicit estimates for the pieces of the decomposition (7.27) we show Conjecture 7.2 (b).

In the two aforementioned special cases when f is isotrivial, or when the smooth fibers are tori or finite quotients, it turns out that all the terms $\gamma_{j,k}(t)$ in (7.27) vanish identically, so the result is stronger (and the proof much easier). In general however these terms do not vanish, and simply identifying them requires very substantial work. Once the pieces of the decomposition (7.27) have been defined, the required *a priori* estimates on all the pieces are proved by an intricate argument by contradiction and blowup, in the spirit of [33]. Furthermore, the estimates proved are actually in stronger parabolic Hölder norms.

7.5. **Gromov-Hausdorff limits.** While the results in the previous subsection give very strong estimates for $\omega(t)$ away from the singular fibers of f, these estimates blow up much too fast near the singular fibers to give us useful information. Nevertheless, Song-Tian [44, 45, 47] conjectured that $(X, \omega(t))$ should have a Gromov-Hausdorff limit as $t \to +\infty$, as follows:

Conjecture 7.4. *In the same setting as Theorem 7.1, we have*

(a) *There is $C>0$ such that for all $t\geqslant 0$,*

$$\mathrm{diam}(X,\omega(t))\leqslant C, \tag{7.28}$$

(b) *As $t\to+\infty$, $(X,\omega(t))$ converges in the Gromov-Hausdorff topology to (Z,d), the metric completion of $(Y^\circ,\omega_{\mathrm{can}})$*

(c) *Z is homeomorphic to Y.*

After much work on these questions, Conjecture 7.4 (a) was completely settled by Jian-Song [35]. As a consequence, they also deduced that sequential Gromov-Hausdorff limits exist, which is not a priori clear. Another breakthrough was achieved by Song-Tian-Zhang [48], who proved Conjecture 7.4 (b) and (c) in the case when $m=1$ and X_y tori, and combining their method with [35] settles Conjecture 7.4 (b) and (c) completely when $m=1$. Furthermore, [48] proved Conjecture 7.4 (c) when Y is smooth or has orbifold singularities. Lastly, Li and the author [38] proved Conjecture 7.4 (b) for arbitrary m assuming that Y is smooth and that the union $D^{(1)}$ of the codimension 1 irreducible components of D is a simple normal crossings divisor This last assumption (which always holds when $m=1$) is used to appeal to the estimates in [22] which show that in this case ω_{can} has conical singularities along $D^{(1)}$ (up to a small logarithmic error), and it extends smoothly across $D\backslash D^{(1)}$.

8. The case $0<m<n$: nef canonical bundle

In this section we assume that (X^n,ω_0) is a compact Kähler manifold with K_X nef and numerical dimension $0<m<n$, but we do not assume that it is semiample (although it is predicted to be so by the Abundance Conjecture 6.1). In this situation much less is known about the long-time behavior of the solution $\omega(t)$ of the Kähler-Ricci flow (7.19).

8.1. **Weak limit of the flow.** To start, we now let $\chi:=-\mathrm{Ric}(\omega_0)$, which is a closed real $(1,1)$-form cohomologous to $-2\pi c_1(X)$, and let

$$\alpha(t)=e^{-t}\omega_0+(1-e^{-t})\chi, \tag{8.29}$$

which is a family of closed real $(1,1)$-forms cohomologous to $\omega(t)$, with no positivity property in general. It is again easy to see [55, (5.30)] that (7.19) holds if and only if we can write

$$\omega(t)=\alpha(t)+i\partial\overline{\partial}\varphi(t), \tag{8.30}$$

with $\varphi(t)$ solving the parabolic Monge-Ampère equation

$$\begin{cases}\dfrac{\partial}{\partial t}\varphi(t)=\log\dfrac{e^{(n-m)t}(\alpha(t)+i\partial\overline{\partial}\varphi(t))^n}{\omega_0^n}-\varphi(t)\\ \varphi(0)=0\\ \alpha(t)+i\partial\overline{\partial}\varphi(t)>0.\end{cases} \tag{8.31}$$

on $X\times[0,+\infty)$. Let also h be the Hermitian metric on K_X naturally induced by ω_0, whose curvature form is equal to χ.

