CONVERGENCE RATE OF THE HYPERSONIC SIMILARITY FOR TWO-DIMENSIONAL STEADY POTENTIAL FLOWS WITH LARGE DATA

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Abstract. We establish the optimal convergence rate of the hypersonic similarity for two-dimensional steady potential flows with *large data* past over a straight wedge in the $BV \cap L^1$ framework, provided that the total variation of the large data multiplied by $\gamma - 1 + \frac{a_\infty^2}{M_\infty^2}$ is uniformly bounded with respect to the adiabatic exponent $\gamma > 1$, the Mach number M_∞ of the incoming steady flow, and the hypersonic similarity parameter a_∞. Our main approach in this paper is first to establish the Standard Riemann Semigroup of the initial-boundary value problem for the isothermal hypersonic small disturbance equations with large data and then to compare the Riemann solutions between two systems with boundary locally case by case. Based on them, we derive the global L^1–estimate between the two solutions by employing the Standard Riemann Semigroup and the local L^1–estimates. We further construct an example to show that the convergence rate is optimal.

Contents

1. Introduction and Main Theorems

We are concerned with the optimal convergence rate of the hypersonic similarity for two-dimensional steady potential flows with *large data* past over a straight wedge in the $BV \cap L^1$ framework. In gas dynamics, hypersonic flows are the flows with a large Mach number (at least larger than five). One of the important properties of the hypersonic flows is the hypersonic

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similarity, which was first developed by Tsien in [27] for the two-dimensional potential flow and the three-dimensional axis-symmetric steady potential flow in 1940s. The convergence without a rate of the hypersonic similarity for two-dimensional steady potential flows past over a straight wedge was rigorously verified in [18]. In this paper, we further develop mathematical analysis on the hypersonic similarity for steady hypersonic potential flow over a two-dimensional straight wedge with large data (see Fig. 1.1) to establish the *optimal* convergence rate rigorously.

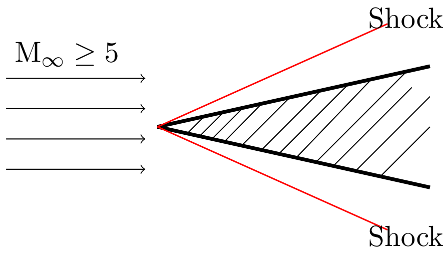

FIGURE 1.1. Hypersonic flow over a two-dimensional slender straight wedge

Physically, the law of the hypersonic similarity is also called the Van Dyke similarity law [30], which states that, for the steady flow around a slender wedge, the flow structures are similar under some scaling if the Mach number of the incoming flow is sufficiently large. More precisely, after scaling, the governed equations of the flow with the same hypersonic similarity parameter can be approximated by the same hypersonic small-disturbance system.

Mathematically, consider a uniform hypersonic flow with velocity $(u_\infty, 0)$ past over a two-dimensional straight wedge with boundaries $\bar{y} = \pm\tau b_0 \bar{x}$, for a fixed constant b_0 and a sufficiently small parameter $\tau > 0$. The two-dimensional steady isentropic irrotational Euler flows are governed by the following equations:

$$\begin{cases} \partial_{\bar{x}}(\bar{\rho}\bar{u}) + \partial_{\bar{y}}(\bar{\rho}\bar{v}) = 0, \\ \partial_{\bar{x}}\bar{v} - \partial_{\bar{y}}\bar{u} = 0, \end{cases} \tag{1.1}$$

where density $\bar{\rho}$ and velocity $(\bar{u}, \bar{v})$ satisfy the following Bernoulli law:

$$\frac{1}{2}(\bar{u}^2 + \bar{v}^2) + \frac{\bar{\rho}^{\gamma-1}}{\gamma-1} = \frac{1}{2}u_\infty^2 + \frac{\rho_\infty^{\gamma-1}}{\gamma-1}. \tag{1.2}$$

Due to the symmetry of the initial-boundary value problem, we constrain ourself to consider the lower half-plane in $\mathbb{R}^2$ with wedge boundary $\bar{y} = \tau b_0 \bar{x}$ for $b_0 < 0$ (see Fig. 1.2). Then, along the wedge boundary, the flow satisfies the impermeable slip boundary condition:

$$(\bar{u}, \bar{v}) \cdot \mathbf{n} = 0, \tag{1.3}$$

where $\mathbf{n} = \frac{(\tau b_0, -1)}{\sqrt{1+\tau^2 b_0^2}}$ is the unit inner normal of the wedge boundary.

Define the hypersonic similarity parameter:

$$a_\infty := \tau \mathrm{M}_\infty = \tau u_\infty \rho_\infty^{-\frac{\gamma-1}{2}}. \tag{1.4}$$

Following the arguments in [2, 27], we define the scaling:

$$\bar{x} = x, \quad \bar{y} = \tau y, \quad \bar{u} = u_\infty(1 + \tau^2 u), \quad \bar{v} = u_\infty \tau v, \quad \bar{\rho} = \rho_\infty \rho, \tag{1.5}$$

where $\rho_\infty = \lim_{\bar{y}\to-\infty} \bar{\rho}(\bar{y})$ so that $\lim_{y\to-\infty} \rho(y) = 1$.

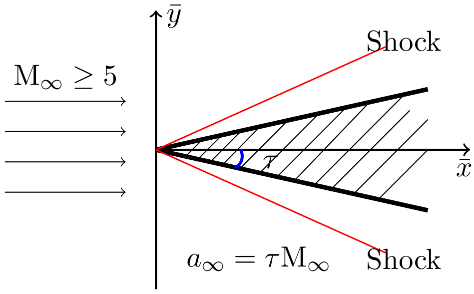

FIGURE 1.2. Hypersonic similarity law

Substituting (1.5) into (1.1)–(1.2), we obtain

$$\begin{cases} \partial_x(\rho(1+\tau^2 u)) + \partial_y(\rho v) = 0, \\ \partial_x v - \partial_y u = 0, \end{cases} \tag{1.6}$$

and

$$u + \frac{1}{2}(v^2 + \tau^2 u^2) + \frac{\rho^{\gamma-1} - 1}{(\gamma-1)a_\infty^2} = 0. \tag{1.7}$$

Now the fluid domain and its boundaries (see Fig. 1.3) are given by

$$\Omega_{\mathrm{w}} = \{(x,y) \,:\, x > 0, y < b_0 x\}$$

and

$$\Gamma_{\mathrm{w}} = \{(x,y) \,:\, x > 0, y = b_0 x\}, \quad \Sigma_0 = \{(x,y) \,:\, x = 0, y < 0\}.$$

Let $\mathbf{n}_{\mathrm{w}} = \frac{(b_0,-1)}{\sqrt{1+b_0^2}}$ be the unit inner normal vector of Γ_{w}. Then the boundary condition (1.3) becomes

$$(1+\tau^2 u^2, v) \cdot \mathbf{n}_{\mathrm{w}} = 0 \qquad \text{on } \Gamma_{\mathrm{w}}. \tag{1.8}$$

In addition, we impose the initial data on Σ_0 as

$$(\rho, u, v) = (\rho_0, \mathrm{u}_0, v_0)(y) \qquad \text{on } \Sigma_0, \tag{1.9}$$

where ρ_0, u_0, and v_0 satisfy (1.7).

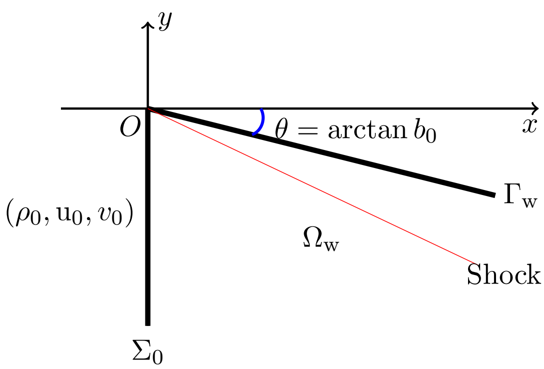

FIGURE 1.3. The initial-boundary value problem (1.6)–(1.9)

Mathematically, the hypersonic similarity means that, for a fixed hypersonic similarity parameter a_∞, the structure of the solution of (1.6)–(1.9) is persistent if M_∞ is large (*i.e.*, τ is small). In practice, when M_∞ is sufficiently large, γ is expected to be near 1. Therefore, if the hypersonic similarity is valid, when τ and $\gamma-1$ are sufficiently small, the solution of the initial-boundary value problem (1.6)–(1.9) should be approximated by the problem via taking $\tau=0$ and $\gamma=1$:

$$\begin{cases} \partial_x\rho+\partial_y(\rho v)=0, \\ \partial_x v-\partial_y u=0, \\ u+\frac{1}{2}v^2+\frac{\ln\rho}{a_\infty^2}=0, \end{cases} \tag{1.10}$$

with the boundary condition:

$$v=b_0 \qquad \text{on } \Gamma_{\mathrm{w}}, \tag{1.11}$$

and the initial data:

$$(\rho,u,v)=(\rho_0,u_0,v_0)(y) \qquad \text{on } \Sigma_0, \tag{1.12}$$

where ρ_0 and (u_0,v_0) satisfy $(1.10)_3$.

System (1.10) is called the hypersonic small-disturbance system. The hypersonic similarity was established in [18] by proving the existence of global entropy solutions of problem (1.6)–(1.9) with large data, provided that $(\gamma-1+\tau^2)(T.V.\{(\rho_0,v_0);\Sigma_0\}+|b_0|)<\infty$, and then showing that the solutions converge pointwise to the solution of problem (1.10)–(1.12) as $\gamma-1+\tau^2$ tends to zero. Therefore, a next natural question is what the convergence rate with respect to parameter $\gamma-1+\tau^2$ should be. The main purpose of this paper is to establish the optimal convergence rate of the solutions of problem (1.6)–(1.9) to the solution of problem (1.10)–(1.12) in L^1 as $\gamma-1+\tau^2\to 0$ with large initial data. To this end, we set

$$\boldsymbol{\mu}=(\epsilon,\tau^2):=(\gamma-1,\tau^2). \tag{1.13}$$

Denoted by $(\rho^{(\boldsymbol{\mu})},u^{(\boldsymbol{\mu})},v^{(\boldsymbol{\mu})})$ the solution of problem (1.6)–(1.9), and denoted by (ρ,u,v) the solution of problem (1.10)–(1.12) (*i.e.*, corresponding to the case: $\boldsymbol{\mu}=\mathbf{0}$). Since the flow moves from the left to the right, then $1+\tau^2u^{(\boldsymbol{\mu})}>0$ so that $u^{(\boldsymbol{\mu})}$ can be solved from equation (1.7):

$$u^{(\boldsymbol{\mu})}=\frac{1}{\tau^2}\Big(\sqrt{1-\tau^2B^{(\epsilon)}(\rho^{(\boldsymbol{\mu})},v^{(\boldsymbol{\mu})},\epsilon)}-1\Big), \tag{1.14}$$

where $B^{(\epsilon)}(\rho,v,\epsilon)$ is given by

$$B^{(\epsilon)}(\rho,v,\epsilon):=\frac{2\big(\rho^\epsilon-1\big)}{a_\infty^2\epsilon}+v^2. \tag{1.15}$$

Substituting (1.14) with (1.15) into equations (1.6), we reformulate problem (1.6)–(1.9) as

$$\begin{cases} \partial_x\big(\rho^{(\boldsymbol{\mu})}\sqrt{1-\tau^2B^{(\epsilon)}(\rho^{(\boldsymbol{\mu})},v^{(\boldsymbol{\mu})},\epsilon)}\big)+\partial_y(\rho^{(\boldsymbol{\mu})}v^{(\boldsymbol{\mu})})=0 & \text{in } \Omega_{\mathrm{w}}, \\ \partial_x v^{(\boldsymbol{\mu})}-\partial_y\Big(\frac{\sqrt{1-\tau^2B^{(\epsilon)}(\rho^{(\boldsymbol{\mu})},v^{(\boldsymbol{\mu})},\epsilon)}-1}{\tau^2}\Big)=0 & \text{in } \Omega_{\mathrm{w}}, \end{cases} \tag{1.16}$$

together with the initial condition:

$$(\rho^{(\boldsymbol{\mu})},v^{(\boldsymbol{\mu})})=(\rho_0,v_0)(y) \qquad \text{on } \Sigma_0, \tag{1.17}$$

and the boundary condition:

$$\Big(\sqrt{1-\tau^2B^{(\epsilon)}(\rho^{(\boldsymbol{\mu})},v^{(\boldsymbol{\mu})},\epsilon)},v^{(\boldsymbol{\mu})}\Big)\cdot\mathbf{n}_{\mathrm{w}}=0 \qquad \text{on } \Gamma_{\mathrm{w}}. \tag{1.18}$$

Similarly, from the third equation of (1.10), we obtain

$$u = -\frac{1}{2}v^2 - \frac{\ln\rho}{a_\infty^2}. \tag{1.19}$$

Then, substituting (1.19) into the other two equations of (1.10), we reformulate problem (1.10)–(1.12) as

$$\begin{cases} \partial_x\rho + \partial_y(\rho v) = 0 & \text{in } \Omega_{\mathrm{w}}, \\ \partial_x v + \partial_y\big(\frac{1}{2}v^2 + \frac{\ln\rho}{a_\infty^2}\big) = 0 & \text{in } \Omega_{\mathrm{w}}, \end{cases} \tag{1.20}$$

with the initial condition:

$$(\rho, v) = (\rho_0, v_0)(y) \qquad \text{on } \Sigma_0, \tag{1.21}$$

and the boundary condition:

$$v = b_0 \qquad \text{on } \Gamma_{\mathrm{w}}. \tag{1.22}$$

Our main results in this paper are stated as follows:

Theorem 1.1 (Main Theorem)**.** *Assume that there exist $\rho^* > \rho_* > 0$ so that $\rho_0 \in [\rho_*, \rho^*]$. Assume that $(\rho_0 - 1, v_0) \in (L^1 \cap BV)(\Sigma_0)$. Moreover, assume that there exists $C_0 > 0$ independent of $\boldsymbol{\mu}$ such that*

$$\|\boldsymbol{\mu}\|\big(T.V.\{(\rho_0, v_0); \Sigma_0\} + |b_0|\big) < C_0$$

for $\|\boldsymbol{\mu}\| := \epsilon + \tau^2$. Let $(\rho^{(\boldsymbol{\mu})}, v^{(\boldsymbol{\mu})})$ and (ρ, v) be the entropy solutions of problem (1.16)*–*(1.18) *and problem* (1.20)*–*(1.22)*, respectively. Then there exists $\boldsymbol{\mu}_0 = (\epsilon_0, \tau_0^2)$ with $\epsilon_0 = \gamma_0 - 1 > 0$ and $\tau_0 > 0$ such that, when $\|\boldsymbol{\mu}\| < \|\boldsymbol{\mu}_0\| := \epsilon_0 + \tau_0^2$,*

$$\|(\rho^{(\boldsymbol{\mu})} - \rho, v^{(\boldsymbol{\mu})} - v)\|_{L^1((-\infty, b_0 x))} \le Cx\|\boldsymbol{\mu}\| \qquad \text{for every } x > 0, \tag{1.23}$$

where $C > 0$ is independent on $\boldsymbol{\mu}$ and x. Moreover, the convergence rate for $\boldsymbol{\mu}$ in (1.23) *is optimal.*

With Theorem 1.1 in hand, we can further show the convergence rate between the entropy solutions $(\rho^{(\boldsymbol{\mu})}, u^{(\boldsymbol{\mu})}, v^{(\boldsymbol{\mu})})$ of problems (1.6)–(1.9) and the entropy solution (ρ, u, v) of problem (1.10)–(1.12) below.

Theorem 1.2. *Under the assumptions in* Theorem 1.1*, let $(\rho^{(\boldsymbol{\mu})}, u^{(\boldsymbol{\mu})}, v^{(\boldsymbol{\mu})})$ and (ρ, u, v) be the entropy solutions of problem* (1.6)*–*(1.9) *and problem* (1.10)*–*(1.12)*, respectively. Let $\boldsymbol{\mu}_0$ be defined in* Theorem 1.1*. Then, for any $\|\boldsymbol{\mu}\| < \|\boldsymbol{\mu}_0\|$, the following optimal convergence rate holds*:

$$\|(\rho^{(\boldsymbol{\mu})} - \rho, u^{(\boldsymbol{\mu})} - u, v^{(\boldsymbol{\mu})} - v)\|_{L^1((-\infty, b_0 x))} \le C(1 + x)\|\boldsymbol{\mu}\| \qquad \text{for every } x > 0, \tag{1.24}$$

where $C > 0$ is a constant independent of $\boldsymbol{\mu}$ and x.

To complete the proof, our main strategy is to further develop the methods used in [4, 5, 12] for the Cauchy problem into the initial-boundary value problem with large data by requiring that one of the two problems can generate a Standard Riemann Semigroup, denoted by *SRS*, while the other admits approximate solutions constructed by the wave-front tracking scheme.

Since there is no theory on the *SRS* for the initial-boundary value problem in general, we identify an affine transformation to transfer the initial-boundary value problem (1.20)–(1.22) to be in a quarter region with the unchanged equations (1.20). Then we can apply the results in [14] to show that the transformed problem admits a unique *SRS*. After that, by applying the inverse transformation, we can establish the L^1-stability and the existence of the *SRS* of the initial-boundary value problem (1.20)–(1.22). Moreover, a new semigroup formula is also established.

On the other hand, for the initial-boundary value problem (1.16)–(1.18) with large data, by employing the path decomposition technique developed in [1] and following the argument in [18], we can also construct the approximate solutions via the wave-front tracking scheme. Based on them, employing the new semigroup formula obtained in this paper, we establish the global L^1–difference estimate between two approximate solutions and obtain estimate (1.23) by taking the corresponding limits. Finally, we construct a simple example to illustrate that the convergence rate obtained in Theorem 1.1 is optimal.

We remark that, recently, the law of the hypersonic similarity without a convergence rate was rigorously justified for the steady potential flow past a straight wedge with large data in [18] and the optimal convergence rate for *small data* was obtained in [19] over a Lipschitz curved wedge, as well as for the full Euler flows with small data in [9]. Meanwhile, a similar but different problem on the hypersonic limit was considered in [24, 25] as the Mach number of the upcoming flow M_∞ tends to infinity with the obstacle being fixed, for which the Radon measure valued solutions were constructed as the limit of the solutions of the steady full Euler flows past a two-dimensional obstacle.

There are also some results on the existence of global entropy solutions with large data in BV for the one-dimensional gas dynamics equations in Lagrange coordinates; see [22, 23, 29] for more details. There are also some results on the steady supersonic flow problems that involve the structural stability of shock waves, rarefaction waves, and contact discontinuities; see [7, 8, 10, 13, 15, 31, 32] and the references cited therein. See also [6, 11].

The remaining context of this paper is organized as follows: In §2, we study the elementary wave curves for systems (1.16) and (1.20) globally, and then compare the Riemann solvers between these two systems with a boundary. In §3, we construct the approximate solutions of the initial-boundary value problem (1.16)–(1.18) via the wave-front tracking scheme and then establish the L^1-stability estimates and the properties of the Standard Riemann Semigroup (SRS) for the initial-boundary value problem (1.20)–(1.22). Based on them, a new semigroup formula is derived. In §4, we complete the proof of Theorem 1.1 by first establishing the local L^1-estimates between two approximate solutions and then applying the semigroup formula and the properties of the approximate solutions. Finally, we present an example to illustrate that the convergence rate obtained in Theorem 1.1 is optimal. In §5, we complete the proof of Theorem 1.2. In the appendix, we show some basic estimates, which are used for establishing the optimal convergence rate in §4.3.

2. Riemann Solvers for Systems (1.16) and (1.20)

In this section, we construct the elementary wave curves for system (1.16) and system (1.20), respectively. Then we make the comparison of the Riemann solvers with a boundary between the initial-boundary value problems (1.16)–(1.18) and (1.20)–(1.22).

2.1. **Wave curves for system** (1.16).

For simplicity, we rewrite $(\rho^{(\boldsymbol{\mu})}, v^{(\boldsymbol{\mu})})$ as (ρ, v) and $B^{(\epsilon)}(\rho^{(\boldsymbol{\mu})}, v^{(\boldsymbol{\mu})}, \epsilon)$ as $B^{(\epsilon)}$. Denote $U := (\rho, v)^\top$. Then, by direct computation, the characteristic polynomial of system (1.16) is

$$\big(1-\tau^2(B^{(\epsilon)}+a_\infty^{-2}\rho^\epsilon)\big)(\lambda^{(\boldsymbol{\mu})})^2 - 2v\sqrt{1-\tau^2 B^{(\epsilon)}}\lambda^{(\boldsymbol{\mu})} + v^2 - a_\infty^{-2}\rho^\epsilon = 0. \tag{2.1}$$

It admits two roots that are the two eigenvalues of system (1.16):

$$\lambda_j^{(\boldsymbol{\mu})}(U,\boldsymbol{\mu}) = \frac{a_\infty^2 v\sqrt{1-\tau^2 B^{(\epsilon)}} + (-1)^j \rho^{\frac{\epsilon}{2}}\sqrt{a_\infty^2 - \tau^2\epsilon^{-1}\big((\epsilon+2)\rho^\epsilon - 2\big)}}{a_\infty^2\big(1-\tau^2(B^{(\epsilon)}+a_\infty^{-2}\rho^\epsilon)\big)} \qquad \text{for } j=1,2. \tag{2.2}$$

The corresponding right-eigenvectors are

$$\mathbf{r}_j^{(\boldsymbol{\mu})}(U,\boldsymbol{\mu}) = (-1)^j\big(\rho,\ \frac{\rho^\epsilon}{a_\infty^2(\sqrt{1-\tau^2B^{(\epsilon)}}\lambda_j^{(\boldsymbol{\mu})}-v)}\big)^\top \qquad \text{for } j=1,2. \tag{2.3}$$

Lemma 2.1. *For any $U\in D=\{(\rho,v):\rho\in(\rho_*,\rho^*),\ |v|<K\}$ with constants $\rho^*>\rho_*>0$ and $K>0$ independent of $\boldsymbol{\mu}$, then*

$$\lambda_j^{(\boldsymbol{\mu})}(U,\boldsymbol{\mu})\Big|_{\boldsymbol{\mu}=\mathbf{0}} = v+(-1)^j a_\infty^{-1} \qquad \textit{for } j=1,2, \tag{2.4}$$

$$\mathbf{r}_j^{(\boldsymbol{\mu})}(U,\boldsymbol{\mu})\Big|_{\boldsymbol{\mu}=\mathbf{0}} = \big((-1)^j\rho,\ a_\infty^{-1}\big)^\top \qquad \textit{for } j=1,2. \tag{2.5}$$

Since $a_\infty>0$, we deduce from Lemma 2.1 that system (1.16) is strictly hyperbolic for any $U\in D$ if $\epsilon>0$ and $\tau>0$ are sufficiently small. Moreover, we have

Lemma 2.2. *For any $U\in D$ with D defined in* Lemma 2.1*, there exists a constant vector $\bar{\boldsymbol{\mu}}_0=(\bar\epsilon_0,\bar\tau_0^2)$ with $\bar\epsilon_0>0$ and $\bar\tau_0>0$ such that, for $\|\boldsymbol{\mu}\|\le\|\bar{\boldsymbol{\mu}}_0\|$,*

$$\nabla_U\lambda_j^{(\boldsymbol{\mu})}(U,\boldsymbol{\mu})\cdot\mathbf{r}_j^{(\boldsymbol{\mu})}(U,\boldsymbol{\mu})>0 \qquad \textit{for } j=1,2, \tag{2.6}$$

where $\|\boldsymbol{\mu}\|=\epsilon+\tau^2$ and $\|\bar{\boldsymbol{\mu}}_0\|=\bar\epsilon_0+\bar\tau_0^2$.

