# On existence of solutions to non-convex minimization problems

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**Abstract**

We provide new sufficient conditions for the finiteness of the optimal value and existence of solutions to a general problem of minimizing a proper closed function over a nonempty closed set. The conditions require an asymptotically bounded decay of a function, a relaxation of $p$-supercoercivity, and a certain relation for the asymptotic cone of the constraint set and the asymptotic function of the objective function. Our analysis combines these conditions with a regularization technique. We refine the notion of retractive directions of a set, extend its definition to functions, and establish some basic relations for such directions for both sets and functions. Using these tools, we provide existence of solutions results that generalize many of the results in the literature for both non-convex and convex problems.

## 1 Introduction

Our interest is in investigating sufficient conditions for the existence of solutions to general non-convex minimization problems. The existence of solutions has been extensively studied starting with seminal work [9] showing that a quadratic function, which is bounded from below, attains its minimum on a polyhedral set. The result has been extended to the problem with a quasi-convex objective function in [12]. The work in [4] has established that a convex polynomial attains its solution on a region described by finitely many convex polynomial inequalities, which in turn generalized the result established in [15] for convex quadratic functions.

The inherent difficulty in establishing the existence of solutions is due to directions in unbounded constraint set along which the function may decrease. A unifying framework to address the problem of unboundedness in both functions and sets is proposed in [1, 2], relying on concepts such as the *asymptotic cone* and *asymptotic function* to show existence and stability results for general classes of optimization problems. Subsequently, in [6], these notions were extended to introduce *retractive directions* and prove existence of solutions via a nonempty level-set intersection approach. Later on, the work in [13] has developed the existence results for problems where the constraint sets are given by functional inequalities. More recently, a solution existence result has been provided in [10] for a general polynomial objective and a closed constrained set under a certain regularity condition.

In this paper, we generalize the aforementioned results by imposing some conditions on the objective function and the constraint set which are weaker than those in the existing literature. In particular, our existence of solution results extend the sufficient conditions of Theorem 3.4.1 in [2] for an unconstrained problem to a constrained problem. While the work in [6] and [13] have been aimed at the same type of extension, our results are more general and, in particular, they recover Propositions 12 and 13 in [6], while extending the class of problems to which Proposition 3.1 in [13] can be applied.

Our development is based on three main concepts, as follows. (1) The class of functions that have asymptotically bounded decay, which is inspired by the super-coercivity [7, 11]. This class of functions is wide and includes, for example, polynomials, convex functions, and functions with Lipschitz continuous $p$-derivative for some $p \geq 0$. (2) The asymptotic cone of a set [2] and the asymptotic cone of a function, which extends such a notion for a proper convex function, as introduced in Definition 2.5.2 of [2], to any proper function. (3) The cone of retractive directions of a set and a function. The notion of retractive direction of a set builds on Definition 2.3.1 in [2], and it is slightly more general than a related Definition 1 in [6]. The cone of retractive directions of a function is a new concept to the best of our knowledge. We explore the basic properties of the retractive directions of a set and a function.

As a first result, we provide a necessary condition for the finiteness of an optimal value for a constrained problem. Then, using the aforementioned main concepts, we establish our main results for the existence of solutions for problems where the constraint set is non-algebraic. Then, we refine the results for convex problems for cases when the constraint set is non-algebraic and algebraic. Finally, we extend the main results to non-convex problems where the constraint set is specified by functional constraints.

This paper is organized as follows: Section 2 provides the necessary background required for the subsequent development. Section 3 introduces the notion of asymptotically bounded decay of a function, the retractive directions of a set and a function, and investigates their properties. Section 4 presents our main results and their proofs for the problems where the constraint set is generic and not specified via inequalities. Section 5 refines our main results for convex problems and compares them with the existing results in the literature. Finally, Section 6 further elaborates our main results for the case when the constraint set is specified by functional inequalities. The results are also compared with the closely related results reported in [6, 13].

# 2 Notation and Terminology

We consider the space $\mathbb{R}^n$ equipped with the standard Euclidean norm $\|\cdot\|$ unless otherwise stated. In the following subsections, we introduce some basic concepts that will be used throughout the remainder of this paper.

## 2.1 Basic Definitions

We consider the functions that take values in the set $\mathbb{R} \cup \{+\infty\}$. For a function $f$, we use $\mathrm{dom} f$ to denote its effective domain, i.e., $\mathrm{dom} f = \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$. The epigraph of a function $f$ is denoted by $\mathrm{epi} f$, i.e., $\mathrm{epi} f = \{(x, c) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq c\}$. The following standard definitions are used repeatedly.

**Definition 2.1.1.** *For any $\gamma \in \mathbb{R}$, the lower-level set of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is given by*

$$L_\gamma(f) = \{x \in \mathbb{R}^n \mid f(x) \leq \gamma\}.$$

Our focus is on proper and closed functions, which are defined as follows.

**Definition 2.1.2.** *The function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be proper if its epigraph is a nonempty set.*

**Definition 2.1.3.** *The function $f$ is said to be closed if its epigraph* $\mathrm{epi} f$ *is a closed set.*

The $p$-supercoercivity has been introduced in [7, 11], which states that: a proper function $f$ is $p$-supercoercive, with $p \geq 1$, if

$$\liminf_{\|x\| \to \infty} \frac{f(x)}{\|x\|^p} > 0.$$

When the preceding relation holds with $p = 0$, the function is coercive, a property that has been widely used.

## 2.2 Asymptotic Cones

We turn our attention to unbounded sets whose behavior at infinity is captured by their *asymptotic cones*. We provide the definitions of a sequence that converges in a direction and the asymptotic cone of a set, as given in [2].

**Definition 2.2.1.** *A sequence $\{x_k\} \subset \mathbb{R}^n$ is said to converge in the direction $d \in \mathbb{R}^n$ if there exists a scalar sequence $\{t_k\} \subset \mathbb{R}$ with $t_k \to +\infty$ such that*

$$\lim_{k \to \infty} \frac{x_k}{t_k} = d.$$

**Definition 2.2.2.** *Let $X \subseteq \mathbb{R}^n$ be a nonempty set. The asymptotic cone of $X$, denoted by $X_\infty$, is the set of vectors $d \in \mathbb{R}^n$ that are limits in the directions of any sequence $\{x_k\} \subset X$ i.e.,*

$$X_\infty = \left\{ d \in \mathbb{R}^n \mid \exists \{x_k\} \subset X, \exists \{t_k\} \subset \mathbb{R}, t_k \to +\infty \ \textit{such that} \ \lim_{k \to \infty} \frac{x_k}{t_k} = d \right\}.$$

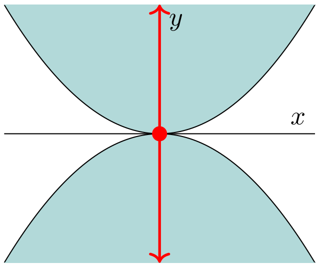


Figure 2.1: The set $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 \leq |y|\}$ and its asymptotic cone $X_\infty = \{(0, y) \mid y \in \mathbb{R}\}$.

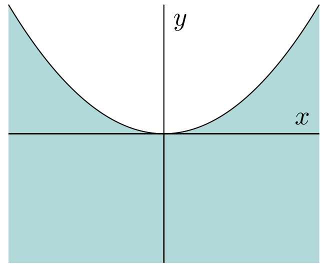


Figure 2.2: The set $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 \geq y\}$ and its asymptotic cone $X_\infty = \mathbb{R}^2$.

For a nonempty set $X$, the set $X_\infty$ is a closed cone by Proposition 2.1.1 in [2]. Asymptotic cones are illustrated in Figure 2.1 and Figure 2.2. We conclude this section with a result given by Proposition 2.1.9 in [2].

**Proposition 2.2.1.** *Let $C_i \subseteq \mathbb{R}^n$, $i \in \mathcal{I}$, where $\mathcal{I}$ an arbitrary index set. Then,*

$$(\cap_{i \in \mathcal{I}} C_i)_\infty \subseteq \cap_{i \in \mathcal{I}} (C_i)_\infty \qquad \textit{whenever } \cap_{i \in \mathcal{I}} C_i \textit{ nonempty.}$$

*The inclusion holds as an equality for closed convex sets $C_i$.*

## 2.3 Asymptotic Functions

Consider the concept of an asymptotic cone applied to the epigraph of a proper function. Doing so allows us to characterize the related notion of *asymptotic functions*. The formal definition is as follows, according to [2].

**Definition 2.3.1.** *For any proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ , there exists a unique function $f_\infty : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ associated with $f$ such that* $\mathrm{epi} f_\infty = (\mathrm{epi} f)_\infty$. *The function $f_\infty$ is said to be the asymptotic function of $f$.*

A useful analytic representation of an asymptotic function $f_\infty$ was originally obtained in [1] and, also, given in Theorem 2.5.1 of [2].

**Theorem 2.3.1.** *For any proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ the asymptotic function $f_\infty$ is given by*

$$f_\infty(d) = \liminf_{\substack{d' \to d \\ t \to +\infty}} \frac{f(td')}{t} \tag{1}$$

*or, equivalently,*

$$f_\infty(d) = \inf \left\{ \liminf_{k \to \infty} \frac{f(t_k d_k)}{t_k} \;\middle|\; t_k \to +\infty, d_k \to d \right\}, \tag{2}$$

*where the infimum is taken over all sequences $\{d_k\} \subset \mathbb{R}^n$ and $\{t_k\} \subset \mathbb{R}$.*

An asymptotic function has some basic properties inherent from its definition, namely that $f_\infty$ is closed and positively homogeneous function since its epigraph is the closed cone $(\mathrm{epi} f)_\infty$. Further, the value $f_\infty(0)$ is either finite or $f_\infty(0) = -\infty$. If $f_\infty(0)$ is finite, then it must be that $f_\infty(0) = 0$ by the positive homogeneity property. As a consequence, for a proper function $f$ we have that $0 \in \{d \mid f_\infty(d) \le 0\}$, implying that

$$\{d \mid f_\infty(d) \le 0\} \neq \emptyset.$$

The directions $d$ such that $f_\infty(d) \le 0$ will be particularly important in our subsequent development. To this end, we will term them as asymptotic directions of a function, and use these directions to define the asymptotic cone of a function, as follows.

**Definition 2.3.2.** *For a proper function $f$, we say that a direction $d$ is an asymptotic direction of $f$ if $f_\infty(d) \le 0$. The asymptotic cone of $f$, denoted by $\mathcal{K}(f)$, is the set of all asymptotic directions of $f$, i.e.,*

$$\mathcal{K}(f) = \{d \mid f_\infty(d) \le 0\}.$$

An asymptotic direction of a function has been given in Definition 3.1.2 of [2], while the asymptotic cone of a function has been defined for a proper convex function in Definition 2.5.2 of [2]. However, we adopt the same definition for an arbitrary proper function.

We next provide a key result for the asymptotic cones of lower-level sets of a proper function, which we will use later on. The result can be found in [2].

**Proposition 2.3.1** (Proposition 2.5.3, [2])**.** *For a proper function $f$ and any $\alpha \in \mathbb{R}$ such that $L_\alpha(f) \neq \emptyset$, one has $(L_\alpha(f))_\infty \subseteq L_0(f_\infty)$ i.e.,*

$$\{x \mid f(x) \le \alpha\}_\infty \subseteq \mathcal{K}(f).$$

*The inclusion is an equality when $f$ is proper, closed, and convex.*

We conclude this section with an existence of solutions result that will also be used in the sequel.

**Proposition 2.3.2.** *Let $X \subseteq \mathbb{R}^n$ be a nonempty closed set, and let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper closed function with $X \cap \mathrm{dom} f \neq \emptyset$. If $f$ is coercive over $X$, i.e.,*

$$\liminf_{\substack{\|x\| \to \infty \\ x \in X}} f(x) = +\infty,$$

*then the problem $\inf_{x \in X} f(x)$ has a finite optimal value and an optimal solution exists.*

*Proof.* The result follows by applying Theorem 2.14 of [3] to the function $f + \delta_X$, where $\delta_X$ is the characteristic function of the set $X$. □ □

# 3 Problem Setup and Basic Concepts

In this section, we introduce the non-convex minimization problem of interest along with new concepts that we will use in the development of our main solution existence results. The problem we consider is

$$\inf_{x\in X} f(x), \tag{$P$}$$

where $X \subseteq \mathbb{R}^n$ is a nonempty closed set and $f$ is a proper closed function. We let $f^*$ denote the optimal value of the problem, and $X^*$ denote the set of optimal solutions.