The following result is a simple consequence of work of Fu-Guo-Song [18] (see also [24, 25, 26]):

Theorem 8.1. *Let (X^n,ω_0) be a compact Kähler manifold with K_X nef and numerical dimension $0<m<n$, and let $\omega(t)$ be the immortal solution of* (7.19)*. Then*

(a) *Given any sequence $t_i\to\infty$ we can find a subsequence and a quasi-psh function $\varphi_\infty : X\to\mathbb{R}\cup\{-\infty\}$, with $\chi_\infty := \chi+i\partial\overline{\partial}\varphi_\infty \geqslant 0$ weakly, such that $\varphi(t_i)\to\varphi_\infty$ in $L^1(X)$ as $t\to\infty$. In particular $\omega(t_i)\to\chi_\infty$ weakly as currents. A priori χ_∞ may depend on the sequence.*

(b) *χ_∞ is a closed positive $(1,1)$-current with minimal singularities in the class $-2\pi c_1(X)$*

(c) *Given any $\ell\geqslant 1$ and any $s\in H^0(X,K_X^\ell)$, we have*

$$\int_X |s|^2_{h^\ell}e^{-\ell\varphi_\infty}\omega_0^n<\infty. \tag{8.32}$$

In part (b) the notion of minimal singularities in the class $[\chi]=-2\pi c_1(X)$, introduced in [12], means that given any quasi-psh function $\eta : X\to\mathbb{R}\cup\{-\infty\}$ with $\chi+i\partial\overline{\partial}\eta\geqslant 0$ weakly, there is $C>0$ such that

$$\varphi_\infty\geqslant\eta-C, \tag{8.33}$$

holds on X.

Property (c) is called "analytic Zariski decomposition" by Tsuji [59]. Note that a priori it may happen that $H^0(X,K_X^\ell)=0$ for all $\ell\geqslant 1$, although this is predicted not to happen as a consequence of the Abundance Conjecture.

Proof. (a) It is well-known (see e.g. [26, Lemma 4.3]) that there is $C>0$ such that for all $t\geqslant 0$,

$$\sup_X\varphi(t)\leqslant C,\quad \sup_X\dot{\varphi}(t)\leqslant C. \tag{8.34}$$

Thanks to the uniform upper bound for $\varphi(t)$, the statement in part (a) follows from basic compactness properties of quasi-psh functions (see e.g. [34, Theorem 3.2.12]), provided we show that φ does not converge to $-\infty$ uniformly on X. To see this, we use the Monge-Ampère equation (8.31) to get

$$\int_X e^{\varphi+\dot{\varphi}}\omega_0^n=e^{(n-m)t}\int_X\omega(t)^n=n!e^{(n-m)t}\mathrm{Vol}(X,\omega(t)), \tag{8.35}$$

and using Stokes we have

$$\begin{aligned}\mathrm{Vol}(X,\omega(t)) &= \frac{1}{n!}\int_X (e^{-t}\omega_0 - 2\pi(1-e^{-t})c_1(X))^n\\ &= e^{-(n-m)t}\frac{(2\pi)^m\binom{n}{m}}{n!}\underbrace{\int_X \omega_0^{n-m}\wedge(-c_1(X))^m}_{>0} + O(e^{-(n-m+1)t}),\end{aligned} \tag{8.36}$$

and so

$$\int_X e^{\varphi+\dot{\varphi}}\omega_0^n \geqslant C^{-1}, \tag{8.37}$$

which implies

$$\sup_X(\varphi(t)+\dot{\varphi}(t)) \geqslant -C. \tag{8.38}$$

But since $\dot{\varphi}(t)\leqslant C$, then (8.38) implies that $\sup_X \varphi(t) \geqslant -C$, as desired.

(b) By part (a) we know that given $t_i\to\infty$ we can find a subsequence and φ_∞ quasi-psh with $\chi+i\partial\overline{\partial}\varphi_\infty\geqslant 0$ such that $\varphi(t_i)\to\varphi_\infty$ in $L^1(X)$. We need to show that φ_∞ has minimal singularities. For this, we consider the envelope

$$V_\infty(x) = \sup\{\psi(x) \mid \chi + i\partial\overline{\partial}\psi \geqslant 0, \psi\leqslant 0\},$$