Proof. For $j=1$, taking the derivatives on both sides of (2.1) with respect to ρ to obtain

$$\begin{aligned}&2\Big(\big(1-\tau^2(B^{(\epsilon)}+a_\infty^{-2}\rho^\epsilon)\big)\lambda_1^{(\boldsymbol{\mu})}-v\sqrt{1-\tau^2B^{(\epsilon)}}\Big)\frac{\partial\lambda_1^{(\boldsymbol{\mu})}}{\partial\rho}\\ &+a_\infty^{-2}\big((2v-(\epsilon+2)\lambda_1^{(\boldsymbol{\mu})})\lambda_1^{(\boldsymbol{\mu})}\tau^2-\epsilon\big)\rho^{\epsilon-1}=0.\end{aligned} \tag{2.7}$$

Substituting (2.2) for $j=1$ into (2.7), we deduce

$$\frac{\partial\lambda_1^{(\boldsymbol{\mu})}}{\partial\rho} = \frac{\big(2v-(\epsilon+2)\lambda_1^{(\boldsymbol{\mu})}\big)\lambda_1^{(\boldsymbol{\mu})}\tau^2-\epsilon}{2\rho^{1-\frac{\epsilon}{2}}\sqrt{a_\infty^2-\tau^2\epsilon^{-1}\big((\epsilon+2)\rho^\epsilon-2\big)}}. \tag{2.8}$$

Similarly, we also obtain

$$\frac{\partial\lambda_1^{(\boldsymbol{\mu})}}{\partial v} = -\frac{a_\infty^2\Big(\big(\sqrt{1-\tau^2B^{(\epsilon)}}-2\tau^2v^2\big)\lambda_1^{(\boldsymbol{\mu})}-v+\tau^2v(\lambda_1^{(\boldsymbol{\mu})})^2\Big)}{2\rho^{\frac{\epsilon}{2}}\sqrt{a_\infty^2-\tau^2\epsilon^{-1}\big((\epsilon+2)\rho^\epsilon-2\big)}}. \tag{2.9}$$

It follows from (2.3) and (2.8)–(2.9) that

$$\begin{aligned}&\nabla_U\lambda_1^{(\boldsymbol{\mu})}(U,\boldsymbol{\mu})\cdot\mathbf{r}_1^{(\boldsymbol{\mu})}(U,\boldsymbol{\mu})\\ &=\frac{2\big((\sqrt{1-\tau^2B^{(\epsilon)}}-2\tau^2v)\lambda_1^{(\boldsymbol{\mu})}-v+\tau^2v(\lambda_1^{(\boldsymbol{\mu})})^2\big)-\big(\sqrt{1-\tau^2B^{(\epsilon)}}\lambda_1^{(\boldsymbol{\mu})}-v\big)\big((2v-(\epsilon+2)\lambda_1^{(\boldsymbol{\mu})})\lambda_1^{(\boldsymbol{\mu})}\tau^2-\epsilon\big)}{2\rho^{-\frac{\epsilon}{2}}\sqrt{a_\infty^2-\tau^2\epsilon^{-1}\big((\epsilon+2)\rho^\epsilon-2\big)}\big(\sqrt{1-\tau^2B^{(\epsilon)}}\lambda_1^{(\boldsymbol{\mu})}-v\big)},\end{aligned}$$

which, by Lemma 2.1, implies that

$$\nabla_U\lambda_1^{(\boldsymbol{\mu})}(U,\boldsymbol{\mu})\cdot\mathbf{r}_1^{(\boldsymbol{\mu})}(U,\boldsymbol{\mu})\Big|_{\boldsymbol{\mu}=\mathbf{0}}=a_\infty^{-1}>0.$$

In the same way, for $j=2$, we have

$$\begin{aligned}&\nabla_U\lambda_2^{(\boldsymbol{\mu})}(U,\boldsymbol{\mu})\cdot\mathbf{r}_2^{(\boldsymbol{\mu})}(U,\boldsymbol{\mu})\\ &=\frac{2\big((\sqrt{1-\tau^2B^{(\epsilon)}}-2\tau^2v)\lambda_2^{(\boldsymbol{\mu})}-v+\tau^2v(\lambda_2^{(\boldsymbol{\mu})})^2\big)-(\sqrt{1-\tau^2B^{(\epsilon)}}\lambda_2^{(\boldsymbol{\mu})}-v)\big((2v-(\epsilon+2)\lambda_2^{(\boldsymbol{\mu})})\lambda_2^{(\boldsymbol{\mu})}\tau^2-\epsilon\big)}{2\rho^{-\frac{\epsilon}{2}}\sqrt{a_\infty^2-\tau^2\epsilon^{-1}((\epsilon+2)\rho^\epsilon-2)}(\sqrt{1-\tau^2B^{(\epsilon)}}\lambda_2^{(\boldsymbol{\mu})}-v)}.\end{aligned}$$

Then, by Lemma 2.1 again, we obtain

$$\nabla_U \lambda_2^{(\boldsymbol{\mu})}(U, \boldsymbol{\mu}) \cdot \mathbf{r}_2^{(\boldsymbol{\mu})}(U, \boldsymbol{\mu})\Big|_{\boldsymbol{\mu}=\mathbf{0}} = a_\infty^{-1} > 0.$$

Therefore, we can choose $\bar{\boldsymbol{\mu}}_0 = (\bar{\epsilon}_0, \bar{\tau}_0^2)$ with small $\bar{\epsilon}_0 > 0$ and $\bar{\tau}_0 > 0$ such that, when $\|\boldsymbol{\mu}\| \leq \|\bar{\boldsymbol{\mu}}_0\|$, $\nabla_U \lambda_2^{(\boldsymbol{\mu})}(U, \boldsymbol{\mu}) \cdot \mathbf{r}_2^{(\boldsymbol{\mu})}(U, \boldsymbol{\mu}) > 0$ for $j = 1, 2$ and $U \in D$. This completes the proof. □

Lemma 2.2 implies that both characteristic fields of system (1.16) are genuinely nonlinear. Thus, the elementary waves are either shock waves $S^{(\boldsymbol{\mu})} = S_1^{(\boldsymbol{\mu})} \cup S_2^{(\boldsymbol{\mu})}$ or rarefaction waves $R^{(\boldsymbol{\mu})} = R_1^{(\boldsymbol{\mu})} \cup R_2^{(\boldsymbol{\mu})}$. Next, we study the shock wave curves and rarefaction wave curves of system (1.16) in the (ρ, v)–plane.

For a given left-state $U_L = (\rho_L, v_L)^\top$, the shock solutions $U = (\rho, v)$ are the Riemann solutions satisfying the following Rankine-Hugoniot conditions on the shock with shock speed $\sigma_j(\boldsymbol{\mu})$:

$$\begin{cases} \rho v - \rho_L v_L = \sigma_j^{(\boldsymbol{\mu})}\Big(\rho\sqrt{1-\tau^2 B^{(\epsilon)}(\rho, v, \epsilon)} - \rho_L\sqrt{1-\tau^2 B^{(\epsilon)}(\rho_L, v_L, \epsilon)}\Big), \\ \frac{1}{\tau^2}\Big(\sqrt{1-\tau^2 B^{(\epsilon)}(\rho, v, \epsilon)} - \sqrt{1-\tau^2 B^{(\epsilon)}(\rho_L, v_L, \epsilon)}\Big) = \sigma_j^{(\boldsymbol{\mu})}(v - v_L), \end{cases} \tag{2.10}$$

and the following Lax geometry entropy conditions:

$$\lambda_1^{(\boldsymbol{\mu})}(U, \boldsymbol{\mu}) < \sigma_1^{(\boldsymbol{\mu})} < \lambda_1^{(\boldsymbol{\mu})}(U_L, \boldsymbol{\mu}), \qquad or \qquad \lambda_2^{(\boldsymbol{\mu})}(U, \boldsymbol{\mu}) < \sigma_2^{(\boldsymbol{\mu})} < \lambda_2(U, \boldsymbol{\mu}). \tag{2.11}$$

Then, for sufficiently small $\|\boldsymbol{\mu}\|$, if U_L and $U \in D$, conditions (2.11) imply that

$$\rho > \rho_L,\ v < v_L, \qquad \text{or} \qquad \rho < \rho_L,\ v < v_L. \tag{2.12}$$

Therefore, it follows from (2.10) that

$$\begin{aligned}(v - v_L)^2 = \ & \frac{2(\rho^\epsilon - \rho_L)(\rho - \rho_L)}{a_\infty^2 \epsilon(\rho + \rho_L)} + \tau^2 B^{(\epsilon)}(\rho_L, v_L, \epsilon) B^{(\epsilon)}(\rho, v, \epsilon) \\ & + \Big(\sqrt{(1 - \tau^2 B^{(\epsilon)}(\rho_L, v_L, \epsilon))(1 - \tau^2 B^{(\epsilon)}(\rho, v, \epsilon))} - 1\Big) \\ & \quad \times \Big(v_L v + \frac{2(\rho^\epsilon - 1)\rho + 2(\rho_L^\epsilon - 1)\rho_L}{a_\infty^2 \epsilon(\rho + \rho_L)}\Big). \end{aligned} \tag{2.13}$$

Set $\alpha := \frac{\rho}{\rho_L}$. Define

$$\begin{aligned} & H_S^{(\boldsymbol{\mu})}(v - v_L, \alpha, U_L, \boldsymbol{\mu}) \\ & = (v - v_L)^2 - \frac{2\rho_L^\epsilon(\alpha^\epsilon - 1)(\alpha - 1)}{a_\infty^2 \epsilon(\alpha + 1)} - \tau^2 B^{(\epsilon)}(\rho_L, v_L, \epsilon) B^{(\epsilon)}(\rho_L \alpha, v, \epsilon) \\ & \quad - \Big(\sqrt{(1 - \tau^2 B^{(\epsilon)}(\rho_L, v_L, \epsilon))(1 - \tau^2 B^{(\epsilon)}(\rho_L \alpha, v, \epsilon))} - 1\Big) \\ & \qquad \times \Big(v_L v + \frac{2\big(\rho_L^\epsilon(\alpha^{\epsilon+1} + 1) - \alpha - 1\big)}{a_\infty^2 \epsilon(\alpha + 1)}\Big). \end{aligned} \tag{2.14}$$

Thus, solving $v - v_L$ from equation (2.13) is equivalent to solving the equation:

$$H_S^{(\boldsymbol{\mu})}(v - v_L, \alpha, U_L, \boldsymbol{\mu}) = 0, \tag{2.15}$$

where $H^{(\boldsymbol{\mu})}$ is defined by (2.14). Its solvability is given by the following lemma:

Lemma 2.3. *Let D be defined in* Lemma 2.1. *There exist both a constant $\delta_0 \in (0, \frac{1}{2})$ and a constant vector $\bar{\boldsymbol{\mu}}_0' = (\bar{\epsilon}_0', \bar{\tau}_0'^2)$ with $\bar{\epsilon}_0' < \bar{\epsilon}_0$ and $\bar{\tau}_0' < \bar{\tau}_0$ such that, for $\|\boldsymbol{\mu}\| \leq \|\bar{\boldsymbol{\mu}}_0'\|$,*

(i) *If $\alpha \in [1, \delta_0^{-1}]$, then equation $H^{(\boldsymbol{\mu})}(v - v_L, \alpha, U_L, \boldsymbol{\mu}) = 0$ admits a unique solution $v - v_L = \varphi_{S_1}^{(\boldsymbol{\mu})}(\alpha; U_L, \boldsymbol{\mu}) \in C^2\big([1, \delta_0^{-1}] \times \bar{D} \times (0, \bar{\epsilon}_0') \times (0, \bar{\tau}_0'^2)\big)$ satisfying*

$$\varphi_{S_1}^{(\boldsymbol{\mu})}\Big|_{\boldsymbol{\mu}=\mathbf{0}} = -\frac{\sqrt{2}}{a_\infty}\sqrt{\frac{(\alpha-1)\ln\alpha}{\alpha+1}}, \qquad \frac{\partial \varphi_{S_1}^{(\boldsymbol{\mu})}}{\partial \alpha} < 0; \tag{2.16}$$

(ii) *If $\alpha \in [\delta_0, 1)$, then equation $H^{(\boldsymbol{\mu})}(v - v_L, \alpha, U_L, \boldsymbol{\mu}) = 0$ admits a unique solution $v - v_L = \varphi_{S_2}^{(\boldsymbol{\mu})}(\alpha; U_L, \boldsymbol{\mu}) \in C^2\big([\delta_0, 1) \times \bar{D} \times (0, \bar{\epsilon}_0') \times (0, \bar{\tau}_0'^2)\big)$ satisfying*

$$\varphi_{S_2}^{(\boldsymbol{\mu})}\Big|_{\boldsymbol{\mu}=\mathbf{0}} = -\frac{\sqrt{2}}{a_\infty}\sqrt{-\frac{(1-\alpha)\ln\alpha}{1+\alpha}}, \qquad \frac{\partial \varphi_{S_2}^{(\boldsymbol{\mu})}}{\partial \alpha} > 0. \tag{2.17}$$

Proof. If $\alpha = 1$, then $\rho = \rho_L$.

For $\alpha \neq 1$, it follows from (2.12)–(2.14) and Lemma 2.1 that, when $\boldsymbol{\mu} = \mathbf{0}$,

$$v - v_L = -\frac{\sqrt{2}}{a_\infty}\sqrt{\frac{(\rho-\rho_L)(\rho-\rho_L)}{(\rho+\rho_L)}} = -\frac{\sqrt{2}}{a_\infty}\sqrt{\frac{(\alpha-1)\ln\alpha}{(\alpha+1)}} \qquad \text{for } \alpha > 1, \tag{2.18}$$

or

$$v - v_L = -\frac{\sqrt{2}}{a_\infty}\sqrt{-\frac{(1-\alpha)\ln\alpha}{(1+\alpha)}} \qquad \text{for } 0 < \alpha < 1. \tag{2.19}$$

Now, we first consider the case that $\alpha > 1$. It follows from (2.18) that

$$H_S^{(\boldsymbol{\mu})}\Big(-\frac{\sqrt{2}}{a_\infty}\sqrt{\frac{(\alpha-1)\ln\alpha}{(\alpha+1)}}, \alpha, U_L, \mathbf{0}\Big) = 0, \tag{2.20}$$

$$\frac{\partial H_S^{(\boldsymbol{\mu})}}{\partial(v - v_L)}\Big|_{\boldsymbol{\mu}=\mathbf{0}} = 2(v - v_L) = -\frac{2\sqrt{2}}{a_\infty}\sqrt{\frac{(\alpha-1)\ln\alpha}{(\alpha+1)}}. \tag{2.21}$$

Next, we set

$$\tilde{H}_S^{(\boldsymbol{\mu})}(v - v_L, \alpha, U_L, \boldsymbol{\mu}) := \frac{H^{(\boldsymbol{\mu})}(v - v_L, \alpha, U_L, \boldsymbol{\mu})}{\alpha - 1} \qquad \text{for } \alpha > 1.$$

Then, by (2.20),

$$\tilde{H}_S^{(\boldsymbol{\mu})}\Big(-\frac{2\sqrt{2}}{a_\infty}\sqrt{\frac{(\alpha-1)\ln\alpha}{(\alpha+1)}}, \alpha, U_L, \mathbf{0}\Big) = 0,$$

and, by (2.21),

$$\frac{\partial \tilde{H}_S^{(\boldsymbol{\mu})}(v - v_L, \alpha, U_L, \boldsymbol{\mu})}{\partial(v - v_L)}\Big|_{\boldsymbol{\mu}=\mathbf{0}} = -\frac{2\sqrt{2}}{a_\infty}\sqrt{\frac{\ln\alpha}{\alpha^2-1}}.$$

Since $\lim_{\alpha\to1+}\frac{\ln\alpha}{\alpha-1} = 1$, we have

$$\lim_{\alpha\to1+}\frac{\partial \tilde{H}_S^{(\boldsymbol{\mu})}(v - v_L, \alpha, U_L, \boldsymbol{\mu})}{\partial(v - v_L)}\Big|_{\boldsymbol{\mu}=\mathbf{0}} = -\frac{2}{a_\infty} < 0.$$

Note that $\frac{\ln\alpha}{\alpha^2-1}$ is monotonically decreasing on $[1,\infty)$ and $\lim_{\alpha\to\infty}\frac{\ln\alpha}{\alpha^2-1}=0$. Then we can choose a small constant $\delta_0\in(0,\frac{1}{2})$ and a constant $C_{\delta_0}\in(0,\frac{\sqrt{2}}{2})$ such that, for $\alpha\in[1,\delta_0^{-1})$,

$$-\frac{2}{a_\infty}\le\frac{\partial\tilde{H}^{(\boldsymbol{\mu})}}{\partial(v-v_L)}\Big|_{\boldsymbol{\mu}=\mathbf{0}}<-\frac{2\sqrt{2}}{a_\infty}C_{\delta_0}<0.$$

Therefore, by the implicit function theorem and the compactness of $\bar{D}$, we deduce that there exists a constant $\boldsymbol{\mu}_0'=(\bar{\epsilon}_0',\bar{\tau}_0'^2)$ with $\bar{\epsilon}_0'<\bar{\epsilon}_0$ and $\bar{\tau}_0'<\bar{\tau}_0$ such that there exists a unique solution:

$$v-v_L=\varphi_{S_1}^{(\boldsymbol{\mu})}(\alpha;U_L,\boldsymbol{\mu})\in C^1\big([1,\delta_0^{-1})\times\bar{D}\times(0,\bar{\epsilon}_0')\times(0,\bar{\tau}_0'^2)\big)$$

so that $H^{(\boldsymbol{\mu})}(\varphi_{S_1}^{(\boldsymbol{\mu})},\alpha,U_L,\boldsymbol{\mu})=0$. Moreover, it follows from (2.18) that the first identity in (2.16) holds.