In what follows, we consider a set $X$ that is unbounded. In Subsection 3.1, we introduce a notion of asymptotically bounded decay of a function by bounding the asymptotic behavior of the ratio $f(x)/g(x)$ for an arbitrary function $g$ on $X$. In Subsections 3.2 and 3.3, we introduce the notions of retractive directions for sets and functions, respectively.

## 3.1 Asymptotically Bounded Decay

In this section, we introduce a condition on the function $f$ that generalizes the coercivity property. The formal definition is as follows.

**Definition 3.1.1.** *A proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to exhibit asymptotically bounded decay with respect to a proper function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ on a set $X \subseteq \mathbb{R}^n$ if*

$$\liminf_{\substack{\|x\|\to\infty \\ x\in X}} \frac{f(x)}{g(x)} > -\infty. \tag{3}$$

*We say a function exhibits asymptotically bounded decay with respect to $g$ if $X = \mathbb{R}^n$.*

Note that if were to choose $g(x) = \|x\|^p$ for $p \geq 1$, and

$$\liminf_{\substack{\|x\|\to\infty \\ x\in X}} \frac{f(x)}{\|x\|^p} > 0,$$

relation (3) implies that $f$ is $p$-supercoercive on $X$ and, thus, coercive on $X$. The condition in (3) prohibits the function $f$ from approaching $-\infty$, along the points in the set $X$, faster than the function $g$. The class of functions that have this property is wide. Below we provide several examples.

***Example* 3.1.** *Let $X \subseteq \mathbb{R}^n$ be a nonempty set. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function with a finite minimum on $X$, i.e., $f^* = \inf_{x\in X} f(x) > -\infty$. Then, we have $f(x) \geq f^*$ for all $x \in X$, implying that*

$$\liminf_{\substack{\|x\|\to\infty \\ x\in X}} \frac{f(x)}{\|x\|} \geq \liminf_{\substack{\|x\|\to\infty \\ x\in X}} \frac{f^*}{\|x\|} = 0.$$

*Hence, any proper function with a finite minimum on a set $X$ satisfies Definition 3.1.1 with $g(x) = \|x\|$.* $\square$

Next, we show that any proper convex function exhibits asymptotically bounded decay with respect to $g(x) = \|x\|$ due to the special linear under-estimation property of a convex function.

***Example* 3.2.** *Let $f$ be a proper convex function such that $X \cap \mathrm{dom} f \neq \emptyset$, and let $x_0$ be a point in the relative interior of $\mathrm{dom} f$. Then, by Theorem 23.4 of [14], the subdifferential set $\partial f(x_0)$ is nonempty. Thus, by the convexity of $f$ we have for a subgradient $s_0$ of $f$ at the point $x_0$ and for all $x \in X$,*

$$f(x) \geq f(x_0) + \langle s_0, x - x_0 \rangle \geq f(x_0) - \|s_0\| \|x - x_0\|,$$

*implying that*

$$\liminf_{\substack{\|x\| \to \infty \\ x \in X}} \frac{f(x)}{\|x\|} \geq \liminf_{\substack{\|x\| \to \infty \\ x \in X}} \frac{f(x_0) - \|s_0\| \|x - x_0\|}{\|x\|} = -\|s_0\|.$$

$\square$

Our next example shows that if a function $f$ satisfies $f_\infty(d) \geq 0$ for all nonzero $d \in \mathbb{R}^n$, then $f$ exhibits asymptotically bounded decay with respect to $g(x) = \|x\|$.

***Example* 3.3.** *Let $X$ be a nonempty set and $f$ be a proper function such that $X \cap \mathrm{dom} f \neq \emptyset$. If $f_\infty(d) \geq 0$ for all nonzero $d \in X_\infty$, then*

$$\liminf_{\substack{\|x\| \to \infty \\ x \in X}} \frac{f(x)}{\|x\|} = \liminf_{\substack{d = x\|x\|^{-1} \\ \|x\| \to \infty \\ x \in X}} \frac{f(\|x\| d)}{\|x\|} \geq \liminf_{\substack{d' \to d \\ \|x\| \to \infty}} \frac{f(\|x\| d')}{\|x\|}.$$

*We further have*

$$\liminf_{\substack{d' \to d \\ \|x\| \to \infty}} \frac{f(\|x\| d')}{\|x\|} \geq \liminf_{\substack{d' \to d \\ t \to \infty}} \frac{f(t d')}{t} = f_\infty(d) \geq 0.$$

*Hence,*

$$\liminf_{\|x\| \to \infty} \frac{f(x)}{\|x\|} \geq 0,$$

*and $f$ exhibits asymptotically bounded decay with respect to $g(x) = \|x\|$.* $\square$

Lastly, we show that a function that is $p$-times differentiable with a Lipschitz continuous $p$th differentials exhibits asymptotically bounded decay with respect to $g(x) = \|x\|^{p+1}$ on $X$.

***Example* 3.4.** *Consider a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with Lipschitz continuous $p$th derivatives on an open convex set containing the set $X$, with $p \geq 0$.*

*When $p = 0$, the function is simply Lipschitz continuous. Let the pth derivative have a Lipschitz constant $L_p > 0$, i.e.,*

$$\|D^p f(x) - D^p f(x')\| \leq L_p \|x - x'\| \qquad \textit{for all } x, x' \in \mathrm{dom} f,$$

*where $D^p f(x)$ denotes the pth derivative of $f$ at a point $x$. Then, by Equation (1.5) of [**nesterov2021**] we have that*

$$|f(x) - \Phi_{x_0,p}(x)| \leq \frac{L_p}{(p+1)!} \|x - x_0\|^{p+1} \qquad \textit{for all } x, x_0 \in \mathrm{dom} f,$$

*where $\Phi_{x_0,p}(x)$ is the pth order Taylor approximation of $f$ at the point $x_0$, i.e.,*

$$\Phi_{x_0,p}(x) = \sum_{i=1}^{p} \frac{1}{i!} D^i f(x_0)[x - x_0]^i,$$

*with $[h]^i$ denoting the vector consisting of $i$ copies of a vector $h$, and with $[h]^0 = 1$ when $i = 0$. Then, for $x_0 \in \mathrm{dom} f$ arbitrary but fixed, we have that*

$$f(x) \geq \Phi_{x_0,p}(x) - \frac{L_p}{(p+1)!} \|x - x_0\|^{p+1} \qquad \textit{for all } x \in \mathrm{dom} f,$$

*implying that*

$$\liminf_{\substack{k\to\infty \\ x \in X}} \frac{f(x)}{\|x\|^{p+1}} \geq \liminf_{\substack{k\to\infty \\ x \in X}} \left\{ \frac{\Phi_{x_0,p}(x)}{\|x\|^{p+1}} - \frac{L_p}{(p+1)!} \frac{\|x - x_0\|^{p+1}}{\|x\|^{p+1}} \right\} = -\frac{L_p}{(p+1)!},$$

*where we used the fact that $\lim_{\|x\|\to\infty} \Phi_{x_0,p}(x)/\|x\|^{p+1} = 0$. Hence, $f$ exhibits asymptotically bounded decay with respect to $g(x) = \|x\|^{p+1}$ on $X$.* □

Note that multivariate polynomials are a special class of functions that fall under Example 3.4. In particular, a multivariate polynomial of order $m$, with $m \geq 1$, has a constant $m$th order derivative so it is bounded by some constant $B$. Thus, the $(m-1)$st derivative is Lipshitz continuous with the constant $B$. According to Example 3.4, a multivariate polynomial exhibits asymptotically bounded decay with respect to the function $g(x) = \|x\|^m$.

## 3.2 Retractive Directions of Sets

The key notion that we use throughout the rest of this paper is that of a retractive direction. For a nonempty set, a retractive direction is defined as follows.

**Definition 3.2.1.** *Given a nonempty set $X$, a direction $d \in X_\infty$ is said to be retractive direction of $X$ if for any sequence $\{x_k\} \subseteq X$ converging in the direction $d$ and for any $\rho > 0$, there exists an index $K$ (depending on $\rho$) such that*

$$x_k - \rho d \in X \quad \textit{for all } k \geq K. \tag{4}$$

*The set of retractive directions of a set $X$ is denoted by $\mathcal{R}(X)$. We say that the set $X$ is retractive if $\mathcal{R}(X) = X_\infty$.*

Note that $\mathcal{R}(X) \subseteq X_\infty$ by definition, and $0 \in \mathcal{R}(X)$. We next provide an example of a convex set that has a non-retractive direction.

***Example* 3.5.** *Consider the epigraph of the scalar function $f(s) = s^2$ i.e., $X = \{(s,\gamma) \in \mathbb{R}^2 \mid s^2 \leq \gamma\}$. Let $\{x_k\} \subseteq X$ be given by $x_k = (\sqrt{k}, k)$. Then, $\|x_k\| \to \infty$ as $k \to \infty$. For any $\lambda > 0$, we have with $t_k = \|x_k\|/\lambda$,*

$$\lim_{k\to\infty} \frac{x_k}{t_k} = \lambda \lim_{k\to\infty} \frac{x_k}{\|x_k\|} = \lambda \lim_{k\to\infty} \left( \frac{1}{\sqrt{k+1}}, \frac{\sqrt{k}}{\sqrt{k+1}} \right) = (0, \lambda).$$

*Thus, $(0,\lambda) \in X_\infty$ for any $\lambda > 0$ and, furthermore,*

$$x_k - (0,\lambda) = (\sqrt{k}, k - \lambda) \notin X \qquad \text{for all } k \geq 1.$$

*Hence, $d = (0,\lambda)$ is not a retractive direction of $X$ for any $\lambda > 0$. Thus, $\mathcal{R}(X) = \{0\}$.* □

We now give an example of a non-convex set which has no nonzero retractive direction.

***Example* 3.6.** *Consider the set $X = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1^2 \leq |x_2|\}$ (see Fig. 2.1). Similar to Example 3.5, we can see that the directions $(0,\lambda)$ and $(0,-\lambda)$ are not retractive directions of $X$ for any $\lambda > 0$. Hence, $\mathcal{R}(X) = \{0\}$.* □

Now we highlight some related definitions. Most notable is Definition 2.3.1 in [2] which defines an *asymptotically linear set*, as follows: a closed set $C \subseteq \mathbb{R}^n$ is said to be asymptotically linear if for every $\rho > 0$ and each sequence $\{x_k\} \subseteq C$ that satisfies $x_k \in C$, $\|x_k\| \to +\infty$ and $x_k\|x_k\|^{-1} \to \overline{x}$, there exists an index $K$ such that $x_k - \rho\overline{x} \in C$ for all $k \geq K$. Note that the directions $\bar{x}$ involved in this definition have unit norm, and such directions are retractive according to our Definition 3.2.1. Further, since $\bar{x}$ cannot be zero, the set of such directions is a subset of $\mathcal{R}(C)$.

From the definition of an asymptotically linear set we can invoke a key example of a retractive set which is a polyhedral set.

***Example* 3.7.** *Consider a nonempty polyhedral set $X$. The inclusion $\mathcal{R}(X) \subseteq X_\infty$ always holds by Definition 3.2.1. By Proposition 2.3.1 in [2], an asymptotically polyhedral set $X \subseteq \mathbb{R}^n$ is asymptotically linear. Since the simplest case of an asymptotically polyhedral set is a polyhedral set, it follows that $X$ is asymptotically linear, i.e., $\mathcal{R}(X) = X_\infty$.* □

Another related definition is Definition 1 in [6], which considers $x_k \in C_k$ for an infinite sequence of nested sets $\{C_k\} \subseteq \mathbb{R}^n$ i.e., $C_{k+1} \subseteq C_k$ for all $k$. The directions $d$ of interest are obtained in the limit, as follows:

$$\lim_{k\to\infty} \frac{x_k}{\|x_k\|} = \frac{d}{\|d\|}.$$

By letting $C = C_k$ for all $k$, a direction $d$ is retractive according to Definition 1 in [6] if, for any associated sequence $\{x_k\} \subseteq C$ with $\|x_k\| \to$ and $x_k/\|x_k\| \to$

$d/\|d\|$, we have that $x_k - d \in C$ for all sufficiently large $k$. According to this definition, given a retractive direction $d$ and its associated sequence $\{x_k\}$, for any $\rho > 0$, we have that

$$\lim_{k\to\infty} \frac{x_k}{\|x_k\|} = \frac{d}{\|d\|} = \frac{\rho d}{\|\rho d\|},$$

implying that $\{x_k\}$ is also associated sequence for the direction $\rho d$ for any $\rho > 0$. Thus, the condition $x_k - d \in C$ for all $k$ large enough can be written as $x_k - \rho d \in C$ for all large enough $k$, implying that $d$ is a retractive direction according to Definition 1 in [6] and, also, according to our Definition 3.2.1 with $t_k = \|x_k\|$ for all $k$.