which satisfies $V_\infty\leqslant 0$, is quasi-psh with $\chi+i\partial\overline{\partial}V_\infty\geqslant 0$, and has minimal singularities: given any quasi-psh function $\eta : X\to\mathbb{R}\cup\{-\infty\}$ with $\chi + i\partial\overline{\partial}\eta\geqslant 0$ weakly, since η is usc there is $C>0$ such that $\eta\leqslant C$ on X, so taking $\psi=\eta-C$ in the definition of the envelope shows that $V_\infty\geqslant\eta-C$, as desired. We empasize here that we do not know whether V_∞ is bounded on X. If we show that

$$\|\varphi_\infty - V_\infty\|_{L^\infty(X)}\leqslant C, \tag{8.39}$$

then it follows immediately that φ_∞ also has minimal singularities. To prove (8.39), we first use (8.34) and see that

$$\omega(t)^n \leqslant Ce^{-(n-m)t}\omega_0^n. \tag{8.40}$$

We can then appeal to [18, Prop. 1.1] (which was recently reproved and improved in [25, Theorem 1]) and obtain that

$$\|\varphi(t) - V_t\|_{L^\infty(X)}\leqslant C, \tag{8.41}$$

for all $t\geqslant 0$, where

$$V_t(x) = \sup\{\psi(x) \mid \alpha(t) + i\partial\overline{\partial}\psi\geqslant 0, \psi\leqslant 0\}, \tag{8.42}$$

which satisfies $V_t\leqslant 0$, is quasi-psh, and $\alpha(t)+i\partial\overline{\partial}V_t\geqslant 0$ weakly. Since the class $\alpha(t)$ is Kähler, the functions V_t are in $C^{1,1}(X)$ by [6, 56].

We now claim that as $t \to \infty$, the functions V_t converge pointwise to V_∞. Indeed, we clearly have that $(1-e^{-t})V_\infty$ participates to the supremum defining V_t, and hence

$$V_\infty \leqslant \frac{V_t}{1-e^{-t}}, \tag{8.43}$$

while on the other hand for $t < s$, if $\psi \leqslant 0$ is quasi-psh and satisfies $\alpha(s) + i\partial\overline{\partial}\psi \geqslant 0$ weakly, then we also have

$$\alpha(t) + \frac{1-e^{-t}}{1-e^{-s}} i\partial\overline{\partial}\psi \geqslant \frac{1-e^{-t}}{1-e^{-s}}(\alpha(s) + i\partial\overline{\partial}\psi) \geqslant 0, \tag{8.44}$$

using that $\frac{e^{-s}}{1-e^{-s}} \leqslant \frac{e^{-t}}{1-e^{-t}}$, and so $\frac{1-e^{-t}}{1-e^{-s}}\psi$ participates to the supremum defining V_t, hence

$$\frac{\psi}{1-e^{-s}} \leqslant \frac{V_t}{1-e^{-t}}, \tag{8.45}$$

and taking here the supremum over all such ψ gives

$$\frac{V_s}{1-e^{-s}} \leqslant \frac{V_t}{1-e^{-t}}. \tag{8.46}$$

This shows that $\frac{V_t}{1-e^{-t}}$ decrease pointwise as $t \to \infty$ to some limit usc function $V'_\infty : X \to [-\infty, +\infty)$ which by (8.43) satisfies

$$V_\infty \leqslant V'_\infty \leqslant 0, \tag{8.47}$$

so in particular it is not identically $-\infty$. Observe that by dominated convergence we have $\frac{V_t}{1-e^{-t}} \to V'_\infty$ in $L^1(X)$, and so $\frac{1}{1-e^{-t}} i\partial\overline{\partial}V_t \to i\partial\overline{\partial}V'_\infty$ weakly as currents. Since

$$\frac{1}{1-e^{-t}}(\alpha(t) + i\partial\overline{\partial}V_t) \geqslant 0, \tag{8.48}$$

weakly, passing this to the weak limit as $t \to +\infty$ shows that $\chi + i\partial\overline{\partial}V'_\infty \geqslant 0$ weakly, so V'_∞ participates to the supremum defining V_∞, and hence

$$V'_\infty \leqslant V_\infty, \tag{8.49}$$

and from this and (8.47) we conclude that $V'_\infty = V_\infty$. We have thus shown that $\frac{V_t}{1-e^{-t}} \to V_\infty$ pointwise as $t \to \infty$, and hence also $V_t \to V_\infty$ pointwise, as claimed.