Taking the derivative with respect to α on both sides of equation (2.15) to obtain

$$\frac{\partial H_S^{(\boldsymbol{\mu})}}{\partial\alpha}+\frac{\partial H_S^{(\boldsymbol{\mu})}}{\partial\varphi_{S_1}^{(\boldsymbol{\mu})}}\frac{\partial\varphi_{S_1}^{(\boldsymbol{\mu})}}{\partial\alpha}=0. \tag{2.22}$$

Taking $\boldsymbol{\mu}=\mathbf{0}$ in (2.22), we see that

$$\begin{aligned}\frac{\partial\varphi_{S_1}^{(\boldsymbol{\mu})}}{\partial\alpha}\Big|_{\boldsymbol{\mu}=\mathbf{0}}&=-\frac{\partial H_S^{(\boldsymbol{\mu})}}{\partial\alpha}\Big|_{\boldsymbol{\mu}=\mathbf{0}}\Big(\frac{\partial H_S^{(\boldsymbol{\mu})}}{\partial\varphi_{S_1}^{(\boldsymbol{\mu})}}\Big|_{\boldsymbol{\mu}=\mathbf{0}}\Big)^{-1}\\&=-\frac{\sqrt{2}}{2a_\infty}\frac{\alpha^2+2\alpha\ln\alpha-1}{\alpha(\alpha+1)^{\frac{3}{2}}\sqrt{(\alpha-1)\ln\alpha}}<0\qquad\text{for }\ \alpha>1.\end{aligned} \tag{2.23}$$

Moreover, we have

$$\lim_{\alpha\to1+}\frac{\partial\varphi_{S_1}^{(\boldsymbol{\mu})}}{\partial\alpha}=-a_\infty^{-1},\qquad\lim_{\alpha\to\infty}\frac{\partial\varphi_{S_1}^{(\boldsymbol{\mu})}}{\partial\alpha}=0.$$

Thus, by choosing $\delta_0>0$, $\bar{\epsilon}_0'>0$, and $\bar{\tau}_0'>0$ sufficiently small, it follows that

$$\frac{\partial\varphi_{S_1}^{(\boldsymbol{\mu})}}{\partial\alpha}<0\qquad\text{for }\alpha\in[1,\delta_0^{-1}),\ \|\boldsymbol{\mu}\|\le\|\bar{\boldsymbol{\mu}}_0'\|,\text{ and }U_L\in\bar{D}.$$

Furthermore, we take the derivative with respect to α on both sides of (2.22) and set $\boldsymbol{\mu}=\mathbf{0}$ to obtain

$$\frac{\partial^2\varphi_{S_1}^{(\boldsymbol{\mu})}}{\partial\alpha^2}\Big|_{\boldsymbol{\mu}=\mathbf{0}}=-\Big(\frac{\partial^2H_S^{(\boldsymbol{\mu})}}{\partial\alpha^2}+2\frac{\partial^2H_S^{(\boldsymbol{\mu})}}{\partial\varphi_{S_1}^{(\boldsymbol{\mu})}\partial\alpha}\frac{\partial\varphi_{S_1}^{(\boldsymbol{\mu})}}{\partial\alpha}+\frac{\partial^2H_S^{(\boldsymbol{\mu})}}{\partial^2\varphi_{S_1}^{(\boldsymbol{\mu})}}\Big(\frac{\partial\varphi_{S_1}^{(\boldsymbol{\mu})}}{\partial\alpha}\Big)^2\Big)\Big|_{\boldsymbol{\mu}=\mathbf{0}}\Big(\frac{\partial H_S^{(\boldsymbol{\mu})}}{\partial\varphi_{S_1}^{(\boldsymbol{\mu})}}\Big|_{\boldsymbol{\mu}=\mathbf{0}}\Big)^{-1}.$$

By direct calculation, we have

$$\frac{\partial^2H_S^{(\boldsymbol{\mu})}}{\partial\alpha^2}\Big|_{\boldsymbol{\mu}=\mathbf{0}}=\frac{2\big((\alpha^2-4\alpha-1)(\alpha+1)+4\alpha^2\ln\alpha\big)}{a_\infty^2\alpha^2(\alpha+1)^3},\quad\frac{\partial^2H_S^{(\boldsymbol{\mu})}}{\partial\varphi_{S_1}^{(\boldsymbol{\mu})}\partial\alpha}\Big|_{\boldsymbol{\mu}=\mathbf{0}}=0,\quad\frac{\partial^2H_S^{(\boldsymbol{\mu})}}{\partial^2\varphi_{S_1}^{(\boldsymbol{\mu})}}\Big|_{\boldsymbol{\mu}=\mathbf{0}}=2.$$

Then

$$\frac{\partial^2\varphi_{S_1}^{(\boldsymbol{\mu})}}{\partial\alpha^2}\Big|_{\boldsymbol{\mu}=\mathbf{0}}=\frac{2(\alpha-1)\big((\alpha^2-4\alpha+1)(\alpha+1)+4\alpha^2\ln\alpha\big)\ln\alpha+(\alpha^2+2\alpha\ln\alpha-1)^2}{2\sqrt{2}a_\infty\alpha^2(\alpha+1)^{\frac{5}{2}}\big((\alpha-1)\ln\alpha\big)^{\frac{3}{2}}}.$$

Note that $\lim_{\alpha\to 1+}\frac{\partial^2\varphi^{(\boldsymbol{\mu})}_{S_1}}{\partial\alpha^2}\Big|_{\boldsymbol{\mu}=\mathbf{0}} = a_\infty^{-1}$. Thus, for $\alpha\in[1,\delta_0^{-1})$ and $\|\boldsymbol{\mu}\|\le\|\bar{\boldsymbol{\mu}}_0\|$, we see that $\frac{\partial\varphi^{(\boldsymbol{\mu})}_{S_1}}{\partial\alpha^2}\in C\big([1,\delta_0^{-1})\times\bar{D}\times(0,\bar{\epsilon}_0')\times(0,\bar{\tau}_0'^2)\big)$. This completes the proof of (i), *i.e.*, for the case that $\alpha\ge 1$. In the same way, by (2.19), we can also prove (ii), *i.e.*, for the case that $\alpha\le 1$. $\square$

Now, we study the rarefaction wave curves with $U_L=(\rho_L,v_L)^\top$ as the left-state. If these curves are parameterized as $U(\alpha)=(\rho(\alpha),v(\alpha))^\top$, then the 1-rarefaction wave satisfies

$$\begin{cases}\frac{\mathrm{d}\rho}{\rho}=\frac{\mathrm{d}\alpha}{\alpha},\\ \mathrm{d}v=\frac{\rho^{\epsilon-1}\mathrm{d}\alpha}{a_\infty^2\left(\sqrt{1-\tau^2B^{(\epsilon)}(\rho(\alpha),v(\alpha),\epsilon)}\lambda_1^{(\boldsymbol{\mu})}(U(\alpha),\boldsymbol{\mu})-v(\alpha)\right)},\end{cases}\qquad\text{when }\alpha\in(0,1],\tag{2.24}$$

with $(\rho,u)\big|_{\alpha=1}=(\rho_L,v_L)$, or the 2-rarefaction wave satisfies

$$\begin{cases}\frac{\mathrm{d}\rho}{\rho}=\frac{\mathrm{d}\alpha}{\alpha},\\ \mathrm{d}v=\frac{\rho^{\epsilon-1}\mathrm{d}\alpha}{a_\infty^2\left(\sqrt{1-\tau^2B^{(\epsilon)}(\rho(\alpha),v(\alpha),\epsilon)}\lambda_2^{(\boldsymbol{\mu})}(U(\alpha),\boldsymbol{\mu})-v(\alpha)\right)},\end{cases}\qquad\text{when }\alpha\in[1,\infty),\tag{2.25}$$

with $(\rho,u)\big|_{\alpha=1}=(\rho_L,v_L)$. Then we have the following lemma.

Lemma 2.4. *Let D be given as in* Lemma 2.1. *Then there exists a constant vector $\bar{\boldsymbol{\mu}}_0''=(\bar{\epsilon}_0'',\bar{\tau}_0''^2)$ with $\bar{\epsilon}_0''<\bar{\epsilon}_0$ and $\bar{\tau}_0''<\bar{\tau}_0$ such that, for $\|\boldsymbol{\mu}\|\le\|\bar{\boldsymbol{\mu}}_0''\|$,*

(i) *when $\alpha\in(0,1]$, equation* (2.24) *admits a unique solution $v-v_L=\varphi^{(\boldsymbol{\mu})}_{R_1}(\alpha,U_L,\boldsymbol{\mu})\in C^2\big((0,1]\times\bar{D}\times(0,\bar{\epsilon}_0'')\times(0,\bar{\tau}_0''^2)\big)$ satisfying*

$$\varphi^{(\boldsymbol{\mu})}_{R_1}\Big|_{\alpha=1}=0,\qquad \varphi^{(\boldsymbol{\mu})}_{R_1}\Big|_{\boldsymbol{\mu}=\mathbf{0}}=-\frac{1}{a_\infty}\ln\alpha,\qquad \frac{\partial\varphi^{(\boldsymbol{\mu})}_{R_1}}{\partial\alpha}\le 0;\tag{2.26}$$

(ii) *when $\alpha\in[1,\infty)$, equation* (2.25) *admits a unique solution $v-v_L=\varphi^{(\boldsymbol{\mu})}_{R_2}(\alpha,U_L,\boldsymbol{\mu})\in C^2\big([1,\infty)\times\bar{D}\times(0,\bar{\epsilon}_0'')\times(0,\bar{\tau}_0''^2)\big)$ satisfying*

$$\varphi^{(\boldsymbol{\mu})}_{R_2}\Big|_{\alpha=1}=0,\qquad \varphi^{(\boldsymbol{\mu})}_{R_1}\Big|_{\boldsymbol{\mu}=\mathbf{0}}=\frac{1}{a_\infty}\ln\alpha,\ \frac{\partial\varphi^{(\boldsymbol{\mu})}_{R_1}}{\partial\alpha}\ge 0.\tag{2.27}$$

Proof. We give the proof of (i) only, since the argument for $\alpha\in[1,\infty)$ is the same. First, by equation $(2.24)_1$, we see that $\rho=\rho_L\alpha$. Then we substitute it into equation $(2.24)_2$ and integrate the resulted equation to derive

$$\int_{v_L}^{v}\left(\sqrt{1-\tau^2B^{(\epsilon)}(\rho_L\alpha,\zeta,\epsilon)}\,\lambda_1^{(\boldsymbol{\mu})}(U(\zeta),\boldsymbol{\mu})-\zeta\right)\mathrm{d}\zeta=\frac{\rho_L^\epsilon(\alpha^\epsilon-1)}{a_\infty^2\epsilon},\tag{2.28}$$

where $U(\zeta)=(\rho_L\alpha,\zeta)^\top$. It follows from (2.28) that $v=v_L$ when $\alpha=1$.

When $\alpha\in(0,1)$, set

$$H^{(\boldsymbol{\mu})}_{R_1}(v-v_L,\alpha,\rho_L,\boldsymbol{\mu}):=\int_{v_L}^{v}\left(\sqrt{1-\tau^2B^{(\epsilon)}(\rho_L\alpha,\zeta,\epsilon)}\,\lambda_1^{(\boldsymbol{\mu})}(U(\zeta),\boldsymbol{\mu})-\zeta\right)\mathrm{d}\zeta-\frac{\rho_L^\epsilon(\alpha^\epsilon-1)}{a_\infty^2\epsilon}.$$

Therefore, solving equation (2.28) is equivalent to solving the following equation:

$$H^{(\boldsymbol{\mu})}_{R_1}(v-v_L,\alpha,\rho_L,\boldsymbol{\mu})=0.\tag{2.29}$$

Notice that $H^{(\boldsymbol{\mu})}_{R_1}\in C^2$, $H^{(\boldsymbol{\mu})}_{R_1}(-\frac{\ln\alpha}{a_\infty},\alpha,\rho_L,\mathbf{0})=0$, and

$$\frac{\partial H^{(\boldsymbol{\mu})}_{R_1}(v-v_L,\alpha,\rho_L,\boldsymbol{\mu})}{\partial(v-v_L)}\Bigg|_{\boldsymbol{\mu}=\mathbf{0}}=-a_\infty^{-1}<0.\tag{2.30}$$

Then, by the implicit function theorem, equation (2.29) has a unique solution $v - v_L = \varphi^{(\boldsymbol{\mu})}_{R_1}(\alpha, \rho_L, \boldsymbol{\mu}) \in C^2$ satisfying $\varphi^{(\boldsymbol{\mu})}_{R_1}(1, \rho_L, \boldsymbol{\mu}) = 0$ and $\varphi^{(\boldsymbol{\mu})}_{R_1}\Big|_{\boldsymbol{\mu}=\mathbf{0}} = -\frac{\ln\alpha}{a_\infty}$.

To find $\frac{\partial \varphi^{(\boldsymbol{\mu})}_{R_1}}{\partial\alpha}$, we take the derivatives with respect to α on both sides of (2.29) and then set $\boldsymbol{\mu} = \mathbf{0}$ to obtain

$$\frac{\partial H^{(\boldsymbol{\mu})}}{\partial\alpha}\Big|_{\boldsymbol{\mu}=\mathbf{0}} + \frac{\partial H^{(\boldsymbol{\mu})}}{\partial\varphi^{(\boldsymbol{\mu})}_{R_1}}\Big|_{\boldsymbol{\mu}=\mathbf{0}} \frac{\partial\varphi^{(\boldsymbol{\mu})}_{R_1}}{\partial\alpha}\Big|_{\boldsymbol{\mu}=\mathbf{0}} = 0. \tag{2.31}$$

Inserting the identity that $\frac{\partial H^{(\boldsymbol{\mu})}}{\partial\alpha}\Big|_{\boldsymbol{\mu}=\mathbf{0}} = -\frac{1}{a_\infty^2\alpha}$ into (2.31) and using (2.30), we obtain

$$\frac{\partial\varphi^{(\boldsymbol{\mu})}_{R_1}}{\partial\alpha}\Big|_{\boldsymbol{\mu}=\mathbf{0}} = -\frac{\partial H^{(\boldsymbol{\mu})}}{\partial\alpha}\Big|_{\boldsymbol{\mu}=\mathbf{0}}\Big(\frac{\partial H^{(\boldsymbol{\mu})}}{\partial\varphi^{(\boldsymbol{\mu})}_{R_1}}\Big|_{\boldsymbol{\mu}=\mathbf{0}}\Big)^{-1} = -\frac{1}{a_\infty\alpha} < 0.$$

This completes the proof. $\square$

For $\delta_0 \in (0, \frac{1}{2})$ and $U_L \in \bar{D}$, we set

$$\varphi^{(\boldsymbol{\mu})}_1(\alpha; U_L, \boldsymbol{\mu}) = \begin{cases} \varphi^{(\boldsymbol{\mu})}_{S_1}(\alpha; U_L, \boldsymbol{\mu}) & \text{for } \alpha \in [1, \delta_0^{-1}), \\ \varphi^{(\boldsymbol{\mu})}_{R_1}(\alpha; U_L, \boldsymbol{\mu}) & \text{for } \alpha \in (0, 1], \end{cases} \tag{2.32}$$

and

$$\varphi^{(\boldsymbol{\mu})}_2(\alpha; U_L, \boldsymbol{\mu}) = \begin{cases} \varphi^{(\boldsymbol{\mu})}_{S_2}(\alpha; U_L, \boldsymbol{\mu}) & \text{for } \alpha \in (\delta_0, 1], \\ \varphi^{(\boldsymbol{\mu})}_{R_2}(\alpha; U_L, \boldsymbol{\mu}) & \text{for } \alpha \in [1, \infty), \end{cases} \tag{2.33}$$

where $\varphi^{(\boldsymbol{\mu})}_{S_j}$ and $\varphi^{(\boldsymbol{\mu})}_{R_j}$, $j = 1, 2$, are given by Lemma 2.3 and Lemma 2.4, respectively.

Using Lemmas 2.3–2.4, we have

Lemma 2.5. *Let D be given in* Lemma 2.1 *and $\delta_0 \in (0, \frac{1}{2})$. Let $\bar{\boldsymbol{\mu}}_0'$ be given in* Lemma 2.3*, and let $\bar{\boldsymbol{\mu}}_0''$ be given in* Lemma 2.4*. Then, for $\|\boldsymbol{\mu}\| \le \min\{\|\bar{\boldsymbol{\mu}}_0'\|, \|\bar{\boldsymbol{\mu}}_0''\|\}$ and $U_L \in \bar{D}$, the following statements hold:*

(i) *$\varphi^{(\boldsymbol{\mu})}_1 \in C^2\big((0, \delta_0^{-1}) \times \bar{D} \times (0, \min\{\|\bar{\boldsymbol{\mu}}_0'\|, \|\bar{\boldsymbol{\mu}}_0''\|\})\big)$ satisfies that $\varphi^{(\boldsymbol{\mu})}_1\Big|_{\alpha=1} = 0$ and $\frac{\partial\varphi^{(\boldsymbol{\mu})}_1}{\partial\alpha} \le 0$ for $\alpha \in (0, \delta_0^{-1})$, and*

$$\varphi^{(\boldsymbol{\mu})}_1\Big|_{\boldsymbol{\mu}=\mathbf{0}} = \begin{cases} -\frac{\ln\alpha}{a_\infty} & \text{for } \alpha \in (0, 1], \\ -\frac{\sqrt{2}}{a_\infty}\sqrt{\frac{(\alpha-1)\ln\alpha}{\alpha+1}} & \text{for } \alpha \in [1, \delta_0^{-1}); \end{cases} \tag{2.34}$$

(ii) *$\varphi^{(\boldsymbol{\mu})}_2 \in C^2\big((\delta_0, \infty) \times \bar{D} \times (0, \min\{\|\bar{\boldsymbol{\mu}}_0'\|, \|\bar{\boldsymbol{\mu}}_0''\|\})\big)$ satisfies that $\varphi^{(\boldsymbol{\mu})}_2\Big|_{\alpha=1} = 0$ and $\frac{\partial\varphi^{(\boldsymbol{\mu})}_2}{\partial\alpha} \ge 0$ for $\alpha \in (\delta_0, \infty)$, and*

$$\varphi^{(\boldsymbol{\mu})}_2\Big|_{\boldsymbol{\mu}=\mathbf{0}} = \begin{cases} -\frac{\sqrt{2}}{a_\infty}\sqrt{-\frac{(1-\alpha)\ln\alpha}{1+\alpha}} & \text{for } \alpha \in (\delta_0, 1], \\ \frac{\ln\alpha}{a_\infty} & \text{for } \alpha \in [1, \infty). \end{cases} \tag{2.35}$$

Based on Lemma 2.5, we define

$$\Phi^{(\boldsymbol{\mu})}_1(\alpha_1; U_L, \boldsymbol{\mu}) = \big(\rho_L\alpha_1, v_L + \varphi^{(\boldsymbol{\mu})}_1(\alpha_1; U_L, \boldsymbol{\mu})\big) \qquad \text{for } \alpha_1 \in (0, \delta_0^{-1}), \tag{2.36}$$

$$\Phi^{(\boldsymbol{\mu})}_2(\alpha_2; U_L, \boldsymbol{\mu}) = \big(\rho_L\alpha_2, v_L + \varphi^{(\boldsymbol{\mu})}_2(\alpha_2; U_L, \boldsymbol{\mu})\big) \qquad \text{for } \alpha_2 \in (\delta_0, \infty). \tag{2.37}$$

Denote

$$\begin{aligned}\Phi^{(\boldsymbol{\mu})}(\boldsymbol{\alpha};U_L,\boldsymbol{\mu}):&=\Phi_2^{(\boldsymbol{\mu})}\big(\alpha_2;\Phi_1^{(\boldsymbol{\mu})}(\alpha_1;U_L,\boldsymbol{\mu}),\boldsymbol{\mu}\big)\\&=(\rho_L\alpha_2\alpha_1,v_L+\varphi_1^{(\boldsymbol{\mu})}(\alpha_1;U_L,\boldsymbol{\mu})+\varphi_2^{(\boldsymbol{\mu})}(\alpha_2;U_M^{(\boldsymbol{\mu})},\boldsymbol{\mu}))^\top,\end{aligned}\tag{2.38}$$

where $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$ and $U_M^{(\boldsymbol{\mu})}=(\rho_L\alpha_1,v_L+\varphi_1^{(\boldsymbol{\mu})}(\alpha_1;U_L,\boldsymbol{\mu}))^\top$.

Next, we consider the elementary wave curves of system (1.20) for $U:=(\rho,v)^\top$. The eigenvalues of system (1.20) are

$$\lambda_1(U)=v-a_\infty^{-1},\qquad \lambda_2(U)=v+a_\infty^{-1},\tag{2.39}$$

and the corresponding two right-eigenvectors are

$$\mathbf{r}_1(U)=(-\rho,a_\infty^{-1})^\top,\qquad \mathbf{r}_2(U)=(\rho,a_\infty^{-1})^\top.\tag{2.40}$$

Notice that, for any $U\in D$, by Lemma 2.1 and (2.39)–(2.40), we know that

$$\lambda_j^{\boldsymbol{\mu}}(U,\boldsymbol{\mu})\Big|_{\boldsymbol{\mu}=\mathbf{0}}=\lambda_j(U),\quad \mathbf{r}_j^{(\boldsymbol{\mu})}(U,\boldsymbol{\mu})\Big|_{\boldsymbol{\mu}=\mathbf{0}}=\mathbf{r}_j(U)\qquad\text{for } j=1,2.\tag{2.41}$$

For $U\in D$, system (1.20) is strictly hyperbolic. Moreover, a direct computation shows that

$$\nabla_U\lambda_j(U)\cdot\mathbf{r}_j(U)=a_\infty^{-1}>0\qquad\text{for } j=1,2.$$

It implies that the two characteristics families are genuinely nonlinear in D. Then, for any two constant states $U,U_L\in D$, we can parameterize the first elementary wave curve (including 1-shock S_1 and 1-rarefaction wave R_1) and the second elementary wave curve (including 2-shock S_2 and 2-rarefaction wave R_2) of system (1.20) which connects U_L to U as

$$U=\Phi_1(\alpha_1;U_L)=(\rho_L\alpha_1,v_L+\varphi_1(\alpha_1))^\top,\tag{2.42}$$

$$U=\Phi_2(\alpha_2;U_L)=(\rho_L\alpha_2,v_L+\varphi_2(\alpha_2))^\top,\tag{2.43}$$

respectively, where

$$\varphi_1(\alpha_1)=\begin{cases}-\frac{\ln\alpha_1}{a_\infty} & \text{for } \alpha_1\in(0,1],\\ -\frac{\sqrt{2}}{a_\infty}\sqrt{\frac{(\alpha_1-1)\ln\alpha_1}{\alpha_1+1}} & \text{for } \alpha_1\in[1,\infty),\end{cases}\tag{2.44}$$

$$\varphi_2(\alpha_2)=\begin{cases}-\frac{\sqrt{2}}{a_\infty}\sqrt{-\frac{(1-\alpha_2)\ln\alpha_2}{1+\alpha_2}} & \text{for } \alpha_2\in(0,1],\\ \frac{\ln\alpha_2}{a_\infty} & \text{for } \alpha_2\in[1,\infty).\end{cases}\tag{2.45}$$

Finally, we set

$$\Phi(\boldsymbol{\alpha};U_L)=(\rho_L\alpha_2\alpha_1,v_L+\varphi_1(\alpha_1)+\varphi_2(\alpha_2))^\top\qquad\text{for } \boldsymbol{\alpha}=(\alpha_1,\alpha_2).\tag{2.46}$$

Then, by direct computation, we have

Lemma 2.6. *$\varphi_k(\alpha)$, $k=1,2$, satisfy*

(i) *$\varphi_k(\alpha)\in C^2(\mathbb{R}_+)$ for $k=1,2$;*
(ii) *$\varphi_1'(\alpha)<0$ and $\varphi_2'(\alpha)>0$ for $\alpha\in\mathbb{R}_+$;*
(iii) *$\varphi_k(1)=0$ and $\varphi_k'(1)=(-1)^k a_\infty^{-1}$ for $k=1,2$;*
(iv) *For any $U_L\in D$,*

$$\varphi_j^{(\boldsymbol{\mu})}\Big|_{\boldsymbol{\mu}=\mathbf{0}}=\varphi_j,\quad \frac{\partial\varphi_j^{(\boldsymbol{\mu})}}{\partial\alpha}\Big|_{\boldsymbol{\mu}=\mathbf{0}}=\varphi_j'(\alpha)\qquad \text{for } j=1,2.\tag{2.47}$$

2.2. **Comparison of the Riemann solvers between systems** (1.16) **and** (1.20)**.** In this subsection, we consider the comparison of the Riemann solvers between system (1.16) and (1.20) with/without a boundary.