Consider now a sequence $\{x_k\} \subset X$ converging in a nonzero direction $d$, i.e., for some scalar sequence $\{t_k\}$ with $t_k \to \infty$, we have

$$\lim_{k\to\infty} \frac{x_k}{t_k} = d \qquad \text{with } d \neq 0.$$

Then,

$$\lim_{k\to\infty} \frac{x_k}{\|x_k\|} = \lim_{k\to\infty} \frac{x_k/t_k}{\|x_k\|/t_k} = \frac{d}{\|d\|}.$$

If $d$ is retractive according to our Definition 3.2.1, then by letting $\rho = \|d\|$, we conclude the $d$ is also retractive according to Definition 1 in [6]. As stated previously, the only direction $d$ for which the equivalence does not hold is $d = 0$. That is, $0 \in \mathcal{R}(X)$ by Definition 3.2.1, but not by Definition 1 of [6].

We now state some general properties of retractive directions of a set.

**Proposition 3.2.1.** *For a nonempty set $X$, the set $\mathcal{R}(X)$ of retractive directions of $X$ is a nonempty cone.*

*Proof.* Let $d \in \mathcal{R}(X)$ and let $\lambda \geq 0$ be arbitrary. Let $\{x_k\} \subset X$ and $\{t_k\} \subseteq \mathbb{R}$ be such that $t_k \to +\infty$ and $x_k \cdot t_k^{-1} \to \lambda d$, as $k \to \infty$, and let $\rho > 0$ be arbitrary. Since $d \in \mathcal{R}(X)$ and $\rho\lambda > 0$, there exists $K$ such that $x_k - \rho\lambda d \in X$ for all $k \geq K$. Therefore, $\lambda d \in \mathcal{R}(X)$. $\square$ $\square$

The cone $\mathcal{R}(X)$ is not necessarily closed, as seen in the following example.

***Example* 3.8.** *Consider the set $X$ given by the epigraph of the function $f(x) = -\sqrt{x}$ for $x \geq 0$, i.e., $X = \{(x,\gamma) \in \mathbb{R}^2 \mid -\sqrt{x} \leq \gamma\}$. The asymptotic cone of $X$ is the non-negative orthant, i.e.,*

$$X_\infty = \{(d_1, d_2) \in \mathbb{R}^2 \mid d_1 \geq 0,\ d_2 \geq 0\}.$$

*We claim that every direction $d = (d_1, d_2) \in X_\infty$ with $d_1 > 0$ and $d_2 > 0$ is a retractive direction of $X$. To see this, let $\{(x_k, \gamma_k)\} \subset X$ and $\{t_k\} \subseteq \mathbb{R}$ be sequences such that $t_k \to \infty$ and $(x_k t_k^{-1}, \gamma_k t_k^{-1}) \to d$. Let $\rho > 0$ be arbitrary. Then, since $(x_k, \gamma_k) \to d$ and $d_1, d_2 > 0$, it follows that $x_k \to +\infty$ and $\gamma_k \to +\infty$. Thus, there is a large enough $K$ such that*

$$x_k - \rho d_1 > 0, \quad \text{and} \quad \gamma_k - \rho d_2 > 0 \qquad \text{for all } k \geq K.$$

*Noting that the positive orthant is contained in the set $X$, we see that $(x_k, \gamma_k) - \rho d \in X$ for all $k \geq K$. Hence, $d$ is a retractive direction of the set $X$.*

*Next we show that $\mathcal{R}(X)$ is not closed. Note that $(1,0) \in X_\infty$ and consider a sequence $\{d_k\} \subset X_\infty$, with $d_{k,i} > 0$ for $i = 1, 2$, and for all $k$, such that*

$$\lim_{k \to \infty} d_k = (1, 0).$$

*As seen above, we have that each $d_k$ is a retractive direction of $X$. However, the limit $(1,0)$ is not a retractive direction of $X$. To show this, we consider a sequence $\{\bar{x}_k\} \subset X$ given by*

$$\bar{x}_k = (k, -\sqrt{k}) \qquad \textit{for all } k \geq 1,$$

*and note that $\bar{x}_k \cdot \|\bar{x}_k\|^{-1} \to (1,0)$. For every $k \geq 1$, we can see that*

$$\bar{x}_k - (1,0) = (k-1, -\sqrt{k}) \notin X.$$

*Hence, $(1,0)$ is not a retractive direction of $X$ and, consequently $\mathcal{R}(X)$ is not closed.* $\square$

The following proposition considers the cone of retractive directions of the intersection of finitely many sets.

**Proposition 3.2.2.** *Let $X = \cap_{i=1}^m X_i$ be a nonempty intersection set of closed sets $X_i$, for some $m \geq 2$. If $X_\infty = \cap_{i=1}^m (X_i)_\infty$, then $\cap_{i=1}^m \mathcal{R}(X_i) \subseteq \mathcal{R}(X)$.*

*Proof.* Suppose $d \in \mathcal{R}(X_i)$ for all $i$. Since $\mathcal{R}(X_i) \subseteq (X_i)_\infty$ for all $i$, it follows that $d \in \cap_{i=1}^m (X_i)_\infty$. By the assumption that $\cap_{i=1}^m (X_i)_\infty = X_\infty$, we have that $d \in X_\infty$. Let $\{x_k\} \subset X$ be any sequence converging in direction $d$ and let $\rho > 0$ be arbitrary. Then, $\{x_k\} \subset X_i$ for all $i$. Since $d \in \mathcal{R}(X_i)$ for all $i$, it follows that for every $i = 1, \ldots, m$, there exists an index $K_i$ such that

$$x_k - \rho d \in X_i \qquad \text{for all } k \geq K_i.$$

Let $K = \max_{1 \leq i \leq m} K_i$. Then, it follows that

$$x_k - \rho d \in X_i \qquad \text{for all } k \geq K \text{ and for all } i = 1, \ldots, m,$$

implying that

$$x_k - \rho d \in \cap_{i=1}^m X_i \qquad \text{for all } k \geq K.$$

Thus, $d \in \mathcal{R}(X)$. $\square$ $\square$

When the sets $X_i$ in Proposition 3.2.2 are closed and convex, the condition $X_\infty = \cap_{i=1}^m (X_i)_\infty$ is always satisfied, as seen from Proposition 2.2.1.

## 3.3 Retractive Directions of Functions

In this section, we introduce the concept of a *retractive direction of a function.*

**Definition 3.3.1.** *An asymptotic direction $d \in \mathcal{K}(f)$ of a proper function $f$ is said to be a retractive direction of $f$ if for every $\{x_k\} \subset \mathrm{dom} f$ converging in direction $d$ and for every $\rho > 0$, there exists an index $K$ such that*

$$f(x_k - \rho d) \le f(x_k) \qquad \text{for all } k \ge K.$$

*The set of directions along which $f$ is retractive is denoted by $\mathcal{R}(f)$.*

By definition, one can show that $\mathcal{R}(f)$ is a cone and $0 \in \mathcal{R}(f)$. Furthermore, for a proper function $f$ and a nonempty lower-level set $L_\gamma(f)$, we have that

$$\mathcal{R}(f) \subseteq \mathcal{K}(f) \quad \text{and} \quad \mathcal{R}(L_\gamma(f)) \subseteq \mathcal{K}(f).$$

It turns out that, in general, there is no special relationship between the cone $\mathcal{R}(f)$ and the cone $\mathcal{R}(L_\gamma(f))$. The following example illustrates that we can have $\mathcal{R}(L_\gamma(f)) \subseteq \mathcal{R}(f)$ for a non-convex function.

***Example* 3.9.** *Let $f(s) = \sqrt{|s|}$. Then, for every $\gamma > 0$, the lower-level set $L_\gamma(f)$ is nonempty and bounded, implying that $\mathcal{R}(L_\gamma(f)) = \{0\}$. However, we have that $(\mathrm{epi} f)_\infty = \{(d, w) \mid d \in \mathbb{R}, w \ge 0\}$, implying that $f_\infty(d) = 0$ for all $d \in \mathbb{R}$. Moreover, in this case $\mathcal{R}(f) = \mathbb{R}$, and we have for any $\gamma > 0$,*

$$\mathcal{R}(L_\gamma(f)) = \{0\} \subset \mathcal{R}(f).$$

□

In the following proposition, we establish some properties of $\mathcal{R}(f)$ for a convex function. Interestingly, in this case $\mathcal{R}(f) \subseteq \mathcal{R}(L_\gamma(f))$ for a nonempty lower-level set $L_\gamma(f)$ (converse inclusion to that of Example 3.9).

**Proposition 3.3.1.** *Let $f$ be a proper closed convex function. Then, the following relations hold:*

*(a)* $\mathcal{R}(f) \subseteq \{d \mid f_\infty(d) = 0\}$.

*(b) For any nonempty lower-level set $L_\gamma(f)$ we have*

$$\mathcal{R}(f) \subseteq \mathcal{R}(L_\gamma(f)).$$

*Proof.* (a) Let $d \in \mathcal{R}(f)$. Then, for any $\{x_k\}$ converging in direction $d$ and any $\rho > 0$, there is a large enough $K$ such that

$$f(x_k - \rho d) \le f(x_k) \qquad \text{for all } k \ge K.$$

Let $y_k = x_k - \rho d$. Then, $\{y_k\}$ also converges in direction $d$, and the preceding relation implies that

$$f(y_k) \le f(y_k + \rho d) \qquad \text{for all } k \ge K.$$

By Proposition 2.5.2 of [2] for a proper closed convex function $f$, we have that

$$f_\infty(d) = \sup_{x \in \mathrm{dom} f} \{f(x+d) - f(x)\}.$$

Therefore, it follows that

$$f_\infty(\rho d) \geq \sup_{k \geq K} \{f(y_k + \rho d) - f(y_k)\} \geq 0.$$

Since $\rho > 0$ and $f_\infty$ is positively homogeneous by Proposition 2.5.1 of [2], it follows that

$$f_\infty(d) \geq 0,$$

which, combined with the fact that $\mathcal{R}(f) \subseteq \mathcal{K}(f)$, implies that $f_\infty(d) = 0$.
(b) Let $L_\gamma(f)$ be nonempty. To arrive at a contradiction, assume that there is $d \in \mathcal{R}(f)$ such that $d \notin \mathcal{R}(L_\gamma(f))$. Since $\mathcal{R}(f) \subseteq \mathcal{K}(f)$, it follows that $d \in \mathcal{K}(f)$. By Proposition 2.3.1, we have that $(L_\gamma(f))_\infty = \mathcal{K}(f)$, implying that $d \in (L_\gamma(f))_\infty$. Since $d \in (L_\gamma(f))_\infty$ but it is a non-retractive direction of the set $L_\gamma(f)$, there is a sequence $\{x_k\} \subset L_\gamma(f)$ converging in direction $d$ and some $\bar{\rho} > 0$ such that

$$f(x_k - \bar{\rho} d) \geq \gamma \qquad \text{for infinitely many indices } k.$$

Without loss of generality, we may assume that

$$f(x_k - \bar{\rho} d) \geq \gamma \qquad \text{for all } k,$$

for otherwise we would just choose a suitable subsequence of $\{x_k\}$. Since $\{x_k\} \subset L_\gamma(f)$, it follows that

$$f(x_k - \bar{\rho} d) \geq \gamma \geq f(x_k) \qquad \text{for all } k.$$

Thus, we have that $\{x_k\}$ converges in direction $d$ and the direction $d$ belongs to $\mathcal{K}(f)$, but $f(x_k - \bar{\rho} d) \geq f(x_k)$ for all $k$. Hence, the direction $d$ is not retractive for the function $f$, i.e., $d \notin \mathcal{R}(f)$, which is a contradiction. □ □

The inclusion in Proposition 3.3.1(b) can be strict, as seen in the following example.