Passing to our sequence $t_i \to \infty$ as in part (a), we have that $\varphi(t_i)$ converges to φ_∞ in $L^1(X)$, and (up to passing to a further subsequence) also pointwise a.e. Passing to the limit in (8.41), and using the above claim, gives

$$|\varphi_\infty - V_\infty| \leqslant C, \tag{8.50}$$

a.e. on X, hence everywhere by elementary properties of psh functions (see e.g. [23, Theorem K.15]), and (8.39) follows.

(c) Given any $\ell \geqslant 1$ and any nontrivial $s \in H^0(X, K_X^\ell)$ (if it exists), since the metric h^ℓ on K_X^ℓ has curvature equal to $\ell\chi$, it follows from the Poincaré-Lelong equation that

$$\chi + \frac{1}{\ell} i\partial\overline{\partial} \log |s|^2_{h^\ell} \geqslant 0, \tag{8.51}$$

weakly. Thus, since φ_∞ has minimal singularities, it follows that there exists $C > 0$ such that

$$\varphi_\infty \geqslant \frac{1}{\ell} \log |s|^2_{h^\ell} - C, \tag{8.52}$$

or equivalently,

$$|s|^2_{h^\ell} e^{-\ell\varphi_\infty} \leqslant C, \tag{8.53}$$

and integrating this against ω_0^n gives (8.32). □

The following is then a very natural expectation (see also [68]):

Conjecture 8.2. *Let (X^n, ω_0) be a compact Kähler manifold with K_X nef and numerical dimension $0 < m < n$, and let $\omega(t)$ be the immortal solution of (7.19). Then there is a quasi-psh function φ_∞ with $\chi_\infty := \chi + i\partial\overline{\partial}\varphi_\infty \geqslant 0$ weakly, such that $\varphi(t) \to \varphi_\infty$ in $L^1(X)$ as $t \to +\infty$. In particular, $\omega(t) \to \chi_\infty$ weakly as currents as $t \to +\infty$. Furthermore, χ_∞ is independent also of the choice of the initial metric ω_0.*

In other words, the subsequential limits φ_∞ in Theorem 8.1 should be unique, and independent of the subsequence, and the current χ_∞ should be even independent of the initial metric. Of course, by Theorem 8.1, such χ_∞ would have minimal singularities. If Conjecture 8.2 is settled, then $he^{-\varphi_\infty}$ would be a singular metric on K_X with semipositive curvature current χ_∞, which up to scaling by a constant would be completely canonical, i.e. depending only on the complex structure of X.

Such a singular metric could be used for L^2 type arguments, especially if one can establish further regularity of φ_∞. When K_X is semiample, it is known by [13] that φ_∞ is continuous on X, and it may even be Hölder continuous. One can then ask about the following weaker properties:

Conjecture 8.3. *In the same setting as Conjecture 8.2, the following hold:*

(a) *φ_∞ has vanishing Lelong number at every point*
(b) *φ_∞ is bounded*

By a result of Skoda [42], if (a) holds then $e^{-A\varphi_\infty} \in L^1(X)$ for all $A > 0$, which implies that (8.32) holds. Thus, condition (a) can be viewed as a strengthening of (8.32). The implication (b)⇒(a) is well-known, and as mentioned above (b) holds when K_X is semiample.

These conditions can also be stated in terms of the smooth potentials $\varphi(t)$. Thanks to a result of Demailly-Kollár [10], condition (a) is equivalent to the fact that for every $A > 0$ there is $C > 0$ such that for all $t \geqslant 0$,

$$\int_X e^{-A\varphi(t)}\omega_0^n \leqslant C, \tag{8.54}$$

while condition (b) is equivalent to the existence of $C > 0$ such that for all $t \geqslant 0$,

$$\inf_X \varphi(t) \geqslant -C. \tag{8.55}$$

8.2. **Singularity type at infinity.** Following Hamilton [29], we say that an immortal solution of (5.15) develops a "Type III" singularity at infinity if there is $C > 0$ such that for all $t \geqslant 0$ we have

$$\sup_X |\mathrm{Rm}(\omega(t))|_{\omega(t)} \leqslant C, \tag{8.56}$$

and develops a "Type IIb" singularity if this fails. In [58], Zhang and the author gave an almost complete classification (complete when $n = 2$) of what the possible singularity type is, under the assumption that K_X is semiample, see also [55, Theorem 6.6]. In all the cases that they considered, the singularity type was actually independent of the initial metric ω_0, and the author conjectured in [55, Conjecture 6.7] that this should always be the case even just assuming that K_X is nef. Zhang [66] proved this conjecture when K_X is semiample, and very recently Wondo-Zhang [63] settled it completely.