First, we are concerned with the Riemann problem for system (1.16) with $(\tilde{x}, \tilde{y}) \in \Omega$ and the following data:

$$U(\tilde{x}, y) = \begin{cases} U_R := (\rho_R, v_R)^\top & \text{for } y > \tilde{y}, \\ U_L := (\rho_L, v_L)^\top & \text{for } y < \tilde{y}. \end{cases} \tag{2.48}$$

Lemma 2.7. *Let domain D be given as in* Lemma 2.1. *For any two given constant states $U_L, U_R \in D$, there exist small constants $\delta_0' \in (\delta_0, \frac{1}{2})$ and $\bar{\boldsymbol{\mu}}_1' = (\bar{\epsilon}_1', \bar{\tau}_1'^2)$ with $\bar{\epsilon}_1' < \min\{\bar{\epsilon}_0', \bar{\epsilon}_0''\}$ and $\bar{\tau}_1' < \min\{\bar{\tau}_0', \bar{\tau}_0''\}$ such that, for $\|\boldsymbol{\mu}\| \le \|\bar{\boldsymbol{\mu}}_1'\|$, the Riemann problem* (1.16) *and* (2.48) *admits a unique constant state $U_M^{(\bar{\boldsymbol{\mu}})}$ satisfying*

$$U_M^{(\boldsymbol{\mu})} = \Phi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}), \qquad U_R = \Phi_2^{(\boldsymbol{\mu})}(\alpha_2; U_M^{(\boldsymbol{\mu})}, \boldsymbol{\mu}), \tag{2.49}$$

where $\alpha_1, \alpha_2 \in (\delta_0', \frac{1}{\delta_0'})$.

Proof. To obtain the solution of the Riemann problem (1.16) and (2.48), it suffices to solve the following equations for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$:

$$U_R = \Phi^{(\boldsymbol{\mu})}(\boldsymbol{\alpha}; U_L, \boldsymbol{\mu}).$$

More precisely, by (2.38), it can be rewritten as

$$\begin{cases} \rho_R =: \Phi^{(\boldsymbol{\mu}),1}(\boldsymbol{\alpha}; U_L, \boldsymbol{\mu}) = \rho_L \alpha_2 \alpha_1, \\ v_R =: \Phi^{(\boldsymbol{\mu}),2}(\boldsymbol{\alpha}; U_L, \boldsymbol{\mu}) = v_L + \varphi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}) + \varphi_2^{(\boldsymbol{\mu})}\big(\alpha_2; \Phi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}), \boldsymbol{\mu}\big). \end{cases} \tag{2.50}$$

For $\boldsymbol{\mu} = \mathbf{0}$, by (2.34)–(2.35), equations (2.50) admits a unique solution $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ for $U_L, U_R \in D$. Next, by (2.47),

$$\det\left(\frac{\partial(\Phi^{(\boldsymbol{\mu}),1}, \Phi^{(\boldsymbol{\mu}),2})}{\partial(\alpha_1, \alpha_2)}\Big|_{\boldsymbol{\mu}=\mathbf{0}}\right) = \rho_L\big(\alpha_2\varphi_2'(\alpha_2) - \alpha_1\varphi_1'(\alpha_1)\big).$$

Thus, it follows from Lemma 2.6 that there exists a small constant $\delta_0' \in (\delta_0, \frac{1}{2})$ such that, for $\alpha_1, \alpha_2 \in (\delta_0', \frac{1}{\delta_0'})$ and $\rho_L \in D$,

$$\rho_L\big(\alpha_2\varphi_2'(\alpha_2) - \alpha_1\varphi_1'(\alpha_1)\big) > C_{\delta_0'} > 0.$$

Then, by the implicit function theorem, there exist constants $\bar{\boldsymbol{\mu}}_1' = (\bar{\epsilon}_1', \bar{\tau}_1'^2)$ with $\bar{\epsilon}_1' < \min\{\bar{\epsilon}_0', \bar{\epsilon}_0''\}$ and $\bar{\tau}_1' < \min\{\bar{\tau}_0', \bar{\tau}_0''\}$ such that, for $\|\boldsymbol{\mu}\| \le \|\bar{\boldsymbol{\mu}}_1'\| = \bar{\epsilon}_1' + \bar{\tau}_1'^2$, equations (2.50) admit a unique solution $(\alpha_1, \alpha_2) \in (\delta_0', \frac{1}{\delta_0'})^2$. □

We now make the comparison of the Riemann solutions between system (1.16) and system (1.20) with the initial-boundary condition (2.48).

Proposition 2.1. *For a given number $k = 1, 2$, assume that two constant states $U_L, U_R \in D$ satisfy*

$$U_R = \Phi(\boldsymbol{\beta}; U_L), \qquad U_R = \Phi_k^{(\boldsymbol{\mu})}(\alpha_k; U_L, \boldsymbol{\mu}), \tag{2.51}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2)$, $\alpha_k \in (\delta_0', \frac{1}{\delta_0'})$, and $\delta_0' > 0$ is given in Lemma 2.7. *Then, for $\|\boldsymbol{\mu}\| \le \|\bar{\boldsymbol{\mu}}_1'\|$,*

$$\beta_k = \alpha_k + O(1)|\alpha_k - 1|\|\boldsymbol{\mu}\|, \qquad \beta_j = 1 + O(1)|\alpha_k - 1|\|\boldsymbol{\mu}\|, \tag{2.52}$$

where $j \neq k$, $j = 1, 2$, *and* $\bar{\boldsymbol{\mu}}_1' = (\bar{\epsilon}_1', \bar{\tau}_1')$ *is given in* Lemma 2.7. *Moreover, if* $|U_R - U_L| = \alpha_{NP}$, *then*

$$|\beta_1 - 1| + |\beta_2 - 1| = O(1)\alpha_{NP}, \tag{2.53}$$

where the bounds of $O(1)$ *are independent on* $\boldsymbol{\mu}$.

Proof. First, consider the following equations that are derived from (2.51):

$$\Phi(\boldsymbol{\beta}; U_L) = \Phi_k^{(\boldsymbol{\mu})}(\alpha_k; U_L, \boldsymbol{\mu}) \qquad \text{for } \boldsymbol{\beta} = (\beta_1, \beta_2).$$

Without loss of the generality, we consider only the case: $k = 1$. By (2.36) and (2.42)–(2.43), for $U_L \in D$, the above equation is equivalent to the following equations:

$$\begin{cases} F_1(\beta_1, \beta_2, \alpha_1, \boldsymbol{\mu}, U_L) = 0, \\ F_2(\beta_1, \beta_2, \alpha_1, \boldsymbol{\mu}, U_L) = 0, \end{cases} \tag{2.54}$$

where

$$\begin{cases} F_1(\beta_1, \beta_2, \alpha_1, \boldsymbol{\mu}, U_L) := \beta_1\beta_2 - \alpha_1, \\ F_2(\beta_1, \beta_2, \alpha_1, \boldsymbol{\mu}, U_L) := \varphi_1(\beta_1) + \varphi_2(\beta_2) - \varphi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}). \end{cases}$$

When $\boldsymbol{\mu} = \mathbf{0}$, by Lemma 2.6, system (2.54) has a unique solution $\beta_1 = \alpha_1$ and $\beta_2 = 1$. When $\boldsymbol{\mu} \neq \mathbf{0}$, by Lemma 2.6,

$$\det\Big(\frac{\partial(F_1, F_2)}{\partial(\beta_1, \beta_2)}\Big)\Big|_{\boldsymbol{\mu}\neq\mathbf{0}, \beta_1=\alpha_1, \beta_2=1} = \varphi_2'(1) - \alpha_1\varphi_1'(\alpha_1) > C_{\delta_0'}' > 0 \qquad \text{for } \alpha_1 \in (\delta_0', \tfrac{1}{\delta_0'}).$$

Thus, by the implicit function theorem, system (2.54) admits a unique solution:

$$(\beta_1, \beta_2) = (\beta_1(\alpha_1, \boldsymbol{\mu}), \beta_2(\alpha_1, \boldsymbol{\mu})) \in C^2.$$

In addition, by Lemma 2.6, $\beta_1(1, \boldsymbol{\mu}) = \beta_2(1, \boldsymbol{\mu}) = 1$, $\beta_1(\alpha_1, 0) = \alpha_1$, and $\beta_2(\alpha_1, 0) = 1$. Then we can apply the Taylor expansion formula to obtain

$$\beta_1(\alpha_1, \boldsymbol{\mu}) = \beta_1(\alpha_1, 0) + \beta_1(1, \boldsymbol{\mu}) - \beta_1(1, 0) + O(1)|\alpha_1 - 1|\|\boldsymbol{\mu}\| = \alpha_1 + O(1)|\alpha_1 - 1|\|\boldsymbol{\mu}\|,$$
$$\beta_2(\alpha_1, \boldsymbol{\mu}) = \beta_2(\alpha_1, 0) + \beta_2(1, \boldsymbol{\mu}) - \beta_2(1, 0) + O(1)|\alpha_1 - 1|\|\boldsymbol{\mu}\| = 1 + O(1)|\alpha_1 - 1|\|\boldsymbol{\mu}\|,$$

which are estimates (2.52).

Since, for $U_L, U_R \in D$, there exists $C > 0$, independent of $\boldsymbol{\mu}$, such that

$$\frac{1}{C}\sum_{j=1,2}|\beta_j - 1| \leq |\Phi(\boldsymbol{\beta}; U_L) - U_L| \leq C\sum_{j=1,2}|\beta_j - 1|,$$

estimate (2.53) follows immediately. □

Following the proof of Proposition 2.1 above, we have the following corollary in a direct way, whose proof is omitted.

Corollary 2.1. *Assume that two constant states* $U_L, U_R \in D$ *satisfy*

$$U_R = \Phi(\boldsymbol{\beta}; U_L), \quad U_R = \Phi^{(\boldsymbol{\mu})}(\boldsymbol{\alpha}; U_L, \boldsymbol{\mu}) \qquad \text{for } \boldsymbol{\beta} = (\beta_1, \beta_2) \text{ and } \boldsymbol{\alpha} = (\alpha_1, \alpha_2), \tag{2.55}$$

where $\alpha_1, \alpha_2 \in (\delta_0', \frac{1}{\delta_0'})$, *and constant* $\delta_0' > 0$ *is given in* Lemma 2.7. *Then, for* $\|\boldsymbol{\mu}\| \leq \|\bar{\boldsymbol{\mu}}_1'\|$,

$$\beta_j = \alpha_j + O(1)\Big(\sum_{k=1,2}|\alpha_k - 1|\Big)\|\boldsymbol{\mu}\| \qquad \text{for } j = 1, 2, \tag{2.56}$$

where $\bar{\boldsymbol{\mu}}_1' = (\bar{\epsilon}_1', \bar{\tau}_1')$ *is given in* Lemma 2.7. *Moreover, if* $\tilde{U}_R \in D$, $|\tilde{U}_R - U_R| = \alpha_{NP}$, *and*

$$U_R = \Phi(\boldsymbol{\beta}; U_L), \quad \tilde{U}_R = \Phi^{(\boldsymbol{\mu})}(\boldsymbol{\alpha}; U_L, \boldsymbol{\mu}) \qquad \text{for } \boldsymbol{\beta} = (\beta_1, \beta_2) \text{ and } \boldsymbol{\alpha} = (\alpha_1, \alpha_2), \tag{2.57}$$

then

$$\beta_j = \alpha_j + O(1)\big(\sum_{k=1,2} |\alpha_k - 1|\big)\|\boldsymbol{\mu}\| + O(1)\alpha_{NP} \qquad \textit{for } j = 1, 2, \tag{2.58}$$

where the bounds of $O(1)$ *are independent of* $\boldsymbol{\mu}$.

Next, we compare the Riemann solutions near the boundary with the following initial boundary value conditions:

$$\begin{cases} v_b^{(\boldsymbol{\mu})} = \sqrt{1-\tau^2 B^{(\epsilon)}(\rho_b^{(\boldsymbol{\mu})}, v_b^{(\boldsymbol{\mu})}, \epsilon)}\, b_0 & \text{on } \{x = \hat{x},\ y = \hat{y} + b_0(x - \hat{x})\}, \\ U(x, y) = U_L & \text{on } \{x = \hat{x},\ y < \hat{y}\}, \end{cases} \tag{2.59}$$

where $U_L = (\rho_L, v_L)^\top$ and $b_0 < 0$.

Lemma 2.8. *For any given constant state* $U_L \in D$ *with* D *defined by* Lemma 2.1*, there exist small constants* $\bar{\epsilon}_1'' \leq \min\{\bar{\epsilon}_0', \bar{\epsilon}_0''\}$*,* $\bar{\tau}_1'' \leq \min\{\bar{\tau}_0', \bar{\tau}_0''\}$*, and* $\delta_0'' \in (\delta_0, \frac{1}{2})$ *such that, for* $\|\boldsymbol{\mu}\| \leq \|\bar{\boldsymbol{\mu}}_1''\|$*, the Riemann problem* (1.16) *and* (2.59) *admits a unique solution* $U_b^{(\boldsymbol{\mu})} = (\rho_b^{(\boldsymbol{\mu})}, v_b^{(\boldsymbol{\mu})})^\top$ *connecting* U_L *by the first-family wave curve with strength* $\alpha_1 \in (\delta_0'', \frac{1}{\delta_0''})$:

$$U_b^{(\boldsymbol{\mu})} = \Phi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}), \qquad U_b^{(\boldsymbol{\mu})} = (\rho_b^{(\boldsymbol{\mu})}, v_b^{(\boldsymbol{\mu})})^\top. \tag{2.60}$$

Proof. It suffices to show that the following equation:

$$v_L + \varphi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}) = b_0\sqrt{1 - \tau^2 B^{(\epsilon)}(\Phi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}), \epsilon)}$$

has a unique solution α_1 when $\|\boldsymbol{\mu}\|$ is small for $U_L \in D$ and $b_0 < 0$.

Let

$$F_b(\alpha_1; \boldsymbol{\mu}, b_0, U_L) = v_L + \varphi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}) - b_0\sqrt{1 - \tau^2 B^{(\epsilon)}(\Phi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}), \epsilon)}.$$

When $\boldsymbol{\mu} = \mathbf{0}$, $F_b(\alpha_1; \boldsymbol{\mu}, b_0, U_L)$ can be reduced as

$$F_b(\alpha_1; \mathbf{0}, b_0, U_L) = \varphi_1(\alpha_1) + v_L - b_0.$$

If $v_L \leq b_0$, then, by (2.44), equation $F_b(\alpha_1; \mathbf{0}, b_0, v_L) = 0$ has a unique solution $\alpha_1 = e^{a_\infty(v_L - b_0)} \in (0, 1]$.

If $v_L > b_0$, it follows from $\lim_{\alpha_1 \to \infty} \frac{(\alpha_1 - 1)\ln\alpha_1}{\alpha_1 + 1} = \infty$ that

$$\lim_{\alpha_1 \to \infty} F_b(\alpha_1; \mathbf{0}, b_0, U_L) = -\infty.$$

Moreover, for $\alpha_1 \in [1, \infty)$,

$$F_b(1; \mathbf{0}, b_0, U_L) = v_L - b_0 > 0$$

and, by Lemma 2.6,

$$\frac{\partial F_b(\alpha_1; \mathbf{0}, b_0, U_L)}{\partial \alpha_1} = \varphi_1'(\alpha_1) < 0.$$

Thus, $F_b(\alpha_1; \mathbf{0}, b_0, U_L) = 0$ has a unique solution $\alpha_1 \in (1, \infty)$. Therefore, if $\boldsymbol{\mu} = \mathbf{0}$, $F_b(\alpha_1; \boldsymbol{\mu}, b_0, U_L) = 0$ has a unique solution α_1 for $U_L \in D$ and $b_0 < 0$.

Next, notice that there exists a constant $\delta_0'' > 0$ such that, for $\alpha_1 \in (\delta_0'', \frac{1}{\delta_0''})$,

$$\frac{\partial F_b}{\partial \alpha_1}\Big|_{\boldsymbol{\mu} = \mathbf{0}} = \varphi_1'(\alpha_1) < -C_{\delta_0''} < 0.$$

Then, by the implicit function theorem, there exist small constants $\bar{\epsilon}_1'' \leq \min\{\bar{\epsilon}_0', \bar{\epsilon}_0''\}$ and $\bar{\tau}_1'' \leq \min\{\bar{\tau}_0', \bar{\tau}_0''\}$ such that, for $\|\boldsymbol{\mu}\| \leq \|\bar{\boldsymbol{\mu}}_1''\|$, $F_b(\alpha_1; \boldsymbol{\mu}, b_0, U_L) = 0$ admits a unique solution $\alpha_1 \in (\delta_0'', \frac{1}{\delta_0''})$. This completes the proof. □

Now we are ready to compare the Riemann solutions between system (1.16) and system (1.20) with a boundary.

Proposition 2.2. *Let $U_L = (\rho_L, v_L)^\top$, $U_b = (\rho_b, v_b)^\top$, and $U_b^{(\boldsymbol{\mu})} = (\rho_b^{(\boldsymbol{\mu})}, v_b^{(\boldsymbol{\mu})})^\top$ be the three constant states in D satisfying*

$$U_b = \Phi_1(\beta_1; U_L), \qquad U_b^{(\boldsymbol{\mu})} = \Phi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}) \quad \text{for } \alpha_1 \in (\delta_0'', \tfrac{1}{\delta_0''}), \tag{2.61}$$

and

$$v_b = b_0, \qquad v_b^{(\boldsymbol{\mu})} = b_0\sqrt{1 - \tau^2 B^{(\epsilon)}(\rho_b^{(\boldsymbol{\mu})}, v_b^{(\boldsymbol{\mu})}, \epsilon)}, \tag{2.62}$$

where $\delta_0'' > 0$ is given in Lemma 2.8. *Then, for $\|\boldsymbol{\mu}\| \leq \|\bar{\boldsymbol{\mu}}_1''\|$,*

$$\beta_1 = \alpha_1 + O(1)\big(1 + |\alpha_1 - 1|\big)\|\boldsymbol{\mu}\|, \tag{2.63}$$

where $\bar{\epsilon}_1''$ and $\bar{\tau}_1''$ are given in Lemma 2.8 *and the bound of $O(1)$ is independent of $\boldsymbol{\mu}$.*

Proof. By (2.36), (2.42), and (2.61)–(2.62), we have the following relation for α_1 and β_1:

$$v_L + \varphi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}) = \sqrt{1 - \tau^2 B^{(\epsilon)}(\rho_b^{(\boldsymbol{\mu})}, v_b^{(\boldsymbol{\mu})}, \epsilon)}\big(\varphi_1(\beta_1) + v_L\big).$$

Let

$$\begin{aligned} \mathcal{F}_b(\beta_1, \alpha_1, \boldsymbol{\mu}, U_L) :&= \sqrt{1 - \tau^2 B^{(\epsilon)}(\rho_b^{(\boldsymbol{\mu})}, v_b^{(\boldsymbol{\mu})}, \epsilon)}\, \varphi_1(\beta_1) - \varphi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}) \\ &\quad + \Big(\sqrt{1 - \tau^2 B^{(\epsilon)}(\rho_b^{(\boldsymbol{\mu})}, v_b^{(\boldsymbol{\mu})}, \epsilon)} - 1\Big) v_L. \end{aligned}$$

For $\boldsymbol{\mu} = \mathbf{0}$, it is direct to see that $\mathcal{F}_b(\beta_1, \alpha_1, \mathbf{0}, U_L) = \varphi_1(\beta_1) - \varphi_1(\alpha_1) = 0$ has a unique solution $\beta_1 = \alpha_1$. In addition, by Lemma 2.6, for $\alpha_1 \in (\delta_0'', 1) \cup (1, \frac{1}{\delta_0''})$, we have

$$\frac{\partial \mathcal{F}_b(\beta_1, \alpha_1, \boldsymbol{\mu}, U_L)}{\partial \beta_1}\bigg|_{\boldsymbol{\mu}=\mathbf{0}, \beta_1=\alpha_1} = -\varphi_1'(\alpha_1) - C_{\delta_0''} < 0. \tag{2.64}$$

Then, by the implicit function theorem, for $\|\boldsymbol{\mu}\| \leq \|\bar{\boldsymbol{\mu}}_1''\|$, there exists a unique solution $\beta_1 = \beta_1(\alpha_1, \boldsymbol{\mu}) \in C^2$ of the equation: $\mathcal{F}_b = 0$. Moreover, when $\alpha_1 = 1$, by Lemmas 2.5–2.6, $\mathcal{F}_b(\beta_1, \alpha_1, \boldsymbol{\mu}, U_L) = 0$ can be reduced to

$$\sqrt{1 - \tau^2 B^{(\epsilon)}(\rho_L, v_L, \epsilon)}\, \varphi_1(\beta_1) + \Big(\sqrt{1 - \tau^2 B^{(\epsilon)}(\rho_L, v_L, \epsilon)} - 1\Big) v_L = 0,$$

so that

$$\beta_1(1, \boldsymbol{\mu}) = 1 + O(1)\|\boldsymbol{\mu}\|,$$

where the bound of $O(1)$ is independent of $\boldsymbol{\mu}$.