***Example* 3.10.** *Let $f(x) = \langle c, x \rangle$ for some $c \in \mathbb{R}^n$, $c \neq 0$. Then, for any $\gamma \in \mathbb{R}$, the lower-level set $L_\gamma(f)$ is nonempty and polyhedral, so by Example 3.7 we have that $\mathcal{R}(L_\gamma(f)) = (L_\gamma(f))_\infty$. Thus, $\mathcal{R}(L_\gamma(f)) = \{d \in \mathbb{R}^n \mid \langle c, d \rangle \leq 0\}$.*

*It can be seen that $f_\infty(d) = \langle c, d \rangle$. By Proposition 3.3.1, if a direction $d$ is retractive, then we must have $\langle c, d \rangle = 0$. Thus,*

$$\mathcal{R}(f) = \{d \mid \langle c, d \rangle = 0\} \subset \mathcal{R}(L_\gamma(f)) \qquad \text{for all } \gamma \in \mathbb{R}\ .$$

□

Given two distinct lower-level sets of a function, there is no particular inclusion relation for the cones of their retractive directions, even for a convex function. The following examples illustrate that either inclusion between the cones of retractive directions of two lower-level sets is possible.

***Example* 3.11.** *Consider the function $f(x_1, x_2) = \|x\| - x_1$, for $x_1 \geq 0$ and $x_2 \in \mathbb{R}$, and level sets $L_0(f)$ and $L_\gamma(f)$ with $\gamma > 0$. For the set $L_0(f)$ we have*

$$L_0(f) = \{(x_1, 0) \mid x_1 \geq 0\},$$

*which is a polyhedral set. Thus, $\mathcal{R}(L_0(f)) = L_0(f)$.*

*The set $L_\gamma(f)$ with $\gamma > 0$ is given by $L_\gamma(f) = \{x \in \mathbb{R}^2 \mid x_2^2 \leq 2x_1\gamma + \gamma^2\}$. The asymptotic cone of $L_\gamma(f)$ is given by $\{(d_1, 0) \in \mathbb{R}^2 \mid d_1 \geq 0\}$ but the retractive direction is only the zero vector, i.e., $\mathcal{R}(L_\gamma(f)) = \{0\}$. Thus, we have $L_0(f) \subset L_\gamma(f)$, while*

$$\mathcal{R}(L_\gamma(f)) = \{0\} \subset \mathcal{R}(L_0(f)).$$

□

***Example* 3.12.** *Consider the function $f(x_1, x_2) = e^{-\sqrt{x_1 x_2}}$ for $x_1 \geq 0$, $x_2 \geq 0$. The lower-level set $L_\gamma(f)$ is nonempty for all $\gamma > 0$. Consider $L_1(f)$ and $L_\gamma(f)$ with $\gamma > 0$. For the set $L_1(f)$ we have*

$$L_1(f) = \mathbb{R}^2_+,$$

*which is a polyhedral set, so we have $\mathcal{R}(L_1(f)) = \mathbb{R}^2_+$. For the set $L_\gamma(f)$ with $\gamma > 0$, we have*

$$L_\gamma(f) = \{x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq (\ln \gamma)^2\}.$$

*The asymptotic cone of $L_\gamma(f)$ is $\mathbb{R}^2$ and the cone of retractive directions is*

$$\mathcal{R}(L_\gamma(f)) = \{(0, 0)\} \cup \{(d_1, d_2) \mid d_1 > 0, d_2 > 0\}.$$

*Thus, for $\gamma < 1$, we have $L_\gamma(f) \subset L_1(f)$, while*

$$\mathcal{R}(L_\gamma(f)) \subset \mathcal{R}(L_1(f)).$$

□

Finally, we consider polynomial functions. A polynomial $h : \mathbb{R}^n \to \mathbb{R}$ of order $p$ has the following representation [4]:

$$h(x) = \sum_{i=0}^{p} \phi_i(x) \qquad \text{for all } x \in \mathbb{R}^n, \tag{5}$$

where each $\phi_i : \mathbb{R}^n \to \mathbb{R}$ is the $i$th order polynomial and $\phi_0$ is a constant. For every $i = 0, \ldots, p$, the polynomial $\phi_i$ has the property that $\phi_i(tx) = t^i \phi_i(x)$ for all $t \in \mathbb{R}$. Asymptotic behavior of polynomials typically depends on their leading order terms. Given a polynomial $h$ of order $p$ and given $x \in \mathbb{R}^n$, we let $\mu(x)$ denote the maximal order $i \in \{1, \ldots, p\}$ such that $\phi_i(x) \neq 0$, i.e.,

$$\mu(x) = \max\{i \mid \phi_i(x) \neq 0, i = 1, \ldots, p\}.$$

The following lemma provides a closed form expression for the asymptotic function of a polynomial.

***Lemma* 3.3.1.** *The asymptotic function of a polynomial $h$ of order $p$ is given by*

$$h_\infty(d) = \begin{cases} -\infty, & \mu(d) \geq 2 \text{ and } \phi_{\mu(d)}(d) < 0, \\ \phi_1(d), & \mu(d) = 1, \\ +\infty, & \mu(d) \geq 2 \text{ and } \phi_{\mu(d)}(d) > 0. \end{cases}$$

*Proof.* Using the relation $h(x) = \sum_{i=0}^{p} \phi_i(x)$, for all $x \in \mathbb{R}^n$, and the alternative characterization of the asymptotic function, as given in Theorem 2.3.1, we have

$$\begin{aligned} h_\infty(d) &= \liminf_{\substack{t\to\infty \\ d'\to d}} \frac{h(td')}{t} \\ &= \liminf_{\substack{t\to\infty \\ d'\to d}} \left( \sum_{i=0}^{p} t^{-1}\phi_i(td') \right) \\ &= \phi_1(d) + \liminf_{t\to\infty} \left( \sum_{i=2}^{p} t^{i-1}\phi_i(d) \right), \end{aligned}$$

where the last equality follows from the fact that $\phi_0/t \to 0$, as $t \to \infty$ and the fact that each $\phi_i$ is a continuous function. When $\mu(d) \geq 2$, if $\phi_{\mu(d)}(d) > 0$, then $h_\infty(d) = +\infty$, while if $\phi_{\mu(d)}(d) < 0$, then $h_\infty(d) = -\infty$. When $\mu(d) = 1$, then we are left with $\phi_1(d)$. □ □

The cone of retractive directions for a convex polynomial is characterized in the following lemma. It uses a notion of *the constancy space of a proper convex function*, defined by $\mathcal{C}(h)$ is the constancy space of $h$ given by

$$\mathcal{C}(h) = \{d \in \mathbb{R}^n \mid f_\infty(d) = f_\infty(-d) = 0\}. \tag{6}$$

***Lemma* 3.3.2.** *Let $h$ be a convex polynomial of order $p \geq 1$. Then, we have*

$$\mathcal{C}(h) = \{d \mid h_\infty(d) = 0\} \quad \text{and} \quad \mathcal{R}(h) = \{d \mid h_\infty(d) = 0\}.$$

*Proof.* By Lemma 3.3.1 we have that

$$\{d \mid h_\infty(d) = 0\} = \{d \mid \phi_1(d) = 0, \phi_2(d) = 0, \ldots, \phi_p(d) = 0\}.$$

Let $d$ be such that $h_\infty(d) = 0$. Then, for every $i = 1, \ldots, p$, we have $\phi_i(-d) = (-1)^i\phi_i(d)$, thus implying that $h_\infty(-d) = 0$. Hence, by Theorem 2.5.3 in [2] it follows that

$$h(x + td) = h(x) \qquad \text{for all } x \in \mathrm{dom} f.$$

Therefore, the direction $d$ lies in the constancy space $\mathcal{C}(h)$, implying that

$$\{d \mid h_\infty(d) = 0\} \subseteq \mathcal{C}(h) \subseteq \mathcal{R}(h), \tag{7}$$

where the last inclusion in the preceding relation holds since every direction $d \in \mathcal{C}(h)$ is retractive. By Proposition 3.3.1(a), we have that $\mathcal{R}(h) \subseteq \{d \mid h_\infty(d) = 0\}$, which implies that the equality holds throughout in (7). □ □

# 4 Main Results

In this section, we focus on the optimization problem $(P)$ and present our main results, including a necessary condition for the finiteness of the optimal value of the problem $(P)$ and sufficient conditions for the existence of solutions.

## 4.1 Necessary Condition for Finiteness of Optimal Value

We provide a result regarding the finiteness of the optimal value of the problem $(P)$. To the best of our knowledge, it appears to be new.

**Proposition 4.1.1.** *Let $X$ be a nonempty set and let $f$ be proper function with $X \cap \operatorname{dom} f \neq \emptyset$. If the problem $(P)$ has a finite optimal value, then*

$$(f + \delta_X)_\infty(d) \geq 0 \qquad \textit{for all } d \in \mathbb{R}^n,$$

*where $\delta_X$ is the characteristic function of the set $X$, i.e., $\delta_X(x) = 0$ when $x \in X$ and $\delta_X(x) = +\infty$ otherwise. Moreover, if $X$ and $f$ are additionally assumed to be convex, then we have*

$$f_\infty(d) \geq 0 \qquad \textit{for all } d \in X_\infty.$$

*Proof.* According to the alternative representation of an asymptotic function in Theorem 2.3.1, we have that for any $d \in \mathbb{R}^n$,

$$(f + \delta)_\infty(d) = \liminf_{\substack{d' \to d \\ t \to +\infty}} \frac{(f + \delta_X)(td')}{t}$$

Since $\delta_X(td') = +\infty$ when $td' \notin X$, it follows that

$$(f + \delta)_\infty(d) = \liminf_{\substack{td' \in X,\, d' \to d \\ t \to +\infty}} \frac{(f + \delta_X)(td')}{t} \geq \liminf_{t \to +\infty} \frac{(\inf_{x \in X} f(x))}{t} = 0,$$

where the inequality follows by using $\delta_X(td') = 0$ and $f(td') \geq \inf_{x \in x} f(x)$ when $td' \in X$.

When $X$ and $f$ are additionally assumed to be convex, we have

$$(f + \delta_X)_\infty(d) = f_\infty(d) + (\delta_X)_\infty(d)$$

by Remark 3.4.3 in [2]. Since $(\delta_X)_\infty = \delta_{X_\infty}$ by Corollary 2.5.1 of [2], it follows that

$$f_\infty(d) + \delta_{X_\infty}(d) \geq 0 \qquad \text{for all } d \in \mathbb{R}^n,$$

implying that $f_\infty(d) \geq 0$ for all $d \in X_\infty$.

□ □

We note that the condition in Proposition 4.1.1 is not sufficient for the finiteness of the optimal value $\inf_{x\in X} f(x)$ even for a convex problem, as seen in the following example.

***Example* 4.1.** *Consider the problem* $\inf_{x\in\mathbb{R}} f(x)$ *with the function* $f(x) = -\sqrt{x}$ *for* $x \geq 0$*, and* $f(x) = +\infty$ *otherwise. We have that* $(\mathrm{epi} f)_\infty = \mathbb{R}^2_+$ *and, thus,* $f_\infty(d) = 0$ *for all* $d \geq 0$*, and* $f_\infty(d) = +\infty$ *otherwise. Hence,* $f + \delta_X \equiv f$ *since* $X = \mathbb{R}$*, and*

$$(f + \delta_X)_\infty(d) = f_\infty(d) \geq 0 \qquad \textit{for all } d \in X_\infty,$$

*showing that the condition of Proposition 4.1.1 is satisfied. However, the optimal value of the problem is* $f^* = -\infty$. □

## 4.2 Conditions for Existence of Solutions to General nonconvex Problems

In this section, we provide some sufficient conditions for the existence of solutions for the problem $(P)$. The first result relies on the condition that the asymptotic cone $X_\infty$ of the set and the asymptotic cone $\mathcal{K}(f)$ of the function have no nonzero vector in common, as given in the following proposition.

**Theorem 4.2.1.** *Let* $X$ *be a closed set and* $f$ *be a proper closed function with* $X \cap \mathrm{dom} f \neq \emptyset$*. Assume that* $X_\infty \cap \mathcal{K}(f) = \{0\}$*. Then, the problem* $(P)$ *has a finite optimal value* $f^*$ *and its solution set* $X^*$ *is nonempty and compact.*

*Proof.* Since $X \cap \mathrm{dom} f \neq \emptyset$, there exist a point $x_0 \in X \cap \mathrm{dom} f$ with a finite value $f(x_0)$. Therefore, for $\gamma = f(x_0)$, the lower-level set $L_\gamma(f)$ is nonempty. By Proposition 2.3.1, we have that $(L_\gamma(f))_\infty \subseteq \mathcal{K}(f)$, thus implying that

$$(X \cap L_\gamma(f))_\infty \subseteq X_\infty \cap (L_\gamma(f))_\infty \subseteq X_\infty \cap \mathcal{K}(f) = \{0\},$$

where the first inclusion follows from Proposition 2.2.1. Thus, $(X \cap L_\gamma(f))_\infty = \{0\}$ and the set $X \cap L_\gamma(f)$ is bounded by Proposition 2.1.2 of [2], and hence compact since $X$ and $f$ are closed. Therefore, the problem $\inf_{X\cap L_\gamma(f)} f(x)$ has a finite optimal value and a solution exists by the Weierstrass Theorem. Since the problem $\inf_{x\in X\cap L_\gamma(f)} f(x)$ is equivalent to the problem $(P)$, it follows that $f^*$ is finite and attained. The compactness of $X^*$ follows by noting that $X^* = X \cap L_{f^*}(f)$ is nonempty, closed, and bounded due to $(X \cap L_{f^*}(f))_\infty = \{0\}$. □ □

For the condition on the asymptotic cones of $X$ and $f$ in Theorem 4.2.1, we note that

$$X_\infty \cap \mathcal{K}(f) = \{0\} \iff f_\infty(0) = 0 \text{ and } f_\infty(d) > 0 \text{ for all nonzero } d \in X_\infty.$$

Theorem 4.2.1 generalizes Theorem 3.1 of [10], which additionally requires that $f$ is bounded below on $X$.