It remains a very interesting problem to complete the classification of [58] in all dimensions, under the assumption that K_X is semiample. The only case which is left to understand is when the fiber space f has some singular fibers, and the smooth fibers X_y are all tori or finite quotients of tori.

8.3. **Diameter bounds.** Lastly, let us mention a recent striking result of Guo-Phong-Song-Sturm [24]: let (X^n, ω_0) be a compact Kähler manifold with K_X nef and numerical dimension $0 < m < n$, and let $\omega(t)$ be the immortal solution of (7.19). If we also suppose that $H^0(X, K_X^\ell) \neq 0$ for some $\ell \geqslant 1$ (i.e. that X has nonnegative Kodaira dimension), then there is $C > 0$ such that for all $t \geqslant 0$ we have

$$\mathrm{diam}(X, \omega(t)) \leqslant C, \tag{8.57}$$

and furthermore, given any $t_i \to +\infty$, up to passing to a subsequence we will have that $(X, \omega(t_i))$ converges in Gromov-Hausdorff to some compact metric space (Z, d). It would be of course very interesting to remove the condition that X have nonnegative Kodaira dimension (which is expected to hold by the Abundance conjecture).

References

[1] T. Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. (2) **102** (1978), no. 1, 63–95.

[2] F. Campana, A. Höring, T. Peternell, *Abundance for Kähler threefolds*, Ann. Sci. École Norm. Sup. (4) **49** (2016), no. 4, 971–1025; Erratum and addendum at `http://math.unice.fr/~hoering/articles/erratum-addendum-abundance.pdf`

[3] H.-D. Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. **81** (1985), no. 2, 359–372.
[4] J. Cao, H. Guenancia, M. Păun, *Variation of singular Kähler-Einstein metrics: Kodaira dimension zero. With an appendix by Valentino Tosatti*, J. Eur. Math. Soc. (JEMS) **25** (2023), no. 2, 633–679.
[5] J. Chu, M.C. Lee, *On the Hölder estimate of Kähler-Ricci flow*, Int. Math. Res. Not. IMRN 2023, no. 6, 4932–4951.
[6] J. Chu, B. Zhou, *Optimal regularity of plurisubharmonic envelopes on compact Hermitian manifolds*, Sci. China Math. **62** (2019), no. 2, 371–380.
[7] T.C. Collins, V. Tosatti, *Kähler currents and null loci*, Invent. Math. **202** (2015), no.3, 1167–1198.
[8] O. Das, W. Ou, *On the log abundance for compact Kähler threefolds*, Manuscripta Math. **173** (2024), no. 1-2, 341–404.
[9] O. Das, W. Ou, *On the log abundance for compact Kähler threefolds II*, preprint, arXiv:2306.00671.
[10] J.-P. Demailly, J. Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 4, 525–556.
[11] J.-P. Demailly, M. Păun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math., **159** (2004), no. 3, 1247–1274.
[12] J.-P. Demailly, T. Peternell, M. Schneider, *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic Geom. **3** (1994), no. 2, 295–345.
[13] S. Dinew, Z. Zhang, *On stability and continuity of bounded solutions of degenerate complex Monge-Ampère equations over compact Kähler manifolds*, Adv. Math. **225** (2010), no. 1, 367–388.
[14] M. Feldman, T. Ilmanen, D. Knopf, *Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons*, J. Differential Geom. **65** (2003), no. 2, 169–209.
[15] W. Fischer, H. Grauert, *Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1965), 89–94.
[16] F.T.-H. Fong, M.C. Lee, *Higher-order estimates of long-time solutions to the Kähler-Ricci flow*, J. Funct. Anal. **281** (2021), no. 11, Paper No. 109235, 34 pp.
[17] F.T.-H. Fong, Z. Zhang, *The collapsing rate of the Kähler-Ricci flow with regular infinite time singularity*, J. Reine Angew. Math. **703** (2015), 95–113.
[18] X. Fu, B. Guo, J. Song, *Geometric estimates for complex Monge-Ampère equations*, J. Reine Angew. Math. **765** (2020), 69–99.
[19] A. Fujiki, G. Schumacher, *The moduli space of extremal compact Kähler manifolds and generalized Weil-Petersson metrics*, Publ. Res. Inst. Math. Sci. 26 (1990), no. 1, 101–183.
[20] M. Gill, *Collapsing of products along the Kähler-Ricci flow*, Trans. Amer. Math. Soc. **366** (2014), no. 7, 3907–3924.
[21] M. Gross, V. Tosatti, Y. Zhang, *Collapsing of abelian fibered Calabi-Yau manifolds*, Duke Math. J. **162** (2013), no. 3, 517–551.
[22] M. Gross, V. Tosatti, Y. Zhang, *Geometry of twisted Kähler-Einstein metrics and collapsing*, Comm. Math. Phys. **380** (2020), no. 3, 1401–1438.
[23] R.C. Gunning, *Introduction to holomorphic functions of several variables. Vol. I. Function theory*, The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1990.
[24] B. Guo, D.H. Phong, J. Song, J. Sturm, *Diameter estimates in Kähler geometry*, to appear in Comm. Pure Appl. Math.
[25] B. Guo, D.H. Phong, F. Tong, C. Wang, *On L^∞ estimates for Monge-Ampère and Hessian equations on nef classes*, to appear in Anal. PDE
[26] B. Guo, J. Song, *Local noncollapsing for complex Monge-Ampère equations*, J. Reine Angew. Math. **793** (2022), 225–238.