Finally, by the Taylor formula and the fact that $\beta_1(1, \mathbf{0}) = 1$,

$$\begin{aligned} \beta_1(\alpha_1, \boldsymbol{\mu}) &= \beta_1(\alpha_1, \mathbf{0}) + \beta_1(1, \boldsymbol{\mu}) - \beta_1(1, \mathbf{0}) + O(1)|\alpha_1 - 1|\|\boldsymbol{\mu}\| \\ &= \alpha_1 + O(1)\big(1 + |\alpha_1 - 1|\big)\|\boldsymbol{\mu}\|, \end{aligned}$$

where the bound of $O(1)$ is independent of $\boldsymbol{\mu}$. □

Based on Propositions 2.1–2.2, we have

Proposition 2.3. *Let $U_L=(\rho_L,v_L)^\top$ and $U_b=(\rho_b,v_b)^\top$ be two constant states in D satisfying*

$$U_b=\Phi(\boldsymbol{\beta};U_L),\quad U_b=\Phi_1^{(\boldsymbol{\mu})}(\alpha_1;U_L,\boldsymbol{\mu})\qquad \text{for } \boldsymbol{\beta}=(\beta_1,\beta_2), \tag{2.65}$$

$$v_b=b_0\sqrt{1-\tau^2B^{(\epsilon)}(\rho_b,v_b,\epsilon)}. \tag{2.66}$$

Then, for $\alpha_1\in(\delta_0'',\frac{1}{\delta_0''})$ and $\|\boldsymbol{\mu}\|\le\|\bar{\boldsymbol{\mu}}_1''\|$,

$$\beta_1=\alpha_1+O(1)\big(1+|\alpha_1-1|\big)\|\boldsymbol{\mu}\|,\quad \beta_2=1+O(1)\big(1+|\alpha_1-1|\big)\|\boldsymbol{\mu}\|, \tag{2.67}$$

where the small constants δ_0'', $\bar{\epsilon}_1''$, and $\bar{\tau}_1''$ are given in Lemmas 2.7–2.8*, and the bound of $O(1)$ is independent of $\boldsymbol{\mu}$.*

3. Existence of Solutions of Problem (1.16)–(1.18) and Well-Posedness of Problem (1.20)–(1.22) with Large Data

In this section, we construct the approximate solutions of the initial-boundary value problem (1.16)–(1.18) and establish the well-posedness of the initial-boundary value problem (1.20)–(1.22) in $BV\cap L^1$, which is the basis to establish the L^1-error estimate between the two respective entropy solutions of problem (1.16)–(1.18) and problem (1.20)–(1.22).

3.1. **Wave front-tracking scheme for problem** (1.16)–(1.18)**.** Let $\nu\in\mathbb{N}_+$ be a given parameter. As in [1, 3] (see also [16]), for given initial data $U_0(y)=(\rho_0,v_0)^\top(y)$ with $y<0$, we can construct a piecewise constant function $U_0^\nu(y)=(\rho_0^\nu,v_0^\nu)^\top(y)$ such that

$$\|U_0^\nu(\cdot)-U_0(\cdot)\|_{L^1(\Sigma_0)}\le 2^{-\nu},\qquad T.V.\{U_0^\nu(\cdot);\Sigma_0\}\le T.V.\{U_0(\cdot);\Sigma_0\}. \tag{3.1}$$

Then the approximate solution $U^{(\boldsymbol{\mu}),\nu}(x,y)$ in Ω is constructed in the following way:

Let $y_N<y_{N-1}<\cdots<y_1<y_0=0$ be the location of the discontinuities of $U_0^\nu(y)$ at $x=0$. At each point $(0,y_k)$ for $1\le k\le N$, we solve the Riemann problem (1.16) and (2.44) with $U_L=U_0^\nu(y_k-)$ and $U_R=U_0^\nu(y_k+)$. At $(0,y_0)$, we solve the Riemann problem (1.16) and (2.59) with $U_L=U_0^\nu(0-)$. Then, by Lemmas 2.7–2.8, the solutions of these two types of Riemann problem may consist of shock waves $S^{(\boldsymbol{\mu})}$ or rarefaction waves $R^{(\boldsymbol{\mu})}$. We further partition the rarefaction waves into several small central rarefaction fans (still denoted by) $R^{(\boldsymbol{\mu})}$ with strength less than ν^{-1}, which propagate with the characteristic speeds. Such a modified solution of the two Riemann problems is called an *Accurate Riemann Solver* (*ARS*). Putting all the modified solutions together, we define an approximate solution $U^{(\boldsymbol{\mu}),\nu}(x,y)$. It is piecewise constant and prolongs until a pair of neighbouring discontinuities interacts at point $(\hat{x},y)\in\Omega$ or a wave front hits boundary $\Gamma_{\rm w}$ at point $(\hat{x}_1,b_0\hat{x}_1)$. At this point, we continue to construct the approximate solution by giving the *ARS* of the Riemann problem (1.16) and (2.44) with initial data $U^{(\boldsymbol{\mu}),\nu}(\hat{x}-,y)$ or of the Riemann problem (1.16) and (2.59) with Riemann data $U^{(\boldsymbol{\mu}),\nu}(\hat{x}_1-,b_0\hat{x}_1-)$. We repeat this construction as long as the number of the wave fronts does not tend to the infinity in a finite *time*. Then, to avoid the case that the number of wave fronts blows up, we introduce a *Simplified Riemann Solver* (*SRS*), in which all the new waves are lumped into a single non-physical wave $NP^{(\boldsymbol{\mu})}$ with a fixed speed $\hat{\lambda}$, which is larger than all the characteristics speeds. To decide when the *SRS* is used, we introduce a threshold parameter $\varrho>0$, depending only on ν^{-1}. When the strengths of the two approaching physical wave fronts α and β satisfy that $|\alpha-1||\beta-1|>\varrho$, the *ARS* is used and, otherwise, the *SRS* is used.

Moreover, we may change some of the speeds of the wave fronts slightly with a quality less than $2^{-\nu}$, in order to make sure that only two wave fronts interact or only one wave front hits boundary Γ at each point. The set of all the fronts are defined by $J(U^{(\boldsymbol{\mu})}):=S^{(\boldsymbol{\mu})}\cup R^{(\boldsymbol{\mu})}\cup NP^{(\boldsymbol{\mu})}$. Then, applying the path decomposition position method developed in [1] and following the arguments in [18], we obtain the following results for the approximate solution $U^{(\boldsymbol{\mu}),\nu}(x,y)$ of problem (1.16)–(1.18).

Proposition 3.1. *Assume that $\rho_* \le \rho_0 \le \rho^*$ for some constants $\rho^* > \rho_* > 0$. Then there exists both a constant vector $\bar{\boldsymbol{\mu}}_0^* = (\bar{\epsilon}_0^*, (\bar{\tau}_0^*)^2)$ and a constant $\bar{C}_0 > 0$ with $\bar{\epsilon}_0^* > 0$ and $\bar{\tau}_0^* > 0$ depending only on $(a_\infty, \rho^*, \rho_*)$ such that, for $\|\boldsymbol{\mu}\| \le \|\bar{\boldsymbol{\mu}}_0^*\|$, if $(\rho_0 - 1, v_0) \in (L^1 \cap BV)(\Sigma_0)$ and*

$$\|\boldsymbol{\mu}\|\big(T.V.\{U_0(\cdot);\ \Sigma_0\} + |b_0|\big) < \bar{C}_0, \tag{3.2}$$

the approximate solution $U^{(\boldsymbol{\mu})}(x,y)$ constructed above can be defined for all $(x,y) \in \Omega$ and satisfies

$$\sup_{x>0} \|U^{(\boldsymbol{\mu}),\nu}(x,\cdot)\|_{L^\infty(-\infty,\, b_0 x)} + \sup_{x>0} T.V.\{U^{(\boldsymbol{\mu}),\nu}(x,\cdot); (-\infty, b_0 x)\} < \bar{C}_1, \tag{3.3}$$

$$\|U^{(\boldsymbol{\mu}),\nu}(x_1, \cdot + b_0 x_1) - U^{(\boldsymbol{\mu}),\nu}(x_2, \cdot + b_0 x_2)\|_{L^1(-\infty,0)} < \bar{C}_2|x_1 - x_2|. \tag{3.4}$$

The strength of each rarefaction wave-front and the total strength of the non-physical front are small:

$$\max_{\alpha \in R^{(\boldsymbol{\mu})}} |\alpha - 1| < \bar{C}_3 \nu^{-1}, \qquad \sum_{\alpha \in NP^{(\boldsymbol{\mu})}} \alpha < \bar{C}_4 2^{-\nu}. \tag{3.5}$$

Moreover, there exists a subsequence $\{\nu_i\}_{i=1}^\infty$ with $\nu_i \to \infty$ as $i \to \infty$ such that

$$U^{(\boldsymbol{\mu}),\nu_i} \to U^{(\boldsymbol{\mu})} \qquad \text{in } L^1_{\mathrm{loc}}(\Omega_{\mathrm{w}}), \tag{3.6}$$

and $U^{(\boldsymbol{\mu})} \in (BV_{\mathrm{loc}} \cap L^1_{\mathrm{loc}})(\Omega_{\mathrm{w}})$ is an entropy solution of problem (1.16)–(1.18)*. Here the positive constants $\bar{C}_k$, $k = 1,2,3,4$, depend only on $(a_\infty, \rho^*, \rho_*)$, but independent of $(\boldsymbol{\mu}, \nu)$.*

3.2. **Well-posedness of the initial-boundary value problem** (1.20)–(1.22)**.** In this subsection, we consider the initial-boundary value problem (1.20)–(1.22), construct the semigroup, and establish the existence and L^1-stability of the solutions. First, we consider the following initial-boundary value problem on $\bar{\Omega} = \{(x,y)\,:\, x > 0, y < 0\}$:

$$\begin{cases} \partial_x \rho + \partial_x(\rho v) = 0 & \text{in } \bar{\Omega}, \\ \partial_x + \partial_x\big(\frac{1}{2}v^2 + \frac{\ln \rho}{a_\infty^2}\big) = 0 & \text{in } \bar{\Omega}, \end{cases} \tag{3.7}$$

with initial data

$$(\rho, v) = (\bar{\rho}_0, \bar{v}_0)(y) \qquad \text{on } \bar{\Sigma}_0 = \{(x,y)\,:\, x = 0, y < 0\}, \tag{3.8}$$

and boundary condition

$$v = 0 \qquad \text{on } \bar{\Gamma} = \{(x,y)\,:\, x > 0, y = 0\}. \tag{3.9}$$

Let $U(x,y) = (\rho, v)^\top(x,y)$, $\bar{U}_0(y) = (\bar{\rho}_0, \bar{v}_0)^\top(y)$, and $\bar{U}_\infty = (\bar{\rho}_\infty, \bar{v}_\infty)^\top$ with $\bar{\rho}_\infty > 0$ and $\bar{v}_\infty > 0$. Following the results in [14, 21], we have

Lemma 3.1. *Assume that $0 < \rho_* < \bar{\rho}_0(y) < \rho^* < \infty$ and $\bar{U}_0(y) - \bar{U}_\infty \in (BV \cap L^1)(\bar{\Sigma}_0)$. Then there is a constant $\bar{C}_0' > 0$ such that, if*

$$T.V.\{\bar{U}_0(y); \bar{\Sigma}_0\} + |\bar{v}_0(0-)| < \bar{C}_0', \tag{3.10}$$

there exist a domain $\bar{\mathcal{D}} \subseteq BV((-\infty, 0))$, an L^1-Lipschitz semigroup $\bar{\mathcal{S}}_x : (0,\infty) \times \bar{\mathcal{D}} \mapsto \bar{\mathcal{D}}$, and a Lipschitz constant $\bar{L} > 0$ so that

(i) *$\bar{\mathcal{D}}$ contains the L^1-closure of the set of those functions $U(\cdot, y) : (-\infty, 0) \mapsto \bar{\Omega}$ satisfying $U - \bar{U}_\infty \in \big(L^1 \cap BV\big)((-\infty, 0))$;*

(ii) $U(x) = \bar{\mathcal{S}}_x(\bar{U}_0)$ *is the entropy solution of the initial-boundary value problem* (3.7)–(3.9) *and satisfies*

$$\|\bar{\mathcal{S}}_{x_1}\bar{U}_{1,0} - \bar{\mathcal{S}}_{x_2}\bar{U}_{2,0}\|_{L^1((-\infty,0))} \leq \bar{L}\big(\|\bar{U}_{1,0} - \bar{U}_{2,0}\|_{L^1(\bar{\Sigma_0})} + |x_1 - x_2|\big); \tag{3.11}$$

(iii) *If* $\bar{U}_0$ *is a piecewise constant function, then, for* $x > 0$ *sufficiently small,* $\bar{\mathcal{S}}_x(U_0)$ *coincides with the solution of the initial-boundary value problem* (3.7)–(3.9) *by piecing together the Riemann solutions at the all jumps of* $\bar{U}_0$.

We now turn to the initial-boundary value problem (1.20)–(1.22).

Proposition 3.2. *Suppose that* $0 < \rho_* < \rho_0(y) < \rho^* < \infty$ *and* $U_0(y) - U_\infty \in (BV \cap L^1)(\Sigma_0)$ *with* $U_\infty = (1, 0)^\top$. *Then there is a constant* $\bar{C}_0'' > 0$ *such that,*

$$T.V.\{U_0(y); \Sigma_0\} + |b_0| < \bar{C}_0'', \tag{3.12}$$

there exist a domain $\mathcal{D} \subset BV((-\infty, b_0x))$, *an* L^1*-Lipschitz semigroup* $\mathcal{S}_x : (0, \infty) \times \mathcal{D} \mapsto \mathcal{D}$, *and a Lipschitz constant* $L > 0$ *so that*

(i) $\mathcal{D}$ *contains the* L^1*-closure of the set of functions* $U : (-\infty, b_0x) \mapsto \Omega_w$ *satisfying* $U - U_\infty \in \big(L^1 \cap BV\big)((-\infty, b_0x))$;

(ii) $U(x, \cdot) = \mathcal{S}_xU_0(\cdot)$ *is the entropy solution of problem* (1.20)–(1.22) *and*

$$\|\mathcal{S}_{x_1}(U_{1,0}(\cdot)) - \mathcal{S}_{x_2}(U_{2,0}(\cdot))\|_{L^1((-\infty,b_0x))} \leq L\|U_{1,0}(\cdot) - U_{2,0}(\cdot)\|_{L^1(\Sigma_0)}; \tag{3.13}$$

(iii) *If* $U(\tilde{x}, \cdot)$ *is a piecewise constant function, then, for* $x > \tilde{x}$ *sufficiently small,* $\mathcal{S}_xU(\tilde{x}, \cdot)$ *coincides with the solution of the initial-boundary value problem* (1.20)–(1.22) *by piecing together all the Riemann solutions at the all jumps of* $U(x)$.

Proof. Let

$$(\hat{x}, \hat{y}) := (x, y - b_0x). \tag{3.14}$$

Define

$$\hat{\rho}(\hat{x}, \hat{y}) := \rho(\hat{x}, \hat{y} + b_0\hat{x}), \quad \hat{v}(\hat{x}, \hat{y}) := v(\hat{x}, \hat{y} + b_0\hat{x}) - b_0. \tag{3.15}$$

Then $(\hat{\rho}, \hat{v})$ satisfies the initial-boundary value problem (3.7)–(3.9) in the $(\hat{x}, \hat{y})$-plane with the initial data:

$$(\hat{\rho}, \hat{v})(0, \hat{y}) = (\rho_0(\hat{y}), v_0(\hat{y}) - b_0).$$

Thus, by applying Lemma 3.1, there exist a domain $\hat{\mathcal{D}} \subset BV((-\infty, 0))$, an L^1-Lipschitz semigroup $\hat{\mathcal{S}}_{\hat{x}} : [0, \infty) \times \hat{\mathcal{D}} \mapsto \hat{\mathcal{D}}$, and a Lipschitz constant $\hat{L} > 0$ so that facts (i)–(iii) in Lemma 3.1 hold. We define the inverse transformation of (3.14)–(3.15):

$$x = \hat{x}, \quad y = \hat{y} + b_0\hat{x}, \quad \rho(x, y) = \hat{\rho}(x, y - b_0x), \quad v(x, y) = \hat{v}(x, y - b_0x) - b_0.$$

Then substituting them into Lemma 3.1, we obtain (i)–(iii). This completes the proof. □

By Proposition 3.2 and [3], we can also derive the following semigroup formula:

Proposition 3.3. *Let* $V(x, y) : [0, \infty) \mapsto \mathcal{D}$ *be a Lipschitz continuous map with a finite number of wave fronts for some* $x > 0$ *and* $V(0, y) = V_0(y)$. *Let* $\mathcal{S}$ *be a semigroup obtained by* Proposition 3.2. *Then*

$$\begin{aligned} &\|\mathcal{S}_x(V_0(\cdot)) - V(x, \cdot)\|_{L^1(-\infty,\, b_0x)} \\ &\leq L\int_0^x \liminf_{h\to 0+} \frac{\|\mathcal{S}_h(V(s, \cdot)) - V(s + h, \cdot)\|_{L^1(-\infty,\, b_0(s+h))}}{h} \mathrm{d}s, \end{aligned} \tag{3.16}$$

where L *and* $\mathcal{D}$ *are given in* Proposition 3.2.

4. Proof of Theorem 1.1

In this section, we prove the main result of this paper, *i.e.*, Theorem 1.1. To complete the proof of Theorem 1.1, we need first to obtain the convergence rate estimate (1.23) that is based on the local L^1-difference between the entropy solutions $U^{(\boldsymbol{\mu})}$ and U, and then show that it is optimal with respect to $\boldsymbol{\mu}$ by constructing a simple example.

4.1. Local L^1 error estimates between the entropy solutions $U^{(\boldsymbol{\mu})}$ and U.

In this subsection, we give some lemmas regarding the local L^1-difference between the entropy solutions $U^{(\boldsymbol{\mu})}$ and U corresponding to problem (1.16)–(1.18) and problem (1.20)–(1.22) with a boundary, respectively.

Lemma 4.1. *Suppose that $U^{(\boldsymbol{\mu}),\nu}$ is an approximate solution of problem* (1.16)–(1.18) *constructed as in* §3.1 *and satisfies* Proposition 3.1 *with a front at point $(\hat{x}, y_{\mathcal{I}}) \in \Omega$. For a given $k = 1, 2$, denote*

$$U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h, y) = \begin{cases} U_R = (\rho_R, v_R)^\top & \text{for } y > y_{\mathcal{I}} + \dot{y}_{\alpha_k} h, \\ U_L = (\rho_L, v_L)^\top & \text{for } y < y_{\mathcal{I}} + \dot{y}_{\alpha_k} h, \end{cases} \tag{4.1}$$

where $U_L := U^{(\boldsymbol{\mu}),\nu}(\hat{x}, y_{\mathcal{I}}-)$, $U_R := U^{(\boldsymbol{\mu}),\nu}(\hat{x}, y_{\mathcal{I}}+)$, $h > 0$, and $|\dot{y}_{\alpha_k}| < \hat{\lambda}$. Let $\mathcal{S}$ be a uniformly Lipschitz continuous semigroup obtained in Proposition 3.3. *If $\|\boldsymbol{\mu}\| \le \|\bar{\boldsymbol{\mu}}_0^*\|$ with $\bar{\boldsymbol{\mu}}_0^*$ given in* Proposition 3.1 *and $h > 0$ is sufficiently small, then*

(i) *when U_L and U_R are connected by a k-th shock wave-font $\alpha_k \in S_k^{(\boldsymbol{\mu})}$ and $|\dot{y}_{\alpha_k} - \sigma_k^{(\boldsymbol{\mu})}(\alpha_k)| < 2^{-\nu}$ for $\sigma_k^{(\boldsymbol{\mu})}(\alpha_k)$ as the speed of k-th shock wave,*

$$\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(y_{\mathcal{I}}-\eta, y_{\mathcal{I}}+\eta)} \le C_{\mathcal{I}}\big(\|\boldsymbol{\mu}\| + 2^{-\nu}\big)|\alpha_k - 1|h; \tag{4.2}$$

(ii) *when U_L and U_R are connected by a k-th rarefaction front $\alpha_k \in R_k^{(\boldsymbol{\mu})}$ with $|\dot{y}_{\alpha_k} - \lambda_k^{(\boldsymbol{\mu})}(U_R, \boldsymbol{\mu})| < 2^{-\nu}$ for $\lambda_k^{(\boldsymbol{\mu})}(U_R, \boldsymbol{\mu})$ as the speed of the rarefaction wave,*

$$\begin{aligned} &\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(y_{\mathcal{I}}-\eta, y_{\mathcal{I}}+\eta)} \\ &\le C_{\mathcal{I}}\big(\|\boldsymbol{\mu}\| + 2^{-\nu} + (1+\|\boldsymbol{\mu}\|)\nu^{-1}\big)|\alpha_k - 1|h; \end{aligned} \tag{4.3}$$

(iii) *when U_L and U_R are connected by a non-physical wave $\alpha_{NP} \in NP^{(\boldsymbol{\mu})}$,*

$$\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(y_{\mathcal{I}}-\eta, y_{\mathcal{I}}+\eta)} \le C_{\mathcal{I}}\alpha_{NP}\, h, \tag{4.4}$$

where constant $C_{\mathcal{I}} > 0$ is independent of $(\boldsymbol{\mu}, h)$, and constant η satisfies $\eta > \hat{\lambda}h$.

Proof. We divide the proof into three steps.

1. Without loss of the generality, we consider the case: $k = 1$ only, since the case: $k = 2$ can be dealt with in the same way.

By Proposition 3.3, we know that, for sufficiently small $h > 0$, $\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot))$ is the Riemann solution of system (1.20), consisting of three constant states U_L, U_M, and U_R with $U_M = (\rho_M, v_M)^\top$. The three states are separated by the elementary waves β_1 and β_2. Then we have

$$\Phi(\boldsymbol{\beta}; U_L) = \Phi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}) \qquad \text{for } \boldsymbol{\beta} = (\beta_1, \beta_2). \tag{4.5}$$

By Proposition 2.1, if $\|\boldsymbol{\mu}\|$ is sufficiently small, equation (4.5) admits a unique solution $\boldsymbol{\beta} = (\beta_1, \beta_2)$ so that

$$\beta_1 = \alpha_1 + O(1)\|\boldsymbol{\mu}\||\alpha_1 - 1|, \qquad \beta_2 = 1 + O(1)\|\boldsymbol{\mu}\||\alpha_1 - 1|. \tag{4.6}$$

Since $\alpha_1 \in S_1^{(\boldsymbol{\mu})}$, β_1 is also a 1-shock wave, while β_2 may be either a 2-shock wave or 2-rarefaction wave (see Fig. 4.1 or Fig. 4.2 below).

Let us first consider the case that both β_1 and β_2 are shock waves. As shown in Fig. 4.1, when $\eta > \hat{\lambda}h$, and $h > 0$ is sufficiently small, interval $(y_I - \eta, y_I + \eta)$ can be divided into two sub-intervals I and II by fronts α_1, β_1, and β_2.