We now focus on a more general case where the intersection $X_\infty \cap \mathcal{K}(f)$ contains nonzero directions. We have the following result for the case when $f$ exhibits asymptotically bounded decay with respect to $g(x) = \|x\|^p$ on the set $X$, for some $p \geq 0$.

**Theorem 4.2.2.** *Let the set $X \subseteq \mathbb{R}^n$ be closed, and let the objective function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and closed with $X \cap \mathrm{dom} f \neq \emptyset$. Assume that $f$ exhibits asymptotically bounded decay with respect to $g(x) = \|x\|^p$ on $X$ for some $p \geq 0$, and assume that*

$$X_\infty \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f).$$

*Then, the problem $(P)$ has a finite optimal value $f^*$ and its solution set $X^*$ is nonempty.*

*Proof.* The proof is organized along three steps. The first is to show that a regularized problem of the form $\inf_{x\in X}\{f(x) + r\|x\|^p\}$ is coercive and, thus, the regularized problem has a solution for every $r > 0$. The second step is to show that any sequence of solutions to the regularized problems, as we vary $r$, is bounded and, thus, has an accumulation point. Finally, we show that every limit point of such a sequence is a solution to the problem $(P)$.

By our assumption that $f$ exhibits asymptotically bounded decay with respect to $g(x) = \|x\|^p$ on $X$ for some $p \geq 0$, we have that (see Definition 3.1.1)

$$c = \liminf_{\substack{\|x\|\to\infty \\ x\in X}} \frac{f(x)}{\|x\|^p} > -\infty. \tag{8}$$

If $c = +\infty$, then $f$ is coercive on $X$ so by Proposition 2.3.2, the problem $(P)$ has a finite optimal value and it has an optimal solution.

Next, we consider the case $c \in \mathbb{R}$.

(*Step 1: The regularized problem has a solution.*) Let $\epsilon > 0$ be arbitrarily small. By the asymptotically bounded decay of $f$ on $X$ in (8), there exists $R > 0$ large enough so that

$$f(x) \geq (c - \epsilon)\|x\|^p \qquad \text{for all } x \in X \text{ with } \|x\| \geq R.$$

Let $r > 0$ be arbitrary and consider the function $f(x) + r\|x\|^{p+1}$. We have

$$f(x) + r\|x\|^{p+1} \geq r\|x\|^{p+1} + (c - \epsilon)\|x\|^p \qquad \text{for all } x \in X \text{ with } \|x\| \geq R,$$

implying that

$$\liminf_{\substack{\|x\|\to\infty \\ x\in X}} \left\{f(x) + r\|x\|^{p+1}\right\} \geq \liminf_{\substack{\|x\|\to\infty \\ x\in X}} \left\{\|x\|^p\left(r\|x\| + c - \epsilon\right)\right\} = +\infty.$$

Thus, for any $\rho > 0$, the regularized function $f(x) + r\|x\|^{p+1}$ is coercive, so by Proposition 2.3.2, the regularized problem

$$\inf_{x \in X}\{f(x) + r\|x\|^{p+1}\}$$

has a finite optimal value and it has a solution for every $r > 0$.
(*Step 2: A sequence of solutions to regularized problems is bounded.*) Now consider a sequence of positive scalars $\{r_k\}$ such that $r_k \to 0$ as $k \to \infty$. For each $k$, let $x_k^* \in X$ be a solution to the regularized problem

$$\inf_{x \in X}\{f(x) + r_k\|x\|^{p+1}\}. \tag{9}$$

We claim that under the conditions of the theorem, the sequence $\{x_k^*\} \subset X$ must be bounded. To prove this, we argue by contradiction. Assume that $\{x_k^*\}$ is unbounded. Without loss of generality, we may assume that $x_k^* \neq 0$ for all $k$, so that the sequence $\{x_k^* \cdot \|x_k^*\|^{-1}\}$ is bounded. Hence, it must have a convergent sub-sequence. Without loss of generality, we let $\{x_k^* \cdot \|x_k^*\|^{-1}\} \to d$. Since $\{x_k^*\} \subset X$, it follows that $d \in X_\infty$. Fixing an arbitrary $x_0 \in X$ we have

$$f(x_k^*) \leq f(x_k^*) + r_k\|x_k^*\|^{p+1} \leq f(x_0) + r_k\|x_0\|^{p+1} \qquad \text{for all } k,$$

which implies

$$\liminf_{k\to\infty} \frac{f(x_k^*)}{\|x_k^*\|} \leq \lim_{k\to\infty} \frac{f(x_0) + r_k\|x_0\|^{p+1}}{\|x_k^*\|} = 0,$$

where in the last equality we use the fact $r_k \to 0$. Using the explicit form for the asymptotic function $f_\infty$ as given in Theorem 2.3.1, we obtain

$$f_\infty(d) \leq \liminf_{k\to\infty} \frac{f(\|x_k^*\|(\|x_k^*\|^{-1}x_k^*))}{\|x_k^*\|} = \liminf_{k\to\infty} \frac{f(x_k^*)}{\|x_k^*\|} \leq 0.$$

Thus, it follows that $d \in X_\infty \cap \mathcal{K}(f)$. By our assumption that $X_\infty \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f)$, we have that $d \in \mathcal{R}(X) \cap \mathcal{R}(f)$, that is, $d$ is a retractive direction for the set $X$ and a retractive direction for the function $f$. By the definition of such directions, for the sequence $\{x_k^* \cdot \|x_k^*\|^{-1}\}$ and any $\rho > 0$, there exists a large enough index $K$ such that

$$x_k^* - \rho d \in X, \qquad f(x_k^* - \rho d) \leq f(x_k^*) \qquad \text{for all } k \geq K.$$

Thus, for any $k \geq K$, we have

$$f(x_k^*) + r_k\|x_k^*\|^{p+1} \leq f(x_k^* - \rho d) + r_k\|x_k^* - \rho d\|^{p+1} \leq f(x_k^*) + r_k\|x_k^* - \rho d\|^{p+1}.$$

The first inequality follows from the optimality of the point $x_k^*$ for the regularized problem (9) and the fact that $x_k - \rho d \in X$. The second inequality follows by $f(x_k^* - \rho d) \leq f(x_k^*)$. Since $r_k > 0$ it follows that for $k \geq K$,

$$\|x_k^*\|^{p+1} \leq \|x_k^* - \rho d\|^{p+1},$$

and therefore,

$$\|x_k^*\|^2 \le \|x_k^* - \rho d\|^2 \qquad \text{for all } k \ge K.$$

Since $\|d\| = 1$, the preceding inequality implies that $2\langle x_k^*, d\rangle \le \rho$ for all $k \ge K$. Therefore, it follows that

$$\lim_{k\to\infty} \frac{\langle x_k^*, d\rangle}{\|x_k^*\|} \le \lim_{k\to\infty} \frac{\rho}{2\|x_k^*\|} = 0.$$

On the other hand, since $\lim_{k\to\infty} x_k^* \cdot \|x_k^*\|^{-1} = d$ and $\|d\| = 1$, we obtain that $1 \le 0$, which is a contradiction. Hence, it must be that $\{x_k^*\}$ is bounded.
(*Step 3: Any accumulation point of sequence $\{x_k^*\}$ is a solution to* $(P)$.) Since $\{x_k^*\}$ is bounded, it must have an accumulation point. Now, without loss of generality, let $\{x_k^*\}$ converge to $x^*$. Since $x_k^* \to x^*$ and $r_k \to 0$, we have that $r_k\|x_k^*\|^{p+1} \to 0$, as $k \to \infty$, implying that

$$\liminf_{k\to\infty} f(x_k^*) = \liminf_{k\to\infty} \left(f(x_k^*) + r_k\|x_k^*\|^{p+1}\right). \tag{10}$$

Due to the optimality of $x_k^*$ for the regularized problem, we have for any $x \in X$,

$$f(x_k^*) + r_k\|x_k^*\|^{p+1} \le f(x) + r_k\|x\|^{p+1} \qquad \text{for all } k.$$

Therefore, since $r_k \to 0$, for any $x \in X$,

$$\liminf_{k\to\infty} \left(f(x_k^*) + r_k\|x_k^*\|^{p+1}\right) \le f(x) + \lim_{k\to\infty} r_k\|x\|^{p+1} = f(x). \tag{11}$$

Finally, since $x_k^* \to x^*$ and the function $f$ is closed, from relations (10) and (11), it follows that

$$f(x^*) \le \liminf_{k\to\infty} f(x_k^*) \le f(x) \quad \text{ for all } x \in X.$$

Since $\{x_k^*\} \subset X$ and $X$ is closed, we have that $x^* \in X$ and $x^*$ is an optimal solution to the problem $(P)$. $\square$ $\square$

The proof of Theorem 4.2.2 is motivated by the proof analysis of Theorem 3.4.1 in [2], which provides necessary and sufficient conditions for the existence of solutions to an unconstrained minimization problem. Unlike Theorem 3.4.1 in [2], we consider a more general constrained problem. However, when applied to an unconstrained minimization problem, our Theorem 4.2.2 is weaker than Theorem 3.4.1, since it establishes only sufficient conditions.

Related results for the existence of solutions to $\inf_{x\in X} f(x)$ have been reported in Proposition 12 and Proposition 13 of [6] for a convex and a non-convex function $f$, respectively. These results, however, have more stringent requirements than that of Theorem 4.2.2, as discussed later on in Section 6.1. Related is also Proposition 3.1 of [13], which is also discussed in Section 6.1.

Next, we provide an example where the condition $X_\infty \cap \mathcal{K}(F) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f)$ is violated and the problem does not have a finite optimal value.

***Example* 4.2.** *Consider the problem of minimizing a convex scalar function $f(x) = -\sqrt{x}$ over its domain $X = \{x \mid x \geq 0\}$. The optimal value is $f^* = -\infty$ and there is no solution. Since the function is convex, by Example 3.2, it exhibits asymptotically bounded decay with respect to $g(x) = \|x\|$ on $X$. The set $X$ is a closed convex cone, and we have $X_\infty = X$. The asymptotic cone of $f$ coincides with $X$ (see Example 4.1), i.e., $\mathcal{K}(f) = X$, implying that*

$$X_\infty \cap \mathcal{K}(f) = X.$$

*The cone of retractive directions of $X$ coincides with $X$, since $X$ is a polyhedral set (see Example 3.7). The cone $\mathcal{R}(f)$ of retractive directions of $f$ contains only the zero vector. To see this note that for $x_k = \lambda k$, with $\lambda > 0$, and $t_k = k$ for all $k \geq 1$, we have that $\{x_k\}$ converges in the direction $d = \lambda$. However, for any $\rho \in (0, 1)$ and any $k \geq 1$,*

$$f(x_k - \rho\lambda) = -\sqrt{k - \rho\lambda} > -\sqrt{k} = f(x_k).$$

*Hence, we have $\mathcal{R}(f) = \{0\}$, and*

$$\mathcal{R}(X) \cap \mathcal{R}(f) = \{0\},$$

*thus implying that the condition $X_\infty \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f)$ of Theorem 4.2.2 is violated.* □

The following example shows that when the condition $X_\infty \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f)$ is violated, the problem can have a finite optimal value but not a solution.