[27] B. Guo, J. Song, B. Weinkove, *Geometric convergence of the Kähler-Ricci flow on complex surfaces of general type*, Int. Math. Res. Not. IMRN 2016, no. 18, 5652–5669.
[28] R.S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306.
[29] R.S. Hamilton, *The formation of singularities in the Ricci flow*, in *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*, 7–136, Int. Press, Cambridge, MA, 1995.
[30] H.-J. Hein, M.C. Lee, V. Tosatti, *Collapsing immortal Kähler-Ricci flows*, preprint.
[31] H.-J. Hein, V. Tosatti, *Remarks on the collapsing of torus fibered Calabi-Yau manifolds*, Bull. Lond. Math. Soc. **47** (2015), no. 6, 1021–1027.
[32] H.-J. Hein, V. Tosatti, *Higher-order estimates for collapsing Calabi-Yau metrics*, Camb. J. Math. **8** (2020), no. 4, 683–773.
[33] H.-J. Hein, V. Tosatti, *Smooth asymptotics for collapsing Calabi-Yau metrics*, preprint, arXiv:2102.03978.
[34] L. Hörmander, *Notions of convexity*, Progress in Mathematics, 127. Birkhäuser Boston, Inc., Boston, MA, 1994.
[35] W. Jian, J. Song, *Diameter estimates for long-time solutions of the Kähler–Ricci flow*, Geom. Funct. Anal. **32** (2022), no. 6, 1335–1356.
[36] Y. Kawamata, *The cone of curves of algebraic varieties*, Ann. of Math. (2) **119** (1984), no. 3, 603–633.
[37] R. Lazarsfeld, *Positivity in algebraic geometry I & II*, Springer-Verlag, Berlin, 2004.
[38] Y. Li, V. Tosatti, *On the collapsing of Calabi-Yau manifolds and Kähler-Ricci flows*, J. Reine Angew. Math. **800** (2023), 155–192.
[39] X. Ma, G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, Progress in Mathematics, 254. Birkhäuser Verlag, Basel, 2007.
[40] B.G. Moishezon, *Singular Kählerian spaces*, in *Manifolds–Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973)*, 343–351. Univ. Tokyo Press, Tokyo, 1975.
[41] D.H. Phong, J. Sturm, *On stability and the convergence of the Kähler-Ricci flow*, J. Differential Geom. **72** (2006), no. 1, 149–168.
[42] H. Skoda, *Sous-ensembles analytiques d'ordre fini ou infini dans* $\mathbb{C}^n$, Bull. Soc. Math. France **100** (1972), 353–408.
[43] J. Song, *Riemannian geometry of Kähler-Einstein currents*, preprint, arXiv:1404.0445.
[44] J. Song, G. Tian, *The Kähler-Ricci flow on surfaces of positive Kodaira dimension*, Invent. Math. **170** (2007), no. 3, 609–653.
[45] J. Song, G. Tian, *Canonical measures and Kähler-Ricci flow*, J. Amer. Math. Soc. **25** (2012), no. 2, 303–353.
[46] J. Song, G. Tian, *Bounding scalar curvature for global solutions of the Kähler-Ricci flow*, Amer. J. Math. **138** (2016), no. 3., 683–695.
[47] J. Song, G. Tian, *The KÄhler-Ricci flow through singularities*, Invent. Math. **207** (2017), no. 2, 519–595.
[48] J. Song, G. Tian, Z. Zhang, *Collapsing behavior of Ricci-flat Kähler metrics and long time solutions of the Kähler-Ricci flow*, preprint, arXiv:1904.08345.
[49] J. Song, B. Weinkove, *Introduction to the Kähler-Ricci flow*, Chapter 3 of 'Introduction to the Kähler-Ricci flow', eds S. Boucksom, P. Eyssidieux, V. Guedj, Lecture Notes Math. 2086, Springer 2013.
[50] G. Tian, *Notes on Kähler-Ricci flow*, in *Ricci flow and geometric applications*, 105–136, Lecture Notes in Math., 2166, Springer, 2016.
[51] G. Tian, *Some progresses on Kähler-Ricci flow*, Boll. Unione Mat. Ital. **12** (2019), no. 1-2, 251–263.
[52] G. Tian, Z. Zhang, *On the Kähler-Ricci flow on projective manifolds of general type*, Chinese Ann. Math. Ser. B **27** (2006), no. 2, 179–192.