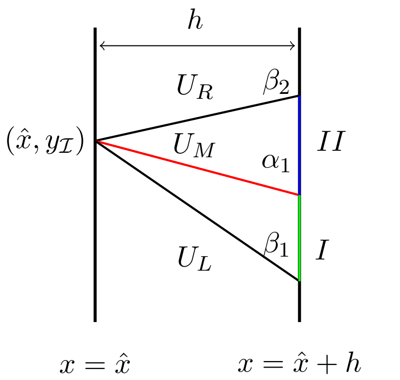

FIGURE 4.1. Comparison of the Riemann solvers for $\alpha_1 \in S_1^{(\boldsymbol{\mu})}$ and β_2 being a 2-shock wave

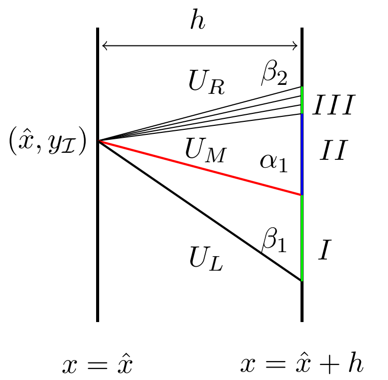

FIGURE 4.2. Comparison of the Riemann solvers for $\alpha_1 \in S_1^{(\boldsymbol{\mu})}$ and β_2 being a 2-rarefaction wave

Denote the speeds of β_1 and β_2 by $\sigma_1(\beta_1)$ and $\sigma_1(\beta_2)$. By the *Rankine-Hugoniot* (R-H) conditions:

$$\sigma_1(\beta_1) = \frac{\rho_M v_M - \rho_L v_L}{\rho_M - \rho_L} = v_L + \frac{\beta_1 \varphi_1(\beta_1)}{\beta_1 - 1},$$
$$\sigma_2(\beta_2) = \frac{\rho_R v_R - \rho_M v_M}{\rho_R - \rho_M} = v_M + \frac{\beta_2 \varphi_2(\beta_2)}{\beta_2 - 1}.$$

Moreover, for $\sigma_1^{(\boldsymbol{\mu})}(\alpha_1)$, when $\|\boldsymbol{\mu}\|$ is sufficiently small, it follows from Lemma 2.6 that

$$\begin{aligned}\sigma_1^{(\boldsymbol{\mu})}(\alpha_1) &= \frac{\rho_R v_R - \rho_L v_L}{\rho_R\sqrt{1-\tau^2 B^{(\epsilon)}(\rho_R, v_R, \epsilon)} - \rho_L\sqrt{1-\tau^2 B^{(\epsilon)}(\rho_L, v_L, \epsilon)}} \\ &= \frac{\alpha_1 \varphi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu}) + (\alpha_1 - 1)v_L}{\alpha_1\sqrt{1-\tau^2 B^{(\epsilon)}(\rho_L\alpha_1, \varphi_1^{(\boldsymbol{\mu})} + v_L, \epsilon)} - \sqrt{1-\tau^2 B^{(\epsilon)}(\rho_L, v_L, \epsilon)}} \\ &= \frac{\alpha_1\varphi_1(\alpha_1)}{\alpha_1 - 1} + v_L + O(1)\|\boldsymbol{\mu}\|.\end{aligned}$$

Therefore, using (4.6) and Propositions 3.1–3.2, we have

$$\sigma_1^{(\boldsymbol{\mu})}(\alpha_1) - \sigma_1(\beta_1) = \frac{\alpha_1\varphi_1(\alpha_1)}{\alpha_1 - 1} + v_L + O(1)\|\boldsymbol{\mu}\| - v_L - \frac{\beta_1\varphi_1(\beta_1)}{\beta_1 - 1} = O(1)\|\boldsymbol{\mu}\|,$$
$$|\sigma_2(\beta_2) - \sigma_1^{(\boldsymbol{\mu})}(\alpha_1)| \le |\sigma_2(\beta_2)| + |\sigma_1^{(\boldsymbol{\mu})}(\alpha_1)| < \infty.$$

Then

$$|I| \le |\sigma_1^{(\boldsymbol{\mu})}(\alpha_1) - \sigma_1(\beta_1)|h + |\dot{y}_{\alpha_1} - \sigma_1^{(\boldsymbol{\mu})}(\alpha_1)|h \le \big(O(1)\|\boldsymbol{\mu}\| + 2^{-\nu}\big)h,$$
$$|II| \le |\sigma_2(\beta_2) - \sigma_1^{(\boldsymbol{\mu})}(\alpha_1)|h + |\dot{y}_{\alpha_1} - \sigma_1^{(\boldsymbol{\mu})}(\alpha_1)|h \le \big(O(1) + 2^{-\nu}\big)h.$$

On the other hand, on interval I,

$$|\mathcal{S}_h U^{(\boldsymbol{\mu}),\nu}(\hat{x},y) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)| = |\rho_R-\rho_L| + |v_R-v_L| \le O(1)|\alpha_1-1|,$$

and, on interval II,

$$\begin{aligned}|\mathcal{S}_h U^{(\boldsymbol{\mu}),\nu}(\hat{x},y) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)| &= |\rho_R-\rho_M| + |v_R-v_M| \\ &\le O(1)|\beta_2-1| \le O(1)\|\boldsymbol{\mu}\||\alpha_1-1|.\end{aligned}$$

Base on these estimates, we finally obtain

$$\begin{aligned}&\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(y_{\mathcal{I}}-\eta,y_{\mathcal{I}}+\eta)} \\ &= \|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(I\cup II)} \\ &\le O(1)|I||\alpha_1-1| + O(1)|II|\|\boldsymbol{\mu}\||\alpha_1-1| \\ &\le O(1)\big(\|\boldsymbol{\mu}\|+2^{-\nu}\big)|\alpha_1-1|h,\end{aligned}$$

which completes the proof of (i) for the case that β_2 is a shock wave.

Next, we consider the case that β_2 is a rarefaction wave as shown in Fig. 4.2. In this case,

$$\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) = \begin{cases} U_R, & \xi\in[\lambda_2(U_R),\frac{\eta}{h}), \\ \Phi_2(\beta_2(\xi);U_M), & \xi\in[\lambda_2(U_M),\lambda_2(U_R)), \\ U_M, & \xi\in[\sigma_1(\beta_1),\lambda_2(U_M)), \\ U_L, & \xi\in(-\frac{\eta}{h},\sigma_1(\beta_1)), \end{cases} \tag{4.7}$$

where $\xi=\frac{y-y_I}{h}$, $\beta_2(\lambda_2(U_M))=1$, and $\beta_2(\lambda_2(U_R))=\beta_2$.

Following the same argument as done for the case that β_2 is a shock wave, we have

$$\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu})(\hat{x}+h,\cdot)\|_{L^1(I)} \le O(1)(\|\boldsymbol{\mu}\|+2^{-\nu})|\alpha_1-1|h. \tag{4.8}$$

For the length of interval II, it follows from Proposition 3.1 that

$$|II| \le |\dot{y}_{\alpha_1}-\xi_0| \le \big(2^{-\nu}+|\sigma_1^{(\boldsymbol{\mu})}-\lambda_2(U_M)|\big)h \le \big(2^{-\nu}+O(1)\big)h.$$

Moreover, on II, by (4.6) and Proposition 3.2, we obtain

$$\begin{aligned}|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)| &= |\rho_R-\rho_M| + |v_R-v_M| \\ &\le O(1)|\beta_2-1| \le O(1)|\alpha_1-1|\|\boldsymbol{\mu}\|,\end{aligned}$$

so that

$$\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(II)} \le O(1)\big(\|\boldsymbol{\mu}\|+2^{-\nu}\big)|\alpha_1-1|h. \tag{4.9}$$

On interval III, it is direct to see

$$\begin{aligned}|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)| &= |\rho(\xi)-\rho_R| + |v(\xi)-v_R| \\ &\le O(1)|\xi_0-\xi_1| = O(1)|\lambda_2(U_M)-\lambda_2(U_R)| \\ &\le O(1)|\beta_2-1| \le O(1)|\alpha_1-1|\|\boldsymbol{\mu}\|,\end{aligned}$$

$$|III| \le O(1)|\xi_0-\xi_1| \le O(1)|\lambda_1(U_M)-\lambda_1(U_R)| \le O(1)h,$$

so that

$$\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(III)} \le O(1)\|\boldsymbol{\mu}\||\alpha_1-1|h. \tag{4.10}$$

Finally, combining estimates (4.8)–(4.10) together, we have

$$\begin{aligned}&\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)\|_{L^1(y_I-\eta,y_I+\eta)}\\&= \|\mathcal{S}_h U^{(\boldsymbol{\mu}),\nu}(\hat{x},y) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)\|_{L^1(I\cup II\cup III)}\\&\le O(1)\big(\|\boldsymbol{\mu}\| + 2^{-\nu}\big)|\alpha_1 - 1|h,\end{aligned}$$

which gives estimate (4.2) for the case that β_2 is a rarefaction wave.

2. Without loss of the generality, similarly, we consider the case: $k = 1$ only. The Riemann solver $\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y))$ consists of three constant states U_L, U_M, and U_R, which are separated by the elementary waves β_1 and β_2 with equation (4.5) and estimates (4.6). Moreover, it follows from (4.6) that β_1 is a 1-rarefaction wave. However, β_2 may be either a 2-shock wave or a 2-rarefaction wave (see Fig. 4.3 or Fig. 4.4 below).

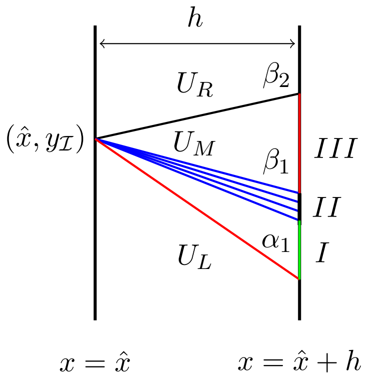

FIGURE 4.3. Comparison of the Riemann solvers for $\alpha_1 \in R_1^{(\boldsymbol{\mu})}$ and β_2 being 2-shock wave

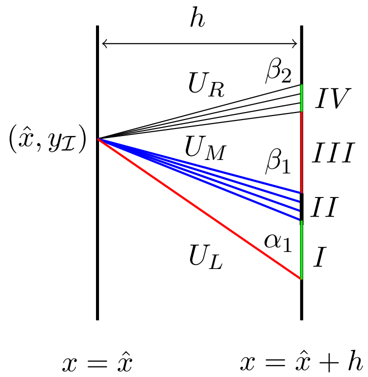

FIGURE 4.4. Comparison of the Riemann solvers for $\alpha_1 \in R_1^{(\boldsymbol{\mu})}$ and β_2 being a 2-rarefaction wave

If β_2 is a shock wave, then

$$\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) = \begin{cases} U_R, & \xi \in [\sigma_2(\beta_2), \frac{\eta}{h}),\\ U_M, & \xi \in [\lambda_1(U_M), \sigma_2(\beta_2)),\\ \Phi_1(\beta_1(\xi); U_L), & \xi \in [\lambda_1(U_L), \lambda_1(U_M)),\\ U_L, & \xi \in (-\frac{\eta}{h}, \lambda_1(U_L)),\end{cases} \tag{4.11}$$

where $\xi = \frac{y-y_{\mathcal{I}}}{h}$, $\beta_1(\lambda_1(U_L)) = 1$, $\beta_1(\lambda_1(U_M)) = \beta_1$, and $\sigma_2(\beta_2)$ is the speed of β_2. Then interval $(y_{\mathcal{I}} - \eta, y_{\mathcal{I}} + \eta)$ is divided into three subintervals I, II, and III.

Using (2.41) and Proposition 3.1, a direct computation leads to

$$\begin{aligned}|I| = |\lambda_1(U_L) - \dot{y}_{\alpha_1}|h &\le \big(|\lambda_1(U_L) - \lambda_1^{(\boldsymbol{\mu})}(U_R,\boldsymbol{\mu})| + 2^{-\nu}\big)h\\&\le \big(|\lambda_1(U_L) - \lambda_1(U_R)| + O(1)\|\boldsymbol{\mu}\| + 2^{-\nu}\big)h\\&\le O(1)\big(|\alpha_1 - 1| + \|\boldsymbol{\mu}\| + 2^{-\nu}\big)h\\&\le O(1)\big(\|\boldsymbol{\mu}\| + \nu^{-1} + 2^{-\nu}\big)h.\end{aligned}$$

Moreover, on I,

$$|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)| = |\rho_R-\rho_L| + |v_R - v_L| \le O(1)|\alpha_1 - 1|.$$

Therefore, we have

$$\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(I)} \le O(1)\big(\|\boldsymbol{\mu}\| + 2^{-\nu} + \nu^{-1}\big)|\alpha_1 - 1|h. \tag{4.12}$$

Next, it follows from (4.6) that

$$|II| = |\lambda_1(U_M) - \lambda_1(U_L)|h \le O(1)|\beta_1 - 1|h = O(1)\big(1+\|\boldsymbol{\mu}\|\big)|\alpha_1 - 1|h,$$

and, on interval II,

$$\begin{aligned}
&|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)| \\
&= |\Phi_1^{(1)}(\beta_1(\xi);U_L) - \rho_R| + |\Phi_1^{(2)}(\beta_1(\xi);U_L) - v_R| \\
&\le |\Phi_1^{(1)}(\beta_1(\xi);U_L) - \rho_M| + |\rho_M - \rho_R| + |\Phi_1^{(2)}(\beta_1(\xi);U_L) - v_M| + |v_M - v_R| \\
&\le O(1)\big(|\xi - \xi_1| + |\beta_2 - 1|\big) \\
&\le O(1)\big(|\beta_1 - 1| + |\beta_2 - 1|\big) \\
&\le O(1)\big(1+\|\boldsymbol{\mu}\|\big)|\alpha_1 - 1|.
\end{aligned}$$

Thus, by Proposition 3.1, we obtain

$$\begin{aligned}
\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(II)} &\le O(1)\big(1+\|\boldsymbol{\mu}\|\big)^2|\alpha_1 - 1|^2 h \\
&\le O(1)\nu^{-1}|\alpha_1 - 1|h.
\end{aligned} \tag{4.13}$$

Finally, it follows from Proposition 3.1 and estimates (4.6) that

$$|III| = |\lambda_1(U_M) - \sigma_2(\beta_2)|h = \Big|v_M - \frac{1}{a_\infty} - \frac{\beta_2 v_R - v_M}{\beta_2 - 1}\Big|h \le O(1)h,$$

and, on interval III,

$$\begin{aligned}
|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)| &= |\rho_M - \rho_R| + |v_M - v_R| \\
&= O(1)|\beta_2 - 1| = O(1)|\alpha_1 - 1|\|\boldsymbol{\mu}\|.
\end{aligned}$$

Then

$$\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(III)} \le O(1)|\alpha_1 - 1|\|\boldsymbol{\mu}\|h. \tag{4.14}$$

Combining estimates (4.12)–(4.14), we finally obtain

$$\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(y_{\mathcal{I}}-\eta, y_{\mathcal{I}}+\eta)} \le O(1)\big(\|\boldsymbol{\mu}\| + \nu^{-1} + 2^{-\nu}\big)|\alpha_1 - 1|h.$$

When β_2 is a rarefaction wave, as shown in Fig. 4.4, interval $(y_{\mathcal{I}} - \eta, y_{\mathcal{I}} + \eta)$ is divided into four subintervals I, II, III, and IV. Then

$$\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) = \begin{cases}
U_R, & \xi \in [\lambda_2(U_R), \frac{\eta}{h}), \\
\Phi_2(\beta_2(\xi);U_M), & \xi \in [\lambda_2(U_M), \lambda_2(U_R)), \\
U_M, & \xi \in [\lambda_1(U_M), \lambda_2(U_M)), \\
\Phi_1(\beta_1(\xi);U_L), & \xi \in [\lambda_1(U_L), \lambda_1(U_M)), \\
U_L, & \xi \in (-\frac{\eta}{h}, \lambda_1(U_L)),
\end{cases} \tag{4.15}$$

where $\xi = \frac{y-y_{\mathcal{I}}}{h}$, $\beta_1(\lambda_1(U_L)) = 1$, $\beta_1(\lambda_1(U_M)) = \beta_1$, $\beta_2(\lambda_2(U_M)) = 1$, and $\beta_2(\lambda_3(U_R)) = \beta_2$. First, by the same argument for the case that β_2 is a shock above,

$$\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(I\cup II\cup III)} \le O(1)\big(\|\boldsymbol{\mu}\| + \nu^{-1} + 2^{-\nu}\big)|\alpha_1 - 1|h. \tag{4.16}$$

Thus, it suffices to consider the estimate of $\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(IV)}$. By Proposition 3.1, we directly have

$$|IV| = |\lambda_2(U_R) - \lambda_2(U_M)|h \le O(1)h,$$

and, on IV,

$$\begin{aligned}|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)| &= |\Phi_2^{(1)}(\beta_2(\xi);U_M) - \rho_R| + |\Phi_2^{(2)}(\beta_2(\xi);U_M) - v_R| \\ &\le O(1)|\xi_2 - \xi_3| \le O(1)|\beta_2 - 1| \le O(1)|\alpha_1 - 1|\|\boldsymbol{\mu}\|,\end{aligned}$$

so that

$$\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)\|_{L^1(IV)} \le O(1)|\alpha_1 - 1|\|\boldsymbol{\mu}\|h. \tag{4.17}$$

Then combining estimate (4.16) with estimate (4.17) yields

$$\begin{aligned}&\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(y_{\mathcal{I}}-\eta,y_{\mathcal{I}}+\eta)} \\ &\le \|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(I\cup II\cup III\cup IV)} \\ &\le O(1)\big(\|\boldsymbol{\mu}\| + \nu^{-1} + 2^{-\nu}\big)|\alpha_1 - 1|h.\end{aligned}$$

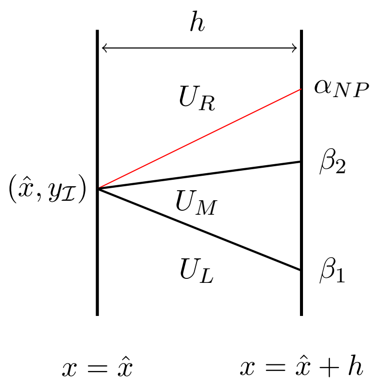

FIGURE 4.5. Comparison of the Riemann solvers for $\alpha_{NP} \in NP^{(\boldsymbol{\mu})}$

3. When the front in $U^{(\nu),\nu}(\hat{x}+h,\cdot)$ is a non-physical wave α_{NP}, as shown in Fig. 4.5, $|U_R - U_L| = \alpha_{NP}$, and the Riemann solution $\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot))$ consists of two waves β_1 and β_2 satisfying

$$U_R = \Phi(\beta_1, \beta_2; U_L). \tag{4.18}$$

Applying Proposition 2.1 leads to

$$|\beta_1 - 1| + |\beta_2 - 1| = O(1)\alpha_{NP}. \tag{4.19}$$

Let U_M be the middle state of $\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot))$. Then it follows from Propositions 3.1–3.2 and estimate (4.19) that

$$\begin{aligned}\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot))-U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(y_{\mathcal{I}}-\eta,y_{\mathcal{I}}+\eta)}&\le O(1)\big(|U_M-U_L|+|U_M-U_R|\big)h\\&\le O(1)\big(|\beta_1-1|+|\beta_2-1|\big)h\\&\le O(1)\alpha_{NP}\,h.\end{aligned}$$

□

Based on Lemma 4.1, it is direct to derive the following corollary for the case that $U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)$ contains more than one discontinuities:

Corollary 4.1. *Let $U^{(\boldsymbol{\mu}),\nu}$ be an approximate solution of problem* (1.16)–(1.18) *constructed in* §3.1 *with a jump at point $(\hat{x},y)\in\Omega$. Let $\mathcal{S}$ be the uniformly Lipschitz continuous semigroup obtained in* Proposition 3.3. *Denote*

$$U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)=\begin{cases}U_R=(\rho_R,v_R)^\top & \text{for } y>y_{\mathcal{I}}+\hat{\lambda}h,\\ \hat{U}_R=(\hat{\rho}_R,\hat{v}_R)^\top & \text{for } y_{\mathcal{I}}+\dot{y}_{\alpha_2}h<y<y_{\mathcal{I}}+\hat{\lambda}h,\\ U_M=(\rho_M,v_M)^\top & \text{for } y_{\mathcal{I}}+\dot{y}_{\alpha_1}h<y<y_{\mathcal{I}}+\dot{y}_{\alpha_2}h,\\ U_L=(\rho_L,v_L)^\top & \text{for } y<y_{\mathcal{I}}+\dot{y}_{\alpha_1}h,\end{cases}\tag{4.20}$$

where $U_L=U^{(\boldsymbol{\mu}),\nu}(\hat{x},y_{\mathcal{I}}-)$, $U_R=U^{(\boldsymbol{\mu}),\nu}(\hat{x},y_{\mathcal{I}}+)$, $|\dot{y}_{\alpha_k}|<\hat{\lambda}$ with $k=1,2$, and $U_L,\hat{U}_R$, and U_R satisfy (3.3). *For $\|\boldsymbol{\mu}\|\le\|\bar{\boldsymbol{\mu}}_0^*\|$ with $\bar{\boldsymbol{\mu}}_0^*$ given in* Proposition 3.1 *and, for sufficiently small $h>0$, if U_L and U_M are connected by a 1-shock wave $\alpha_1\in S_1^{(\boldsymbol{\mu})}$ with $|\dot{y}_{\alpha_1}-\sigma_1^{(\boldsymbol{\mu})}(\alpha_1)|<2^{-\nu}$ (or a 1-rarefaction front $\alpha_1\in R_1^{(\boldsymbol{\mu})}$ with $|\dot{y}_{\alpha_1}-\lambda_1(U_M,\boldsymbol{\mu})|<2^{-\nu}$), if U_M and $\hat{U}_R$ are connected by a 2-shock wave $\alpha_2\in S_2^{(\boldsymbol{\mu})}$ with $|\dot{y}_{\alpha_2}-\sigma_2^{(\boldsymbol{\mu})}(\alpha_2)|<2^{-\nu}$ (or a 2-rarefaction front $\alpha_2\in R_2^{(\boldsymbol{\mu})}$ with $|\dot{y}_{\alpha_2}-\lambda_2(\hat{U}_R,\boldsymbol{\mu})|<2^{-\nu}$), and if $\hat{U}_R$ and U_R are connected by a non-physical wave front $\alpha_{\mathcal{NP}}\in NP^{(\boldsymbol{\mu})}$, then*

$$\begin{aligned}&\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot))-U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(y_{\mathcal{I}}-\eta,y_{\mathcal{I}}+\eta)}\\&\le C_{\mathcal{II}}\big(\|\boldsymbol{\mu}\|+2^{-\nu}+\nu^{-1}\big)\Big(\sum_{k=1,2}|\alpha_k-1|+\alpha_{NP}\Big)h,\end{aligned}\tag{4.21}$$

where constant $\eta>0$ satisfies $\eta>\hat{\lambda}h$, and constant $C_{\mathcal{II}}>0$ is independent of $\boldsymbol{\mu}$ and h.