***Example* 4.3.** *Consider the problem of minimizing a convex scalar function $f(x) = e^x$ over its domain $X = \mathbb{R}$. The optimal value is $f^* = 0$ and there is no solution. The function is convex, so by Example 3.2, it exhibits asymptotically bounded decay with respect to $g(x) = \|x\|$ on $X$. We have $X_\infty = X$ and $\mathcal{R}(X) = X$. The asymptotic cone of $f$ coincides is given by*

$$\mathcal{K}(f) = \{x \mid x \leq 0\},$$

*while the cone $\mathcal{R}(f)$ of retractive directions of $f$ contains only the zero vector. To see this, observe that for $x_k = -\lambda k$, with $\lambda > 0$, and $t_k = k$ for all $k \geq 1$, we have that $\{x_k\}$ converges in the direction $d = -\lambda$. However, for any $\rho \in (0, 1)$ and any $k \geq 1$,*

$$f(x_k + \rho\lambda) = e^{k+\rho\lambda} > e^k = f(x_k).$$

*Hence, we have $\mathcal{R}(f) = \{0\}$ and $\mathcal{R}(X) \cap \mathcal{R}(f) = \{0\}$, implying that the condition $X_\infty \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f)$ of Theorem 4.2.2 is violated.* □

In Theorem 4.2.2, we used $g(x) = \|x\|^p$ for $p \geq 0$. We now show the existence result in the case of a coercive function $g$.

**Theorem 4.2.3.** *Let the feasible set $X \subseteq \mathbb{R}^n$ be closed, and let the objective function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and closed. Suppose that $f$ exhibits* *asymptotically bounded decay with respect to a coercive function $g$ on $X$ and that $X \cap \mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$. Assume further that*

$$X_\infty \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f).$$

*Let $g$ have Lipschitz continuous gradients on $\mathbb{R}^n$ and, for any sequence $\{x_k\} \subseteq X$ converging in any nonzero direction $d \in \mathcal{R}(X) \cap \mathcal{R}(f)$, let the following relation hold*

$$\limsup_{k\to\infty} \frac{\langle \nabla g(x_k), d\rangle}{t_k} > 0 \qquad \textit{where} \ \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

*Then, the problem (P) has a finite optimal value and its solution set is nonempty.*

*Proof.* Consider the regularized problem for any $r > 0$,

$$\inf_{x\in X} \{f(x) + rg^2(x)\}. \tag{12}$$

By our assumption that $f$ exhibits asymptotically bounded decay with respect to $g$ on $X$, we have by Definition 3.1.1 that

$$c = \liminf_{\substack{\|x\|\to\infty \\ x\in X}} \frac{f(x)}{g(x)} > -\infty. \tag{13}$$

If $c = +\infty$, then $f$ is coercive since $g$ is coercive, and the result follows by Proposition 2.3.2.

Next, we consider the case $c \in \mathbb{R}$.

(*Step 1: The regularized problem has a solution.*) Since $g$ is coercive, there exists an $R_g > 0$ such that

$$g(x) > 0 \qquad \text{for all } x \in X \text{ with } \|x\| \geq R_g.$$

By (13), for any $\epsilon > 0$, there exists large enough $R_\epsilon > R_g > 0$ such that

$$f(x) \geq (c-\epsilon) g(x) \qquad \text{for all } x \in X \text{ with } \|x\| \geq R_\epsilon.$$

Thus, for all $x \in X$ with $\|x\| \geq R_\epsilon$,

$$f(x) + rg^2(x) \geq (c-\epsilon)g(x) + rg^2(x),$$

implying that

$$\liminf_{\substack{\|x\|\to\infty \\ x\in X}} \{f(x) + rg^2(x)\} \geq \liminf_{\substack{\|x\|\to\infty \\ x\in X}} g(x)\left(rg(x) + c - \epsilon\right)) = +\infty,$$

where the equality is due to $r > 0$ and the coercivity of $g$ on $X$. Hence, the regularized objective function itself is also coercive, and by Proposition 2.3.2 the problem $\inf_{x\in X}\{f(x) + rg^2(x)\}$ has a solution for any $r > 0$.

(*Step 2: A sequence of solutions to regularized problems is bounded.*) Now, consider a sequence of positive scalars $\{r_k\}$ such that $r_k \to 0$ as $k \to \infty$. For each $k$, let $x_k^* \in X$ be a solution to the regularized problem in (12) with $r = r_k$. Towards a contradiction, assume that the sequence $\{x_k^*\} \subseteq X$ is unbounded.

Without loss of generality, let $x_k^* \neq 0$ for all $k$, and consider the sequence $\{x_k^*/\|x_k^*\|\}$. This sequence is bounded and, hence, must have a convergent subsequence. Again without loss of generality, let $\{x_k^*/\|x_k^*\|\} \to d$ as $k \to \infty$. Thus, $d \in X_\infty$. Since $x_k^* \in X$ is a solution of the regularized problem, we have

$$f(x_k^*) \leq f(x_k^*) + r_k g^2(x_k^*) \leq f(x) + r_k g^2(x) \quad \text{ for all } x \in X \text{ and all } k. \quad (14)$$

Thus, it follows that for an arbitrary fixed $x_0 \in X$,

$$\liminf_{k\to\infty} \frac{f(x_k^*)}{\|x_k\|} \leq \lim_{k\to\infty} \frac{f(x_0) + r_k g^2(x_0)}{\|x_k\|} = 0,$$

where we use the fact $r_k \to 0$. Employing Theorem 2.3.1 which gives the explicit form for the asymptotic function $f_\infty$, we can see that

$$f_\infty(d) \leq \liminf_{k\to\infty} \frac{f(\|x_k^*\|(\|x_k^*\|^{-1}x_k^*))}{\|x_k^*\|} = \liminf_{k\to\infty} \frac{f(x_k^*)}{\|x_k^*\|} \leq 0.$$

Hence $d \in X_\infty \cap \mathcal{K}(f)$. By the assumption that $X_\infty \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f)$, it follows that $d \in \mathcal{R}(X) \cap \mathcal{R}(f)$. Since $d$ is a retractive direction of the set $X$ and the function $f$, for the sequence $\{x_k^*/\|x_k^*\|\}$ converging in the direction $d$ and any for $\rho > 0$, there exists a sufficiently large index $K$ so that

$$x_k^* - \rho d \in X \quad \text{ and } \quad f(x_k^* - \rho d) \leq f(x_k) \quad \text{ for all } k \geq K.$$

Thus, for any $k \geq K$,

$$f(x_k^*) + r_k g^2(x_k^*) \leq f(x_k^* - \rho d) + r_k g^2(x_k^* - \rho d) \leq f(x_k^*) + r_k g^2(x_k^* - \rho d)^2, \quad (15)$$

where the first inequality follows from the optimality of the point $x_k^*$ for the regularized problem and the fact that $x_k - \rho d \in X$, while the second inequality follows by $f(x_k^* - \rho d) \leq f(x_k^*)$. Since $r_k > 0$ for all $k$, relation (15) implies that

$$g^2(x_k^*) \leq g^2(x_k^* - \rho d) \quad \text{ for all } k \geq K.$$

Assume that $K$ is large enough so that $g(x_k) > 0$ for all $k \geq K$. Then, the preceding relation implies that

$$0 < g(x_k^*) \leq g(x_k^* - \rho d) \leq g(x_k^*) - \rho\langle \nabla g(x_k^*), d\rangle + \frac{\rho^2}{2L} \quad \text{ for all } k \geq K. \quad (16)$$

where in the last inequality we use the Lipshitz continuity of $\nabla g$ and $\|d\| = 1$. Relation (16) and the fact that $\rho > 0$ imply that

$$\langle \nabla g(x_k^*), d\rangle \leq \frac{\rho}{2L} \quad \text{ for all } k \geq K.$$

Hence, it follows that

$$\limsup_{k\to\infty} \frac{\langle \nabla g(x_k^*), d\rangle}{\|x_k^*\|} \leq \limsup_{k\to\infty} \frac{\rho}{2L\|x_k^*\|} = 0,$$

which contradicts the assumption on $\nabla g$ that $\limsup_{k\to\infty} \frac{\langle \nabla g(x_k^*), d\rangle}{\|x_k^*\|} > 0$. Thus, the sequence $\{x_k^*\}$ must be bounded.

(*Step 3: Any accumulation point of the sequence* $\{x_k^*\}$ *is a solution to* $(P)$) Since the sequence $\{x_k^*\}$ is bounded, it must have an accumulation point. Without loss of generality, let $\{x_k^*\} \to x^*$ as $k \to \infty$. Taking the limit inferior in (14) yields

$$\liminf_{k\to\infty} f(x_k^*) \le \liminf_{k\to\infty} \{f(x) + r_k g^2(x)\} \quad \text{for all } x \in X.$$

Since $x_k^* \to x^*$ and $r_k \to 0$, by the closedness of $f$ it follows that

$$f(x^*) \le f(x) \quad \text{for all } x \in X.$$

Since $\{x_k^*\} \subseteq X$ and the set $X$ closed, we have that $x^* \in X$ is an optimal solution to the problem $(P)$. $\square$ $\square$

We have the following result as a special consequence of Theorem 4.2.2 (or Theorem 4.2.3) for the case when the set $X$ is polyhedral.

***Corollary* 4.2.1.** *Let $X$ be a nonempty polyhedral set and $f$ be a proper closed function with $X \cap \mathrm{dom} f \neq \emptyset$. Under assumptions of Theorem 4.2.2 (or Theorem 4.2.3) on the asymptotically bounded decay of $f$ on $X$, if*

$$X_\infty \cap \mathcal{K}(f) \subseteq X_\infty \cap \mathcal{R}(f),$$

*then the problem $(P)$ has a finite optimal value and a solution exists.*

*Proof.* The result follows by Theorem 4.2.2 (or Theorem 4.2.3) since a polyhedral set is retractive i.e., $\mathcal{R}(X) = X_\infty$ (see Example 3.7). $\square$ $\square$

# 5 Implications for Convex Problems

In this section, we discuss the relationship between the asymptotic cone and the lineality space [5] of a closed convex set. We also discuss properties of $\mathcal{R}(f)$ for a closed, proper function $f$. Then, we apply our main results from the preceding section to a convex minimization problem and show that our results are more general than those prior.

## 5.1 Asymptotic Cone, Lineality Space, and Cone of Retractive Directions

For a nonempty closed convex set $X \subseteq \mathbb{R}^n$, the asymptotic cone $X_\infty$ has two simple characterizations, as follows (Proposition 2.1.5 of [2]):

$$X_\infty = \{d \in \mathbb{R}^n \mid \exists x \in X \text{ such that } x + td \in X, \forall t \ge 0\},$$

$$X_\infty = \{d \in \mathbb{R}^n \mid x + td \in X, \forall x \in X, \forall t \ge 0\}.$$

The cone $X_\infty$ is often referred to as a recession cone of $X$ [14, 5]. The lineality space of a nonempty closed convex set $X$ is the set defined as follows [14, 5]:

$$\mathrm{Lin}(X) = X_\infty \cap (-X_\infty),$$

or equivalently

$$\mathrm{Lin}(X) = \{d \in \mathbb{R}^n \mid \exists x \in \mathbb{R}^n \text{ such that } x + td \in X,\ \forall t \in \mathbb{R}\}.$$

For a nonempty closed convex set $X$, by the definition of the cone $\mathcal{R}(X)$ of retractive directions of $X$, we have

$$\mathrm{Lin}(X) \subseteq \mathcal{R}(X).$$

The inclusion can be strict. For example, if $X = \{x \in \mathbb{R}^n \mid Ax \le b\}$ for some matrix $A$ and a vector $b$, then $\mathrm{Lin}(X) = \{d \mid Ad = 0\}$ and $\mathcal{R}(X) = X_\infty = \{d \mid Ad \le 0\}$ by the polyhedrality of $X$.

The constancy space $\mathcal{C}(f)$ (cf. (6)) of a proper closed convex function $f$ satisfies the following relations (see Theorem 2.5.3 in [2]):

$$\mathcal{C}(f) = \{d \in \mathbb{R}^n \mid \exists x \in \mathrm{dom} f \text{ such that } f(x + td) = f(x),\ \forall t \in \mathbb{R}\},$$

$$\mathcal{C}(f) = \{d \in \mathbb{R}^n \mid f(x + td) = f(x),\ \forall x \in \mathrm{dom} f, t \in \mathbb{R}\}.$$

For a proper closed convex function $f$, by the definition of the cone $\mathcal{R}(f)$ of retractive directions of $f$, it follows that

$$\mathcal{C}(f) \subseteq \mathcal{R}(f).$$

## 5.2 Sufficient Conditions for Existence of Solutions

We next consider the special case of Theorem 4.2.2 as applied to a general convex problem of the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_j(x) \le 0, j = 1, \ldots, m,\ x \in C. \end{array} \tag{17}$$

We modify Theorem 4.2.2 to obtain the following result.