[53] G. Tian, Z.L. Zhang, *Convergence of Kähler-Ricci flow on lower dimensional algebraic manifolds of general type*, Int. Math. Res. Not. IMRN 2016, no. 21, 6493–6511.
[54] V. Tosatti, *Adiabatic limits of Ricci-flat Kähler metrics*, J. Differential Geom. **84** (2010), no.2, 427–453.
[55] V. Tosatti, *KAWA lecture notes on the Kähler-Ricci flow*, Ann. Fac. Sci. Toulouse Math. **27** (2018), no. 2, 285–376.
[56] V. Tosatti, *Regularity of envelopes in Kähler classes*, Math. Res. Lett. 25 (2018), no. 1, 281–289.
[57] V. Tosatti, B. Weinkove, X. Yang, *The Kähler-Ricci flow, Ricci-flat metrics and collapsing limits*, Amer. J. Math. **140** (2018), no. 3, 653–698.
[58] V. Tosatti, Y. Zhang, *Infinite time singularities of the Kähler-Ricci flow*, Geom. Topol. **19** (2015), no. 5, 2925–2948.
[59] H. Tsuji, *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. **281** (1988), no. 1, 123–133.
[60] H. Tsuji, *Analytic Zariski decomposition*, in *International Symposium "Holomorphic Mappings, Diophantine Geometry and Related Topics" (Kyoto, 1992)*, Sūrikaisekikenkyūsho Kōkyūroku No. **819** (1993), 203–217.
[61] B. Wang, *The local entropy along Ricci flow Part A: the no-local-collapsing theorems*, Camb. J. Math. **6** (2018), no. 3, 267–346.
[62] B. Weinkove, *The Kähler-Ricci flow on compact Kähler manifolds*, in *Geometric analysis*, 53–108, IAS/Park City Math. Ser., 22, Amer. Math. Soc., Providence, RI, 2016.
[63] H. Wondo, Z. Zhang, *Independence of singularity type for numerically effective Kähler-Ricci flows*, preprint, arXiv:2308.12527.
[64] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), 339–411.
[65] Y. Zhang, *Collapsing limits of the Kähler-Ricci flow and the continuity method*, Math. Ann. **374** (2019), no. 1-2, 331–360.
[66] Y. Zhang, *Infinite-time singularity type of the Kähler-Ricci flow*, J. Geom. Anal. **30** (2020), no. 2, 2092–2104.
[67] Z. Zhang, *Scalar curvature bound for Kähler-Ricci flows over minimal manifolds of general type*, Int. Math. Res. Not. IMRN 2009, no. 20, 3901–3912.
[68] Z. Zhang, *Globally existing Kähler-Ricci flows*, Rev. Roumaine Math. Pures Appl. **60** (2015), no. 4, 551–560.

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