Next, we consider the comparison of the approximate solution $U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)$ of problem (1.16)–(1.18) and the entropy solution $\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot))$ of problem (1.20)–(1.22) near boundary $\Gamma_{\rm w}$ with $(\hat{x},b_0\hat{x})$ as a discontinuity point on it. Denote

$$U_L=(\rho_L,v_L)^\top:=U^{(\boldsymbol{\mu}),\nu}(\hat{x},b_0\hat{x}-),\quad U_b=(\rho_b,v_b)^\top:=U^{(\boldsymbol{\mu}),\nu}(\hat{x},b_0\hat{x}),\tag{4.22}$$

$$U_b^{(\boldsymbol{\mu})}:=(\rho_b^{(\boldsymbol{\mu})},v_b^{(\boldsymbol{\mu})})^\top=U^{(\boldsymbol{\mu}),\nu}(x,b_0x)\qquad\text{for } x\in(\hat{x},\hat{x}+h),\ h>0,\tag{4.23}$$

$$U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},y):=\begin{cases}U_b & \text{for } y=b_0\hat{x},\\ U_L & \text{for } y<b_0\hat{x}.\end{cases}\tag{4.24}$$

Then we have the following lemma:

Lemma 4.2. *Let $U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},y)$ be defined by (4.24) with $v_b=b_0$. Let $\mathcal{S}$ be a uniform Lipschitz continuous semigroup obtained in* Proposition 3.2. *Define*

$$U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y):=\begin{cases}U_b^{(\boldsymbol{\mu})} & \text{for } b_0\hat{x}+\dot{y}_{\alpha_1}h<y<b_0h,\\ U_L & \text{for } y<b_0\hat{x}+\dot{y}_{\alpha_1}h,\end{cases} \tag{4.25}$$

where $\dot{y}_{\alpha_1}\in(-\hat{\lambda},b_0)$, and $U_b^{(\boldsymbol{\mu})}$ satisfies

$$v_b^{(\boldsymbol{\mu})}=b_0\sqrt{1-\tau^2B^{(\epsilon)}(\rho_b^{(\boldsymbol{\mu})},v_b^{(\boldsymbol{\mu})},\epsilon)}. \tag{4.26}$$

For $\|\boldsymbol{\mu}\|\le\|\bar{\boldsymbol{\mu}}_0^*\|$ with $\bar{\boldsymbol{\mu}}_0^*$ given in Proposition 3.1 *and for sufficiently small $h>0$,*

(i) *if $U_b^{(\boldsymbol{\mu})}$ and U_L are connected by a 1-shock wave $\alpha_1\in S^{(\boldsymbol{\mu})}$ with $|\dot{y}_{\alpha_1}-\sigma_1^{(\boldsymbol{\mu})}(\alpha_1)|<2^{-\nu}$ for $\sigma_1^{(\boldsymbol{\mu})}(\alpha_1)$ as the speed of α_1, then*

$$\begin{aligned}&\|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot))-U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(b_0\hat{x}-\eta,b_0(\hat{x}+h))}\\&\le C_b(\|\boldsymbol{\mu}\|+2^{-\nu})(|\alpha_1-1|+1)h;\end{aligned} \tag{4.27}$$

(ii) *if $U_b^{(\boldsymbol{\mu})}$ and U_L are connected by a 1-rarefaction wave $\alpha_1\in R^{(\boldsymbol{\mu})}$ with $|\dot{y}_{\alpha_1}-\lambda_1^{(\boldsymbol{\mu})}(U_b^{(\boldsymbol{\mu})},\boldsymbol{\mu})|<2^{-\nu}$ for $\lambda_1^{(\boldsymbol{\mu})}(U_b^{(\boldsymbol{\mu})},\boldsymbol{\mu})$ as the speed of α_1, then*

$$\begin{aligned}&\|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot))-U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(b_0\hat{x}-\eta,b_0(\hat{x}+h))}\\&\le C_b(\|\boldsymbol{\mu}\|+\nu^{-1}+2^{-\nu})(|\alpha_1-1|+1)h,\end{aligned} \tag{4.28}$$

where η satisfies $\eta<-\hat{\lambda}h$, and $C_b>0$ is independent of $\boldsymbol{\mu}$, ν and h.

Proof. We divide the proof into two steps accordingly.

1. We know that $U_b^{(\boldsymbol{\mu})}$, U_L, and α_1 satisfy

$$U_b^{(\boldsymbol{\mu})}=\Phi_1^{(\boldsymbol{\mu})}(\alpha_1;U_L,\boldsymbol{\mu}).$$

$\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},y))$ near the boundary Γ satisfies

$$U_b=\Phi_1(\beta_1;U_L).$$

Then

$$v_L+\Phi_1^{(\boldsymbol{\mu}),(2)}(\alpha_1;U_L,\boldsymbol{\mu})=\sqrt{1-\tau^2B^{(\epsilon)}(\Phi_1^{(\boldsymbol{\mu}),(1)},\Phi_1^{(\boldsymbol{\mu}),(2)},\boldsymbol{\mu})}\big(\Phi_1^{(2)}(\beta_1;U_L)+v_L\big), \tag{4.29}$$

where $\Phi_1^{(\boldsymbol{\mu}),(k)}$ is the k-th component of $\Phi_1^{(\boldsymbol{\mu})}$ for $k=1,2$, and $\Phi_1^{(2)}$ is the 2-nd component of Φ_1. Thus, by Proposition 2.2, when $\|\boldsymbol{\mu}\|\le\|\boldsymbol{\mu}_0^*\|$, equation (4.29) admits a unique solution $\beta_1=\beta_1(\alpha_1,\boldsymbol{\mu})\in C^2$ such that

$$\beta_1=\alpha_1+O(1)\big(1+|\alpha_1-1|\big)\|\boldsymbol{\mu}\|. \tag{4.30}$$

Hence, β_1 is also a 1-shock wave.

To estimate $\|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot))-U_b^{(\boldsymbol{\mu}),\nu}(\hat{h}+h,\cdot)\|_{L^1(b_0\hat{x}-\eta,b_0(\hat{x}+h))}$, it suffices to consider it on intervals I_{b1} and I_{b2} as shown in Fig. 4.6. Let $\sigma_1(\beta_1)$ be the speed of β_1. Then

$$\sigma_1(\beta_1)=\frac{\rho_bv_b-\rho_Lv_L}{\rho_b-\rho_L}=\frac{\beta_1\varphi_1(\beta_1)}{\beta_1-1}+v_L.$$

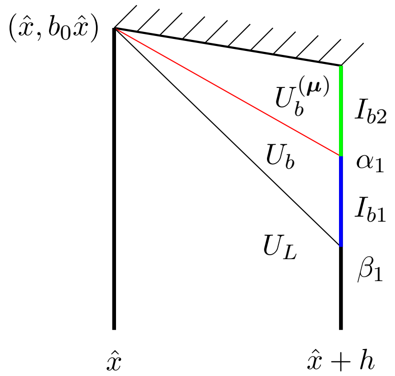

FIGURE 4.6. Comparison of the Riemann solvers for $\alpha_1 \in S_1^{(\boldsymbol{\mu})}$ and β_1 is a 1-shock wave

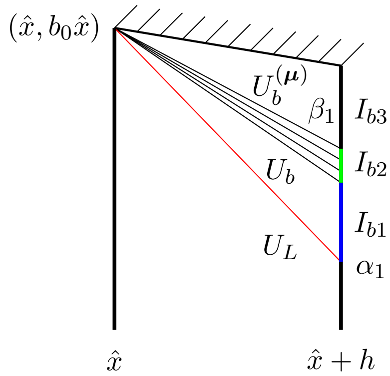

FIGURE 4.7. Comparison of the Riemann solvers for $\alpha_1 \in R_1^{(\boldsymbol{\mu})}$ and β_1 is a 1-rarefaction wave

Note that

$$\begin{aligned}
\sigma_1^{(\boldsymbol{\mu})}(\alpha_1) &= \frac{\rho_b^{(\boldsymbol{\mu})} v_b^{(\boldsymbol{\mu})} - \rho_L v_L}{\rho_b^{(\boldsymbol{\mu})}\sqrt{1-\tau^2 B^{(\epsilon)}(\rho_b^{(\boldsymbol{\mu})}, v_b^{(\boldsymbol{\mu})}, \epsilon)} - \rho_L\sqrt{1-\tau^2 B^{(\epsilon)}(\rho_L, v_L, \epsilon)}} \\
&= \frac{\rho_b^{(\boldsymbol{\mu})} v_b^{(\boldsymbol{\mu})} - \rho_L v_L}{\rho_b^{(\boldsymbol{\mu})} - \rho_L} + O(1)\|\boldsymbol{\mu}\| \\
&= \frac{\alpha_1 \varphi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu})}{\alpha_1 - 1} + v_L + O(1)\|\boldsymbol{\mu}\|.
\end{aligned}$$

Then, by Lemma 2.5 and Proposition 3.1, we have

$$\begin{aligned}
|I_{b1}| &\le |\sigma_1^{(\boldsymbol{\mu})}(\alpha_1) - \sigma_1(\beta_1)|h + 2^{-\nu}h \\
&\le \Big(\Big|\frac{\alpha_1\varphi_1^{(\boldsymbol{\mu})}(\alpha_1; U_L, \boldsymbol{\mu})}{\alpha_1 - 1} - \frac{\beta_1\varphi_1(\beta_1)}{\beta_1 - 1}\Big| + O(1)\|\boldsymbol{\mu}\| + 2^{-\nu}\Big)h \\
&\le O(1)\big(\|\boldsymbol{\mu}\| + 2^{-\nu}\big)h.
\end{aligned}$$

On interval I_{b1}, it follows from (4.30) and Proposition 3.1 that

$$\begin{aligned}
|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}, y)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h, y)| &= |\rho_b - \rho_L| + |v_b - v_L| \\
&= O(1)|\beta_1 - 1| = O(1)\big(|\alpha_1 - 1| + 1\big),
\end{aligned}$$

so that

$$\|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}, \cdot)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h, \cdot)\|_{L^1(I_{b1})} \le O(1)\big(|\alpha_1 - 1| + 1\big)\big(\|\boldsymbol{\mu}\| + 2^{-\nu}\big)h. \tag{4.31}$$

Similarly, on I_{b2}, we have

$$
\begin{aligned}
|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)| &= |\rho_b^{(\boldsymbol{\mu})} - \rho_b| + |v_b^{(\boldsymbol{\mu})} - v_b| \\
&= O(1)|\beta_1 - \alpha_1| + O(1)\|\boldsymbol{\mu}\| \\
&= O(1)\big(|\alpha_1 - 1| + 1\big)\|\boldsymbol{\mu}\|,
\end{aligned}
$$

$$
|I_{b2}| \le |\sigma_1^{(\boldsymbol{\mu})}(\alpha_1) - b_0|h + 2^{-\nu}h \le (O(1) + 2^{-\nu})h,
$$

so that

$$
\|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(I_{b2})} \le O(1)\big(|\alpha_1 - 1| + 1\big)\|\boldsymbol{\mu}\|h. \tag{4.32}
$$

Finally, with estimates (4.31)–(4.32), we arrive at

$$
\begin{aligned}
&\|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(b_0\hat{x}-\eta,b_0(\hat{x}+h))} \\
&\le \sum_{j=1,2} \|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(I_{bj})} \\
&\le O(1)\big(|\alpha_1 - 1| + 1\big)\big(\|\boldsymbol{\mu}\| + 2^{-\nu}\big)h,
\end{aligned}
$$

which gives estimate (4.27).

2. In this case, we know that relation (4.29) and estimate (4.30) hold. Then β_1 is a rarefaction wave, and

$$
\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) = \begin{cases} U_b & \text{for } \xi \in (\lambda_1(U_b), b_0), \\ \Phi_1(\beta_1(\xi); U_L) & \text{for } \xi \in (\lambda_1(U_L), \lambda_1(U_b)], \\ U_L, & \text{for } \xi \in (-\frac{\eta}{h}, \lambda_1(U_L)], \end{cases} \tag{4.33}
$$

where $\xi = \frac{y - b_0\hat{x}}{h}$, $\beta_1(\lambda_1(U_L)) = 1$, and $\beta_1(\lambda_1(U_b)) = \beta_1$.

To show estimate (4.28), we only consider the case as shown in Fig. 4.7, since the other case can be treated in the same way. Interval $(b_0\hat{x} - \eta, b_0(\hat{x}+h))$ is divided into three subintervals, *i.e.*, I_{bj} with $1 \le j \le 3$. Using (2.41), Proposition 3.1, and Lemma 2.5, we have

$$
\begin{aligned}
|I_{b1}| &\le \big(|\lambda_1^{(\boldsymbol{\mu})}(U_b^{(\boldsymbol{\mu})}, \boldsymbol{\mu}) - \lambda_1(U_L)| + 2^{-\nu}\big)h \\
&\le O(1)\big(\|\boldsymbol{\mu}\| + |\alpha_1 - 1| + 2^{-\nu}\big)h \\
&\le O(1)\big(\|\boldsymbol{\mu}\| + \nu^{-1} + 2^{-\nu}\big)h,
\end{aligned}
$$

and, on interval I_{b1},

$$
|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)| = |\rho_b^{(\boldsymbol{\mu})} - \rho_L| + |v_b^{(\boldsymbol{\mu})} - v_L| \le O(1)|\alpha_1 - 1|,
$$

so that

$$
\|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(I_{b1})} \le O(1)|\alpha_1 - 1|\big(\|\boldsymbol{\mu}\| + \nu^{-1} + 2^{-\nu}\big)h. \tag{4.34}
$$

Next, it follows from (4.30) and (4.33) that

$$
|I_{b2}| = |\lambda_1(U_b) - \lambda_1(U_L)|h = O(1)|U_b - U_L|h = O(1)|\alpha_1 - 1|h,
$$

and, on I_{b2},

$$\begin{aligned}|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)| &= |\Phi_1(\beta_1(\xi);U_L) - U_b^{(\boldsymbol{\mu})}| \\ &\le |\Phi_1(\beta_1(\xi);U_L) - U_L| + |U_L - U_b^{(\boldsymbol{\mu})}| \\ &\le O(1)|\xi_0-\xi_1| + O(1)|\alpha_1-1| \\ &\le O(1)|\alpha_1-1|,\end{aligned}$$

so that

$$\|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(I_{b2})} \le O(1)|\alpha_1-1|^2 h \le O(1)|\alpha_1-1|\nu^{-1}h. \tag{4.35}$$

On interval I_{b3}, by Lemma 2.6 and estimate (4.30), we obtain

$$\begin{aligned}|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)| &= |\rho_b^{(\boldsymbol{\mu})} - \rho_b| + |v_b^{(\boldsymbol{\mu})} - v_b| \\ &\le O(1)\big(|\beta_1-\alpha_1| + \|\boldsymbol{\mu}\|\big) \\ &\le O(1)\big(|\alpha_1-1|+1\big)\|\boldsymbol{\mu}\|,\end{aligned}$$

and $|I_{b3}| = |\lambda_1(U_b) - b_0|h \le O(1)h$, so that

$$\begin{aligned}\|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(I_{b3})} &= |\rho_b^{(\boldsymbol{\mu})} - \rho_b| + |v_b^{(\boldsymbol{\mu})} - v_b| \\ &\le O(1)\big(|\beta_1-\alpha_1| + \|\boldsymbol{\mu}\|\big) \\ &\le O(1)\big(|\alpha_1-1|+1\big)\|\boldsymbol{\mu}\|h.\end{aligned} \tag{4.36}$$

Finally, combining estimates (4.34)–(4.36), we obtain

$$\begin{aligned}&\|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(b_0\hat{x}-\eta,b_0(\hat{x}+h))} \\ &\le \sum_{j=1}^{3} \|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(I_{bj})} \\ &\le O(1)\big(|\alpha_1-1|+1\big)\big(\|\boldsymbol{\mu}\| + \nu^{-1} + 2^{-\nu}\big)h.\end{aligned}$$

This completes the proof. □

Next, we consider the comparison between the Riemann solutions with a boundary but away from the reflection points. Using Proposition 2.3 and following the procedure of the proof in Lemma 4.1, we have

Lemma 4.3. *Let $U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},y)$ be a piecewise constant function defined by* (4.24) *with $y \neq b_0\hat{x}$. Let $\mathcal{S}$ be a uniform Lipschitz continuous semigroup obtained by* Proposition 3.2. *Define*

$$U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y) := \begin{cases} U_b & \text{for } b_0\hat{x} + \dot{y}_{\alpha_1}h < y < b_0(\hat{x}+h), \\ U_L & \text{for } y < b_0\hat{x} + \dot{y}_{\alpha_1}h, \end{cases} \tag{4.37}$$

where $\dot{y}_{\alpha_1} \in (-\hat{\lambda}, b_0)$, and U_b is defined by (4.22) *with*

$$v_b = b_0\sqrt{1-\tau^2 B^{(\epsilon)}(\rho_b, v_b, \epsilon)}. \tag{4.38}$$

For $\|\boldsymbol{\mu}\| \le \|\bar{\boldsymbol{\mu}}_0^*\|$ with $\bar{\boldsymbol{\mu}}_0^*$ given in Proposition 3.1 *and for sufficiently small $h > 0$,*

(i) *if U_b and U_L are connected by a 1-shock wave $\alpha_1 \in S^{(\boldsymbol{\mu})}$ with $|\dot{y}_{\alpha_1} - \sigma_1^{(\boldsymbol{\mu})}(\alpha_1)| < 2^{-\nu}$ for $\sigma_1^{(\boldsymbol{\mu})}(\alpha_1)$ as the exact speed of α_1, then*

$$\begin{aligned}&\|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(b_0\hat{x}-\eta, b_0(\hat{x}+h))}\\&\le C_{bb}(\|\boldsymbol{\mu}\| + 2^{-\nu})(|\alpha_1 - 1| + 1)h;\end{aligned}\tag{4.39}$$

(ii) *if U_b and U_L are connected by a 1-rarefaction wave front $\alpha_1 \in R^{(\boldsymbol{\mu})}$ with $|\dot{y}_{\alpha_1} - \lambda_1^{(\boldsymbol{\mu})}(U_b^{(\boldsymbol{\mu})}, \boldsymbol{\mu})| < 2^{-\nu}$ for $\lambda_1^{(\boldsymbol{\mu})}(U_b^{(\boldsymbol{\mu})}, \boldsymbol{\mu})$ as the exact speed of α_1, then*

$$\begin{aligned}&\|\mathcal{S}_h(U_b^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U_b^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(b_0\hat{x}-\eta, b_0(\hat{x}+h))}\\&\le C_{bb}(\|\boldsymbol{\mu}\| + \nu^{-1} + 2^{-\nu})(|\alpha_1 - 1| + 1)h,\end{aligned}\tag{4.40}$$

where $\eta < -\hat{\lambda}h$ and constant $C_{bb} > 0$ is independent on $\boldsymbol{\mu}$, ν and h.

4.2. **Proof of Theorem 1.1 for the convergence rate estimate** (1.23)**.** Now, we are ready to prove estimate (1.23) that is completed by the following two steps:

1. *Estimate for* $\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(-\infty, b_0(\hat{x}+h))}$. Let $b_0\hat{x} = y_0 > y_1 > \cdots > y_N$ (or $b_0\hat{x} > y_0 > y_1 > \cdots > y_N$) be the jumps of $U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)$ on line $x = \hat{x}$. Suppose that there is no wave interaction on the stripe between $x = \hat{x}$ and $x = \hat{x}+h$, and there is no reflection on boundary (x, b_0x) for $x \in (\hat{x}, \hat{x}+h)$. Let $S^{(\boldsymbol{\mu})}$ (or $R^{(\boldsymbol{\mu})}$) be the set of indices α_i with $i \in \{1, 2, \cdots, N\}$ such that $U^{(\boldsymbol{\mu}),\nu}(\hat{x}, y_\alpha+)$ and $U^{(\boldsymbol{\mu}),\nu}(\hat{x}, y_\alpha-)$ are connected by a shock wave front (or a rarefaction wave front) with strength α_i. Let $NP^{(\boldsymbol{\mu})}$ be the set of indices α_{NP} such that $U^{(\boldsymbol{\mu}),\nu}(\hat{x}, y_\alpha+)$ and $U^{(\boldsymbol{\mu}),\nu}(\hat{x}, y_\alpha-)$ are connected by a non-physical front with strength α_{NP}.