**Theorem 5.2.1.** *Let $X$ be a closed convex set, and let $f$ and each $g_j$ be a proper closed convex function such that $C \cap (\cap_{j=1}^m \mathrm{dom} g_j) \cap \mathrm{dom} f \neq \emptyset$. Assume that*

$$C_\infty \cap \left(\cap_{j=1}^m \mathcal{K}(g_j)\right) \cap \mathcal{K}(f) \subseteq \mathcal{R}(C) \cap \left(\cap_{j=1}^m \mathcal{R}(L_0(g_j))\right) \cap \mathcal{R}(f). \tag{18}$$

*Then, the problem* (17) *has a finite optimal value and its solution set is nonempty. Moreover, if the set $C$ is polyhedral, then the result holds under a weaker condition that*

$$C_\infty \cap \left(\cap_{j=1}^m \mathcal{K}(g_j)\right) \cap \mathcal{K}(f) \subseteq C_\infty \cap \left(\cap_{j=1}^m \mathcal{R}(L_0(g_j))\right) \cap \mathcal{R}(f). \tag{19}$$

*Proof.* Since the objective function is convex, it exhibits asymptotically bounded decay with respect to $g(x) = \|x\|$ (see Example 3.2). Furthermore, let

$$X = \{x \in C \mid g_j(x) \leq 0,\ j = 1, \ldots, m\}.$$

The set $X$ is nonempty, closed, and convex, so by Proposition 2.2.1, we have

$$X_\infty = C_\infty \cap \left(\cap_{j=1}^m \{x \in \mathbb{R}^n \mid g_j(x) \leq 0\}_\infty\right).$$

By Proposition 2.3.1, we have that $\{x \mid g_j(x) \leq 0\}_\infty = \mathcal{K}(g_j)$ for all $j = 1, \ldots, m$, thus implying that

$$X_\infty = C_\infty \cap \left(\cap_{j=1}^m \mathcal{K}(g_j)\right).$$

Moreover, by Proposition 3.2.2, we have that

$$\mathcal{R}(C) \cap \left(\cap_{j=1}^m \mathcal{R}(L_0(g_j))\right) \subseteq \mathcal{R}\left(C \cap \left(\cap_{j=1}^m L_0(g_j)\right)\right) = \mathcal{R}(X).$$

The preceding two relations combined with (18) show that the condition $X_\infty \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f)$ of Theorem 4.2.2 is satisfied, and the result follows.

When the set $C$ is polyhedral, the result follows by Corollary 4.2.1. □ □

Theorem 5.2.1 is more general than Proposition 6.5.4 in [5], which requires that the problem (17) has a finite optimal value and that

$$C_\infty \cap \left(\cap_{j=1}^m \mathcal{K}(g_j)\right) \cap \mathcal{K}(f) \subseteq \mathrm{Lin}(C) \cap \left(\cap_{j=1}^m \mathcal{C}(g_j)\right) \cap \mathcal{C}(f).$$

The preceding condition implies that the condition (18) holds since $\mathrm{Lin}(C) \subseteq \mathcal{R}(C)$ and the analogous relation holds for the constancy space and the cone of retractive directions for the functions $f$ and $g_j$.

For the case of a polyhedral set $C$, Theorem 5.2.1 is more general than Proposition 6.5.5 in [5], which requires that the optimal value of the problem (17) is finite and that a stronger condition than (19) holds, namely that

$$C_\infty \cap \left(\cap_{j=1}^m \mathcal{K}(g_j)\right) \cap \mathcal{K}(f) \subseteq \left(\cap_{j=1}^m \mathcal{C}(g_j)\right) \cap \mathcal{C}(f).$$

Finally, when the set $C$ is polyhedral and $g_j \equiv 0$ for all $j$, the condition (19) reduces to

$$C_\infty \cap \mathcal{K}(f) \subseteq C_\infty \cap \mathcal{R}(f).$$

In this case, Theorem 5.2.1 provides a weaker sufficient condition than that of Theorem 27.3 in [14] requiring that $C_\infty \cap \mathcal{K}(f) \subseteq \mathcal{C}(f)$.

# 6 Implications for Non-convex Problems

In this section, we consider the implications of our main results for several types of non-convex problems for the case where the constraint set $X$ is generic and the case when $X$ is given by non-convex functional inequalities.

## 6.1 Generic Constraint Set

We consider the problem $(P)$ for the case when $f$ is convex, for which we have a special case of Theorem 4.2.2.

**Theorem 6.1.1.** *Let $X$ be a nonempty closed set and $f$ be a proper closed convex function with $X \cap \mathrm{dom} f \neq \emptyset$. Assume that*

$$X_\infty \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{C}(f).$$

*Then, the problem $(P)$ has a finite optimal value and a solution exists.*

*Proof.* A convex function exhibits asymptotically bounded decay with respect to $g(x) = \|x\|$ as seen in Example 3.2. Thus, the result follows from Theorem 4.2.2 and the fact that $\mathcal{C}(f) \subseteq \mathcal{R}(f)$ when $f$ is convex. □ □

To the best of our knowledge the result of Theorem 6.1.1 is new. An existing result that considers convex objective and a non-convex constraint set is Proposition 12 in [6], which relies on the stringent assumption that:

(A1) Every nonzero direction $d \in X_\infty$ is retractive and, for all $x \in X$, there exists an $\bar{\alpha} \geq 0$ such that $x + \alpha d \in X$ for all $\alpha \geq \bar{\alpha}$.

In the following example Assumption (A1) fails to hold, so Proposition 12 in [6] cannot be applied to assert the existence of solutions. However, Theorem 6.1.1 can be used.

***Example* 6.1.** *Consider minimizing a proper closed convex function $f$ on the set $X$ given by $X = \{x \in \mathbb{R}^2 \mid x_2 \leq x_1^2\}$ (cf. Figure 2.2). The complement of $X$ is open and convex. Hence, by Proposition 4 of [6], we have that $\mathcal{R}(X) = X_\infty$. However, the set $X$ does not satisfy Assumption (A1) since, for the direction $d = (0, 1) \in X_\infty$ and any $x$ that lies on the boundary of $X$ (i.e., $x_1^2 = x_2$), it is not the case that $x + \alpha d \in X$ for any $\alpha > 0$. Thus, Proposition 12 of [6] cannot be used to claim the existence of solutions in this case. However, if $\mathcal{K}(f) = \mathrm{Lin}(f)$ (such as, for example, when $f(x_1, x_2) = |x_1|$) then by Theorem 6.1.1, the problem $\inf_{x \in X} f(x)$ has a solution.* □

Now, we consider the problem $\inf_{x \in X} f(x)$ with a non-convex set $X$ and a non-convex function $f$ and compare our Theorem 4.2.2 with Proposition 13 in [6]. We first recast the assumptions of Proposition 13 in [6] and then prove that this proposition is a special case of Theorem 4.2.2.

The problem that [6] considers is exactly $(P)$ under Assumption (A1) and an additional assumption for a sequence of sets $\{S_k\}$, such that $S_k = X \cap L_{\gamma_k}(f)$ where $\{\gamma_k\} \subseteq \mathbb{R}$ is a decreasing scalar sequence, which requires that

(A2) For every asymptotic direction[1] $d$ of $\{S_k\}$ and for each $x \in X$,

$$\text{either } \lim_{\alpha \to \infty} f(x + \alpha d) = -\infty \text{ or } f(x - d) \leq f(x).$$

[1] A (nonzero) direction $d$ such that for some unbounded sequence $\{x_k\}$, with $x_k \in S_k$ for all $k$, we have $\lim_{k \to \infty} x_k / \|x_k\| = d / \|d\|$.

The statement of Proposition 13 in [6] is as follows: Suppose that Assumptions (A1) and (A2) hold for problem $(P)$, where the set $X$ is closed and the function $f$ is proper and closed with $X \cap \mathrm{dom} f \neq \emptyset$. Then, the objective function $f$ attains a minimum over $X$ if and only if the optimal value $f^* = \inf_{x \in X} f(x)$ is finite.

We have already discussed the limitations of Assumption (A1) in Example 6.1 that also apply to Proposition 13 in [6]. Now we show that Assumptions (A1) and (A2) imply the assumptions of Theorem 4.2.2.

**Proposition 6.1.1.** *Let $X$ be a closed set and $f$ be a proper closed function with $X \cap \mathrm{dom} f \neq \emptyset$. Suppose that problem $(P)$ satisfies Assumptions (A1) and (A2). If the optimal value $f^* = \inf_{x \in X} f(x)$ is finite, then $f$ exhibits asymptotically bounded decay with respect to $g(x) = \|x\|$ on the set $X$. Furthermore, the following relation holds*

$$X_\infty \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f).$$

*Proof.* Since $f$ is assumed to have a finite minimum on $X$, by Example 3.1 we have that $f$ exhibits asymptotically bounded decay with respect to the function $g(x) = \|x\|$. Since $\{\gamma_k\}$ is a decreasing scalar sequence, we have that

$$S_k = X \cap L_{\gamma_k}(f) \subseteq X \cap L_{\gamma_1}(f) = S_1 \qquad \text{for all } k \geq 1.$$

Therefore, we have that $d \in (X \cap L_{\gamma_1}(f))_\infty$. By Proposition 2.2.1, the retractive directions of the intersection set is contained in the intersection of retractive directions of the sets, so we have for any asymptotic direction $d$ of $\{S_k\}$,

$$d \in (X \cap L_{\gamma_1}(f))_\infty \subseteq X_\infty \cap (L_{\gamma_1}(f))_\infty.$$

Moreover, by Proposition 2.3.1 it holds that $(L_{\gamma_1}(f))_\infty \subseteq \mathcal{K}(f)$. Thus, for any asymptotic direction $d$ of $\{S_k\}$, we have

$$d \in X_\infty \cap \mathcal{K}(f) = \mathcal{R}(X) \cap \mathcal{K}(f), \tag{20}$$

where the equality holds since $X$ is a retractive set by Assumption (A1).

Next, by Assumption (A2), for every $x \in X$, either $\lim_{\alpha \to \infty} f(x + \alpha d) = -\infty$ or $f(x - d) \leq f(x)$. Suppose that the former case holds for some $x \in X$. Then, as $\alpha \to \infty$, we have that $f(x + \alpha d) \to -\infty$ but this is a contradiction since, by Assumption (A1), $x + \alpha d \in X$ for all $\alpha \geq \bar{\alpha}$ and $f$ was assumed to have a finite minimum on $X$. Hence, we must have

$$f(x - d) \leq f(x) \qquad \text{for all } x \in X. \tag{21}$$

Let $\{x_k\}$ be a sequence associated with an asymptotic direction $d$ of $\{S_k\}$ (i.e., $x_k \in S_k$ for all $k$ with $x_k/\|x_k\|^{-1} \to d/\|d\|$). Note that if $x_k/\|x_k\| \to d/\|d\|$, then we also have for any $\rho > 0$,

$$\lim_{k \to \infty} \frac{x_k}{\|x_k\|} = \frac{d}{\|d\|} = \frac{\rho d}{\|\rho d\|}.$$

Therefore, $\rho d$ is also an asymptotic direction of $\{S_k\}$ and relation (21) must also hold for any asymptotic direction $d$ of $\{S_k\}$ and for any $\rho$,

$$f(x - \rho d) \leq f(x) \qquad \text{for all } x \in X.$$

In particular, it holds for every point $x_k$ of the sequence $\{x_k\}$, thus implying that $d$ is a retractive direction of the function $f$ according to Definition 3.3.1. Hence, every asymptotic direction $d$ of $\{S_k\}$ also lies in the set $\mathcal{R}(f)$. This and relation (20), imply that $d \in \mathcal{R}(X) \cap \mathcal{R}(f)$. □ □

## 6.2 Constraint Set given by Functional Inequalities

In this section, we consider problems of the following form:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_j(x) \leq 0,\ j \in \{1, \ldots, m\}, \quad x \in C, \end{array} \tag{22}$$

where $C$ is a closed set, and $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and each $g_j : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are proper closed functions. Existence of solutions to this problem has been studied in both general settings [13] as well as special settings [4, 8, 9, 12, 15].

We provide conditions for the existence of solutions based on Theorem 4.2.2 and Theorem 4.2.3 combined.

**Theorem 6.2.1.** *Let $C$ be a closed set, and let $f$ and each $g_j$ be proper closed functions with $C \cap \left(\cap_{j=1}^m \mathrm{dom} g_j\right) \cap \mathrm{dom} f \neq \emptyset$. Let $X = \{x \in C \mid g_j(x) \leq 0, j = 1, \ldots, m\}$, and assume that $f$ exhibits asymptotically bounded decay on the set $X$ with respect to $g(x) = \|x\|^p$ for some $p \geq 0$ (or with respect to a coercive function $g$ satisfying the assumptions of Theorem 4.2.3). Then, the problem* (22) *has a finite optimal value and an optimal solution exists under any of the following conditions:*
*(C1)* $\quad C_\infty \cap (\cap_{j=1}^m (L_0(g_j))_\infty) \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f)$,
*(C2)* $\quad C_\infty \cap (\cap_{j=1}^m \mathcal{K}(g_j)) \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f)$.

*Proof.* By Proposition 2.2.1, we have that

$$X_\infty \subseteq C_\infty \cap (\cap_{j=1}^m (L_0(g_j))_\infty). \tag{23}$$

Thus, if the condition (C1) holds, then it follows that $X_\infty \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f)$ and the result follows by Theorem 4.2.2 (or Theorem 4.2.3). If condition (C2) holds, then by Proposition 2.3.1 we have

$$(L_0(g_j))_\infty \subseteq \mathcal{K}(g_j) \qquad \text{for all } j = 1, \ldots, m.$$

By combining these relations with (23), again we have that $X_\infty \cap \mathcal{K}(f) \subseteq \mathcal{R}(X) \cap \mathcal{R}(f)$ and the result follows as in the preceding case. □ □

In Theorem 6.2.1, we could not write the cone $\mathcal{R}(X)$ in terms of such cones of the individual sets defining the set $X$, as there is no particular rule that can be applied here, in general. In the case when $X$ and the functions $g_j$ are convex, the conditions (C1) and (C2) of Theorem 6.2.1 coincide, since $(L_0(g_j))_\infty = \mathcal{K}(g_j)$ for all $j$ by Proposition 2.3.1.

We next provide another result for the case when the constraint set $C$ is convex and the functions $g_j$ are convex.

**Theorem 6.2.2.** *Let assumptions of Theorem 6.2.1 hold. Additionally, assume that the set $C$ is convex and that each $g_j$ is a convex function. Then, the problem (22) has a finite optimal value and an optimal solution exists under any of the following conditions:*
*(C3)* $C_\infty \cap (\cap_{j=1}^m \mathcal{K}(g_j)) \cap \mathcal{K}(f) \subseteq \mathcal{R}(C) \cap (\cap_{j=1}^m \mathcal{R}(L_0(g_j)) \cap \mathcal{R}(f)$,
*(C4)* $C_\infty \cap (\cap_{j=1}^m \mathcal{K}(g_j)) \cap \mathcal{K}(f) \subseteq \mathrm{Lin}(C) \cap (\cap_{j=1}^m \mathcal{C}(g_j)) \cap \mathcal{R}(f)$.

*Proof.* Let the condition (C3) hold. Since $X$ is convex and each $g_j$ is convex, we have that $X_\infty = C_\infty \cap (\cap_{j=1}^m \mathcal{K}(g_j))$. Moreover, by Proposition 3.2.2 we have

$$\mathcal{R}(C) \cap (\cap_{j=1}^m \mathcal{R}(L_0(g_j)) \subseteq \mathcal{R}(X).$$

Thus, the condition (C2) of Theorem 6.2.1 is satisfied and the result follows. Suppose that the condition (C4) holds. Then, since

$$\mathrm{Lin}(C) \cap (\cap_{j=1}^m \mathcal{C}(g_j)) = \mathrm{Lin}(X) \subseteq \mathcal{R}(X),$$

it follows that the condition (C2) of Theorem 6.2.1 is satisfied. □ □

Now consider problem (22) where each $g_j$ is a convex polynomial. As a corollary of Theorem 6.2.2, we have the following result which extends Theorem 3 in [4], where $f$ is also convex polynomial.

***Corollary* 6.2.1.** *Consider the problem (22), where $C = \mathbb{R}^n$, the objective function $f$ is proper and closed, and each $g_j$ is a convex polynomial. Further, assume that*

$$\left(\cap_{j=1}^m \mathcal{K}(g_j)\right) \cap \mathcal{K}(f) \subseteq \cap_{j=1}^m \mathcal{R}(g_j) \cap \mathcal{R}(f).$$

*Then, the problem has a finite optimal value and a solution exists.*

*Proof.* By Example 3.4, a polynomial of order $p$ asymptotically decays with respect to the function $g(x) = \|x\|^p$. Since each $g_j$ is a polynomial, by Lemma 3.3.2 we have that $\mathcal{R}(g_j) = \mathcal{C}(g_j)$ for all $j$. Therefore, the condition (C4) of Theorem 6.2.2 is satisfied, with $C = \mathbb{R}^n$, and the result follows. □ □

The following example shows that when the condition on the asymptotic cones of the functions $g_j$ and $f$ of Corollary 6.2.1 is violated, the problem (22) may not have a solution.

***Example* 6.2** (Example 2 in [12])**.** *Consider the following problem*

$$\begin{aligned} &\textit{minimize} && f(x) = -2x_1x_2 + x_3x_4 + x_1^2 \\ &\textit{subject to} && g_1(x) = x_1^2 - x_3 \le 0 \\ &&& g_2(x) = x_2^2 - x_4 \le 0. \end{aligned} \tag{24}$$

*The objective polynomial $f$ is non-convex while both $g_1$ and $g_2$ are convex polynomials. By the convexity of the constraint sets, we have*

$$\mathcal{K}(g_1) = \{(0, d_2, d_3, d_4) \mid d_2 \in \mathbb{R}, d_3 \geq 0, d_4 \in \mathbb{R}\},$$

$$\mathcal{K}(g_2) = \{(d_1, 0, d_3, d_4) \mid d_1 \in \mathbb{R}, d_3 \in \mathbb{R}, d_4 \geq 0\}.$$

*Therefore,*

$$\mathcal{K}(g_1) \cap \mathcal{K}(g_2) = \{(0, 0, d_3, d_4) \mid d_3 \geq 0, d_4 \geq 0\}.$$

*Moreover, by Lemma 3.3.2 we have $\mathcal{C}(g_j) = \{0\}$ for $j = 1, 2$ so that $\mathcal{C}(g_1) \cap \mathcal{C}(g_2) = \{0\}$. The function $f$ is a polynomial of order 2, so by Lemma 3.3.1 we have $\mathcal{K}(f) = \{d \mid f(d) \leq 0\}$. Then, it follows that*

$$\mathcal{K}(g_1) \cap \mathcal{K}(g_2) \cap \mathcal{K}(f) = \{0, 0, d_3, d_4) \mid d_3 d_4 = 0, d_3 \geq 0, d_4 \geq 0\}.$$

*Since $\mathcal{C}(g_1) \cap \mathcal{C}(g_2) = \{0\}$, we must have $\mathcal{C}(g_1) \cap \mathcal{C}(g_2) \cap \mathcal{R}(f) = \{0\}$. Therefore, the condition*

$$\mathcal{K}(g_1) \cap \mathcal{K}(g_2) \cap \mathcal{K}(f) \subseteq \mathcal{C}(g_1) \cap \mathcal{C}(g_2) \cap \mathcal{R}(f)$$

*does not hold, since a nonzero direction $d = (0, 0, d_3, 0)$ with $d_3 > 0$ belongs to $\mathcal{K}(g_1) \cap \mathcal{K}(g_2) \cap \mathcal{K}(f)$ but not to $\mathcal{C}(g_1) \cap \mathcal{C}(g_2) \cap \mathcal{R}(f)$. However, the optimal value of the problem is $f^* = -1$ which is not attained, as shown in [12].* □

We conclude this section by comparing our results with the main result of [13], which is Proposition 3.1 therein, and we demonstrate on an example that our results are more general.

Consider the following problem

$$\begin{array}{ll} \text{minimize} & f(x_1, x_2) = x_1^2 + x_2 \\ \text{subject to} & g_1(x_1, x_2) = \sqrt{|x_1|} - x_2 \leq 0. \end{array} \tag{25}$$

The optimal value of the problem is $f^* = 0$ and the optimal point is (0,0). The objective function is a convex polynomial, which exhibits asymptotically bounded decay with respect to $g(x) = \|x\|^2$. By Lemma 3.3.1 we have $\mathcal{K}(f) = \{(0, d_2) \mid d_2 \leq 0\}$, and $\mathcal{R}(f) = \{(0,0)\}$ by Lemma 3.3.2. For the set $X = \{x \in \mathbb{R}^2 \mid \sqrt{|x_1|} \leq x_2\}$, we have $X_\infty = \{(d_1, d_2) \mid d_1 \in \mathbb{R}, d_2 \geq 0\}$ and $\mathcal{R}(X) = X_\infty$. Hence, $X_\infty \cap \mathcal{K}(f) = \{(0,0)\}$ and $\mathcal{R}(X) \cap \mathcal{R}(f) = \{(0,0)\}$, implying that the conditions of Theorem 4.2.2 are satisfied. Hence, by Theorem 4.2.2, the problem has a finite optimal value and a solution exists.

Proposition 3.1 of [13] assumes that the optimal value of the problem (25) is 0, which is the case. Proposition 3.1 requires that $\{x \mid f(x) \leq 0\}$ is contained in the domain of the function $g_1(x_1, x_2) = \sqrt{|x_1|} - x_2$ (assumption (A1)(b) of [13]) and that $L_\gamma(f) \cap L_0(g_1) \neq \emptyset$ for all $\gamma > 0$, which are both satisfied.

Additionally, Assumption (A1)(a) of [13] needs to be satisfied, which requires some special notions. One of them is *an asymptotically nonpositive direction $d$*

of a function $h$ requiring that for a sequence $\{x_k\} \subset \mathrm{dom}(h)$, with $\|x_k\| \to \infty$ and $x_k/\|x_k\| \to d$, it holds that

$$\limsup_{k\to\infty} h(x_k) \leq 0.$$

Another one is a direction $d$ along which the function $h$ *recedes below 0 on a set* $S \subset \mathrm{dom}h$ requiring that

$$\text{for every } x \in S \text{ there is } \bar{\alpha} \geq 0 \text{ such that } \quad f(x + \alpha d) \leq 0 \qquad \text{for all } \alpha \geq \bar{\alpha}.$$

Lastly, for an asymptotically nonpositive direction $d$ of a function $h$, it is said that $h$ *retracts along* $d$ *on a set* $S \subset \mathrm{dom}h$ if for any sequence $\{x_k\} \subset S$, with $\|x_k\| \to \infty$ and $x_k/\|x_k\| \to d$, there exists $\bar{k}$ such that

$$f(x_k - d) \leq \max\{0, f(x_k)\} \qquad \text{for all } k \geq \bar{k}.$$

Assumption (A1)(a) as applied to problem (25) with a single constraint, requires that the following conditions hold:

(D1) For every asymptotically nonpositive direction $d$ of $f$, the function $f$ either (i) recedes below 0 along $d$ on $\mathrm{dom}f$ or (ii) retracts along $d$ on $\mathrm{dom}f$.

(D2) For every asymptotically nonpositive direction $d$ of $g_1$, the function $g_1$ either (i) recedes below 0 along $d$ on its domain or (ii) retracts along $d$ on its domain and recedes below 0 on the level set $L_0(g_1)$.

If a direction $d$ is asymptotically nonpositive direction of a function $h$, then we must have $h_\infty(d) \leq 0$, which follows from the definition of the asymptotic function. Since $\mathcal{K}(f) = \{(0, d_2) \mid d_2 \leq 0\}$, we can see that any $d \in \mathcal{K}(f)$ is asymptotically nonpositive direction. Thus, $f$ satisfies the condition (D1).

Next, consider the constraint function $g_1$ and a sequence $x_k = (k, \sqrt{k})$ for $k \geq 1$. We have $\|x_k\| \to \infty$, $x_k/\|x_k\| \to (1, 0)$, and $g_1(x_k) = 0$ for all $k$. Thus, $d = (1, 0)$ is an asymptotically nonpositive direction of $g_1$. However, the function does not recede below 0 neither on $\mathrm{dom}g_1$ nor on $L_0(g_1)$. To see this, note that $x = 0$ belongs to $L_0(g_1)$ and $\mathrm{dom}g_1$, while for any $\alpha > 0$,

$$g_1(0 + \alpha d) = g_1(\alpha, 0) = \sqrt{\alpha} > 0.$$

Thus, $g_1$ does not satisfy the condition (D2) so Proposition 3.1 of [13] cannot be used to assert the existence of solutions to problem (25), while our Theorem 4.2.2 can be used.

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