By Lemmas 4.1–4.3 and Proposition 3.1, for sufficiently small $h > 0$, if $\|\boldsymbol{\mu}\| \le \|\boldsymbol{\mu}_0^*\|$, then

$$\begin{aligned}&\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(-\infty, b_0(\hat{x}+h))}\\&\le \sum_{\alpha\in S^{(\boldsymbol{\mu})}\cup R^{(\boldsymbol{\mu})}\cup NP^{(\boldsymbol{\mu})}} \|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},y)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,y)\|_{L^1(y_\alpha-\eta,\, y_\alpha+\eta)}\\&\le C\big(\|\boldsymbol{\mu}\| + 2^{-\nu} + \nu^{-1}\big)\Big(\sum_{\alpha\in S^{(\boldsymbol{\mu})}\cup R^{(\boldsymbol{\mu})}} |\alpha - 1| + 1\Big)h + O(1)\Big(\sum_{\alpha_{NP}\in NP^{(\boldsymbol{\mu})}} \alpha_{NP}\Big)h\\&\le C\big(\|\boldsymbol{\mu}\| + 2^{-\nu} + \nu^{-1}\big)\big(T.V.\{U^{(\boldsymbol{\nu}),\nu}(\hat{x},\cdot)\} + 1\big)h + O(1)2^{-\nu}h\\&\le C\big(\|\boldsymbol{\mu}\| + 2^{-\nu} + \nu^{-1}\big)h,\end{aligned}\tag{4.41}$$

where C is independent of (μ, h), and $\eta = \frac{1}{2}\min_{1\le j\le N}\{y_{j-1} - y_j\}$.

2. *Estimate on* $\|U^{(\boldsymbol{\mu})}(x,\cdot) - U(x,\cdot)\|_{L^1(-\infty, b_0x)}$. Let $U^{(\boldsymbol{\mu}),\nu}(x,y)$ be an approximate solution of (1.16)–(1.18) with initial data U_0^ν satisfying (3.1). Let $\mathcal{S}$ be the uniformly Lipschitz semigroup given by Proposition 3.2. Then, by the triangle inequality, we have

$$\begin{aligned}\|U^{(\boldsymbol{\mu})}(x,\cdot) - U(x,\cdot)\|_{L^1(-\infty,b_0x)} &\le \|U^{(\boldsymbol{\mu})}(x,\cdot) - U^{(\boldsymbol{\mu}),\nu}(x,\cdot)\|_{L^1(-\infty, b_0x)}\\&\quad + \|U^{(\boldsymbol{\mu}),\nu}(x,\cdot) - \mathcal{S}_x(U_0^\nu(\cdot))\|_{L^1(-\infty, b_0x)}\\&\quad + \|\mathcal{S}_x(U_0^\nu(\cdot)) - U(x,\cdot)\|_{L^1(-\infty, b_0x)}\\&=: J_1 + J_2 + J_3.\end{aligned}\tag{4.42}$$

For J_1, by Proposition 3.1, we can choose a subsequence (still denoted as) $\{U^{(\boldsymbol{\mu}),\nu}\}_\nu$ such that $U^{(\boldsymbol{\mu}),\nu} \to U^{(\boldsymbol{\mu})}$ in $L^1_{\rm loc}(\Omega)$ as $\nu \to \infty$. Then $J_1 \to 0$ as $\nu \to \infty$.

Next, for J_3, by the Lipschitz property of $\mathcal{S}$ and (3.1), we have

$$\begin{aligned} J_3 &\le \|\mathcal{S}_x(U_0^\nu(\cdot)) - \mathcal{S}_x(U_0(\cdot))\|_{L^1(-\infty,b_0x)} \\ &\le L\|U_0^\nu(\cdot) - U_0(\cdot)\|_{L^1(-\infty,b_0x)} \to 0 \qquad \text{as } \nu\to\infty. \end{aligned} \tag{4.43}$$

For J_2, thanks to Proposition 3.3 and *Step 1*, we obtain that, when $\|\boldsymbol{\mu}\| \le \|\boldsymbol{\mu}_0^*\|$,

$$\begin{aligned} J_2 &\le L\int_0^x \liminf_{h\to0+}\frac{\|\mathcal{S}_h(U^{(\boldsymbol{\mu}),\nu}(\hat{x},\cdot)) - U^{(\boldsymbol{\mu}),\nu}(\hat{x}+h,\cdot)\|_{L^1(-\infty,\,b_0(\hat{x}+h))}}{h}\,\mathrm{d}\hat{x} \\ &\le O(1)x\big(2^{-\nu}+\nu^{-1}+\|\boldsymbol{\mu}\|\big). \end{aligned} \tag{4.44}$$

Then it follows from (4.41)–(4.44) that we can choose a constant vector $\boldsymbol{\mu}_0 = (\epsilon_0,\tau_0^2)$ with $\epsilon_0>0,\tau_0>0$, and a constant $C_1>0$, independent of $(\boldsymbol{\mu},\nu,x)$ such that, as $\nu\to\infty$,

$$\|U^{(\boldsymbol{\mu})}(x,\cdot) - U(x,\cdot)\|_{L^1(-\infty,b_0x)} \le C_1x\|\boldsymbol{\mu}\|,$$

which gives estimate (1.23).

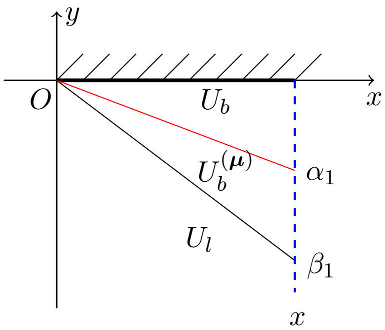

FIGURE 4.8. An example for the optimal convergence rate

4.3. **Proof of Theorem 1.1 for the optimal convergence rate** (1.23)**.** In this subsection, we further show that our convergence rate with respect to $\boldsymbol{\mu}$ in estimate (1.23) is optimal. To achieve this, it suffices to calculate an accurate convergence rate of a special Riemann solution in the following: As shown in Fig. 4.8, $b_0\equiv0$. We consider the Riemann problem for system (1.20) with the following Riemann data:

$$U|_{x=0} = \begin{cases} v_b = 0 & \text{for } y=0,\\ U_l = (\rho_l,v_l)^\top & \text{for } y<0, \end{cases} \tag{4.45}$$

where $\rho_l=1$ and $v_l=\delta>0$. Set $\underline{U} := U_l\big|_{\delta=0} = (1,0)^\top$. Then, by (2.42), the following relation holds:

$$0 = \varphi_1(\alpha_1)+\delta. \tag{4.46}$$

For $\delta=0$, by (4.46) and Lemma 2.6, we see that $\alpha_1=1$. Since $\varphi_1'(1) = -a_\infty^{-1}<0$ from Lemma 2.6, by the implicit function theorem, equation (4.46) admits a unique solution $\alpha_1=\alpha_1(\delta)$ for $\delta>0$ sufficiently small. Moreover, by direct computation, we obtain from (4.46) that

$$\alpha_1'(0) = -\frac{1}{\varphi_1'(1)} = a_\infty. \tag{4.47}$$

Therefore, by the Taylor formula, we have

$$\alpha_1(\delta) = \alpha_1(0) + \alpha_1'(0)\delta + O(1)\delta^2 = 1 + a_\infty\delta + O(1)\delta^2, \tag{4.48}$$

where the bound of $O(1)$ depends only on $\underline{U}$. It implies that α_1 is a shock wave.

Denote by ρ_b the density on boundary $y = 0$. Then, by (2.42), ρ_b satisfies

$$\rho_b = \alpha_1\rho_l = 1 + a_\infty\delta + O(1)\delta^2. \tag{4.49}$$

Therefore, the Riemann problem (1.20) and (4.45) admits a unique solution that consists of only one shock wave α_1 issuing from point $(0,0)$ and belonging to the 1-st family with $U_l = (1,\delta)$ and $U_b := (\rho_b, v_b)^\top$ as its *left*-state and *right*-state for some $\delta > 0$.

Now, we turn to the Riemann solution of problem (1.16) with $U_l = (1,\delta)^\top$ as the *left*-state and $v_b^{(\boldsymbol{\mu})}$ as the velocity on the boundary. Let β_1 be the elementary wave in the Riemann solution. Then

$$v_b^{(\boldsymbol{\mu})} - v_l = \varphi_1^{(\boldsymbol{\mu})}(\beta_1, U_l, \boldsymbol{\mu}), \qquad \rho_b^{(\boldsymbol{\mu})} = \rho_l\beta_1. \tag{4.50}$$

It follows from $b_0 = 0$ and the boundary condition (4.26) in Lemma 4.2 that $v_b^{(\boldsymbol{\mu})} \equiv 0$. Thus, combining (4.46) with (4.50) yields

$$\varphi_1^{(\boldsymbol{\mu})}(\beta_1, U_l, \boldsymbol{\mu}) = \varphi_1(\alpha_1). \tag{4.51}$$

By Lemma 2.6, we have

$$\frac{\partial\varphi_1^{(\boldsymbol{\mu})}(\beta_1, U_l, \boldsymbol{\mu})}{\partial\beta_1}\Big|_{\beta_1=1,\boldsymbol{\mu}=\mathbf{0}} = -a_\infty^{-1} < 0.$$

Thus, by applying the implicit function theorem again, equations (4.50) admit a unique solution β_1 that is a function of $(\alpha_1, \boldsymbol{\mu})$, *i.e.*, $\beta_1 = \beta_1(\alpha_1, \boldsymbol{\mu})$. Let $\rho_b^{(\boldsymbol{\mu})}$ be the density state on boundary $y = 0$. Then, by (2.38), $\rho_b^{(\boldsymbol{\mu})}$ satisfies $\rho_b^{(\boldsymbol{\mu})} = \beta_1\rho_l = \delta\beta_1$.

Moreover, by (4.50), we see that

$$\beta_1(1, \boldsymbol{\mu}) = 0, \qquad \beta_1(\alpha_1, \mathbf{0}) = \alpha_1. \tag{4.52}$$

Then, by the Taylor formula and (4.48), we have

$$\begin{aligned}\beta_1(\alpha_1, \boldsymbol{\mu}) &= \beta_1(\alpha_1, \mathbf{0}) + \beta_1(1, \boldsymbol{\mu}) + O(1)|\alpha_1 - 1|\|\boldsymbol{\mu}\| \\ &= \alpha_1 + O(1)|\alpha_1 - 1|\|\boldsymbol{\mu}\| = 1 + a_\infty\delta + O(1)\|\boldsymbol{\mu}\|\delta,\end{aligned} \tag{4.53}$$

where $\|\boldsymbol{\mu}\| = \epsilon + \tau^2$. Thus, for $\delta > 0$ sufficiently small, the Riemann problem (1.16) with Riemann data $U_l = (1,\delta)^\top$ and $v_b^{(\boldsymbol{\mu})} = 0$ admits a unique solution that also consists of only one shock wave β_1 issuing from point $(0,0)$ and belonging to the 1-st family with $U_l = (1,\delta)$ and $U_b^{(\boldsymbol{\mu})} := (\rho_b^{(\boldsymbol{\mu})}, v_b^{(\boldsymbol{\mu})})^\top$ as its *left*-state and *right*-state for some $\delta > 0$. Moreover, $\beta_1 = 1$ is equivalent to $\delta = 0$.

Next, we compute $\frac{\partial^2\beta_1}{\partial\alpha_1\partial\epsilon}\Big|_{\alpha_1=1,\boldsymbol{\mu}=\mathbf{0}}$ and $\frac{\partial^2\beta_1}{\partial\alpha_1\partial\tau^2}\Big|_{\alpha_1=1,\boldsymbol{\mu}=\mathbf{0}}$. We first take the derivative on (4.52) with respect to α_1 to deduce that

$$\frac{\partial\varphi_1^{(\boldsymbol{\mu})}(\beta_1, U_l, \boldsymbol{\mu})}{\partial\beta_1}\frac{\partial\beta_1}{\partial\alpha_1} = \varphi_1'(\alpha_1). \tag{4.54}$$

Taking $\alpha_1 = 1$ and $\boldsymbol{\mu} = \mathbf{0}$ in (4.54), by Lemma 2.6, we obtain

$$\frac{\partial\beta_1}{\partial\alpha_1}\Big|_{\alpha_1=1,\boldsymbol{\mu}=\mathbf{0}} = \frac{\varphi_1'(1)}{\frac{\partial\varphi_1^{(\boldsymbol{\mu})}(\beta_1, U_l, \boldsymbol{\mu})}{\partial\beta_1}\Big|_{\alpha_1=1,\boldsymbol{\mu}=\mathbf{0}}} = 1. \tag{4.55}$$

Based on this, we further take the derivatives on both sides of (4.54) with respect to ϵ and τ^2 to obtain

$$\begin{aligned}
&\frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1}\frac{\partial^2 \beta_1}{\partial \alpha_1 \partial \epsilon}+\frac{\partial^2 \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1^2}\frac{\partial \beta_1}{\partial \epsilon}\frac{\partial \beta_1}{\partial \alpha_1}+\frac{\partial^2 \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1 \partial \epsilon}\frac{\partial \beta_1}{\partial \alpha_1}=0,\\
&\frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1}\frac{\partial^2 \beta_1}{\partial \alpha_1 \partial \tau^2}+\frac{\partial^2 \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1^2}\frac{\partial \beta_1}{\partial \tau^2}\frac{\partial \beta_1}{\partial \alpha_1}+\frac{\partial^2 \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1 \partial \tau^2}\frac{\partial \beta_1}{\partial \alpha_1}=0.
\end{aligned}$$

These imply that

$$\frac{\partial^2 \beta_1}{\partial \alpha_1 \partial \epsilon}=-\frac{\Big(\frac{\partial^2 \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1^2}\frac{\partial \beta_1}{\partial \epsilon}+\frac{\partial^2 \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1 \partial \epsilon}\Big)\frac{\partial \beta_1}{\partial \alpha_1}}{\frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1}},\qquad \frac{\partial^2 \beta_1}{\partial \alpha_1 \partial \tau^2}=-\frac{\Big(\frac{\partial^2 \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1^2}\frac{\partial \beta_1}{\partial \tau^2}+\frac{\partial^2 \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1 \partial \tau^2}\Big)\frac{\partial \beta_1}{\partial \alpha_1}}{\frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1}}. \tag{4.56}$$

In the following, we are devoted to the estimates of all the terms on both the right-hand sides of $\frac{\partial^2 \beta_1}{\partial \alpha_1 \partial \epsilon}$ and $\frac{\partial^2 \beta_1}{\partial \alpha_1 \partial \tau^2}$ in (4.56), respectively. Since β_1 is a shock wave, states $v^{(\boldsymbol{\mu})}-v_l$ and U_l satisfy equation (2.15):

$$\begin{aligned}
\mathcal{H}_S^{(\boldsymbol{\mu})}:=&(\varphi_1^{(\boldsymbol{\mu})})^2-\frac{2(\beta_1^\epsilon-1)(\beta_1-1)}{a_\infty^2\epsilon(\beta_1+1)}\\
&-\Big(\sqrt{(1-\tau^2\delta^2)(1-\tau^2\mathcal{B}^{(\epsilon)})}-1\Big)\Big((\delta+\varphi_1^{(\boldsymbol{\mu})})\delta+\frac{2\beta_1\big(\beta_1^\epsilon-1\big)}{a_\infty^2\epsilon(\beta_1+1)}\Big)-\tau^2\delta^2\mathcal{B}^{(\epsilon)}\\
=&0,
\end{aligned}\tag{4.57}$$

where $\mathcal{B}^{(\epsilon)}$ is given by

$$\mathcal{B}^{(\epsilon)}=\frac{2(\beta_1^\epsilon-1)}{a_\infty^2\epsilon}+\big(\delta+\varphi_1^{(\boldsymbol{\mu})}\big)^2. \tag{4.58}$$

Using the Taylor formula and estimates (i)–(ii) in Lemma A.1 and applying Lemma A.2, we obtain

$$\begin{aligned}
\frac{\partial \mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial \varphi_1^{(\boldsymbol{\mu})}}=&2\varphi_1^{(\boldsymbol{\mu})}-\Big(\big(1-\tau^2\delta^2\big)^{\frac12}-(1-\tau^2\mathcal{B}^{(\epsilon)})^{-\frac12}\Big)\delta\\
&+2(\delta+\varphi_1^{(\boldsymbol{\mu})})\Big(\frac{1-\tau^2\delta^2}{1-\tau^2\mathcal{B}^{(\epsilon)}}\Big)^{\frac12}\Big((\delta+\varphi_1^{(\boldsymbol{\mu})})\delta+\frac{2\beta_1\big(\beta_1^\epsilon-1\big)}{a_\infty^2\epsilon(\beta_1+1)}\Big)\tau^2-2(\delta+\varphi_1^{(\boldsymbol{\mu})})\tau^2\delta^2\\
=&-2\delta+O(1)(\delta+\epsilon+\tau^2)\delta,\\
\frac{\partial \mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial \beta_1}=&\frac{(\epsilon+1)\beta_1^\epsilon+\epsilon\beta_1^{\epsilon-1}-1}{a_\infty^2(\beta_1+1)^2\epsilon}-\frac{2(1-\tau^2\delta^2)^{\frac12}(1-\tau^2\mathcal{B}^{(\epsilon)})^{\frac12}}{a_\infty^2(\beta_1+1)^2}\frac{(\epsilon+1)\beta_1^\epsilon+\epsilon\beta_1^{\epsilon-1}-1}{\epsilon}\\
&+\Big(\frac{1-\tau^2\delta^2}{1-\tau^2\mathcal{B}^{(\epsilon)}}\Big)^{\frac12}\Big((\delta+\varphi_1^{(\boldsymbol{\mu})})\delta+\frac{2\beta_1(\beta_1^\epsilon-1)}{a_\infty^2(\beta_1+1)\epsilon}\Big)\frac{\tau^2}{a_\infty^2}\beta_1^{\epsilon-1}-\frac{2\tau^2\delta^2}{a_\infty^2}\beta_1^{\epsilon-1},
\end{aligned}\tag{4.59}$$

and

$$\begin{aligned}
\frac{\partial \mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial \epsilon} &= -\Big\{ \frac{\beta_1 - 1}{a_\infty^2(\beta_1+1)} - \Big(\frac{1-\tau^2\delta^2}{1-\tau^2\mathcal{B}^{(\epsilon)}}\Big)^{\frac12}\Big((\delta+\varphi_1^{(\boldsymbol{\mu})})\delta + \frac{2\beta_1(\beta_1^\epsilon - 1)}{a_\infty^2(\beta_1+1)\epsilon}\Big)\frac{\tau^2}{a_\infty^2} \\
&\qquad + \frac{\big((1-\tau^2\delta^2)^{\frac12}(1-\tau^2\mathcal{B}^{(\epsilon)})^{\frac12} - 1\big)\beta_1}{a_\infty^2(\beta_1+1)} - \frac{2}{a_\infty^2}\tau^2\delta^2 \Big\} \frac{(\epsilon\ln\beta_1 - 1)\beta_1^\epsilon + 1}{\epsilon^2} \\
&= -\Big(\frac{1}{a_\infty}\delta + O(1)(\delta+\epsilon+\tau^2)\delta\Big)\Big(a_\infty^2\delta^2 + O(1)(\delta+\epsilon+\tau^2)\delta^2\Big) \\
&= -a_\infty\delta^3 + O(1)(\delta+\epsilon+\tau^2)\delta^3, \\
\frac{\partial \mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial \tau^2} &= \frac12\Big(\Big(\frac{1-\tau^2\mathcal{B}^{(\epsilon)}}{1-\tau^2\delta^2}\Big)^{\frac12}\delta^2 + \Big(\frac{1-\tau^2\delta^2}{1-\tau^2\mathcal{B}^{(\epsilon)}}\Big)^{\frac12}\mathcal{B}^{(\epsilon)}\Big)\Big((\delta+\varphi_1^{(\boldsymbol{\mu})})\delta + \frac{2\beta_1\big(\beta_1^\epsilon - 1\big)}{a_\infty^2\epsilon(\beta_1+1)}\Big) - \delta^2\mathcal{B}^{(\epsilon)} \\
&= \frac12\Big(\frac{2}{a_\infty}\delta + O(1)(\delta+\epsilon+\tau^2)\delta\Big)\Big(\frac{1}{a_\infty}\delta + O(1)(\delta+\epsilon+\tau^2)\delta\Big) \\
&= \frac{2}{a_\infty}\delta^2 + O(1)(\delta+\epsilon+\tau^2)\delta^2.
\end{aligned} \tag{4.60}$$

Now, taking the derivatives on both sides of equation (4.57) with respect to ϵ and τ^2 respectively, we have

$$\frac{\partial \mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial \varphi_1^{(\boldsymbol{\mu})}}\frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \epsilon} + \frac{\partial \mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial \epsilon} = 0, \qquad \frac{\partial \mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial \varphi_1^{(\boldsymbol{\mu})}}\frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \tau^2} + \frac{\partial \mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial \tau^2} = 0. \tag{4.61}$$

Then it follows from estimates (4.59)–(4.60) that

$$\begin{aligned}
\frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \epsilon} &= -\frac{\frac{\partial \mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial \epsilon}}{\frac{\partial \mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial \varphi_1^{(\boldsymbol{\mu})}}} = -\frac{-a_\infty\delta^3 + O(1)(\delta+\epsilon+\tau^2)\delta^3}{-2\delta + O(1)(\delta+\epsilon+\tau^2)\delta} \\
&= -\frac{a_\infty}{2}\delta^2 + O(1)(\delta+\epsilon+\tau^2)\delta^2, \\
\frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \tau^2} &= -\frac{\frac{\partial \mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial \tau^2}}{\frac{\partial \mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial \varphi_1^{(\boldsymbol{\mu})}}} = -\frac{\frac{2}{a_\infty}\delta^2 + O(1)(\delta+\epsilon+\tau^2)\delta + O(1)(\delta+\epsilon+\tau^2)\delta^2}{-2\delta + O(1)(\delta+\epsilon+\tau^2)\delta} \\
&= \frac{1}{2a_\infty}\delta + O(1)(\delta+\epsilon+\tau^2)\delta,
\end{aligned}$$

which leads to

$$\frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \epsilon}\Big|_{\alpha_1=1,\epsilon=\tau=0} = \frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \tau^2}\Big|_{\alpha_1=1,\epsilon=\tau=0} = 0. \tag{4.62}$$

Therefore, we obtain from (4.50) and (4.62) that

$$\frac{\partial \beta_1}{\partial \epsilon}\Big|_{\alpha_1=1,\epsilon=\tau=0} = -\frac{\frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \epsilon}\Big|_{\alpha_1=1,\epsilon=\tau=0}}{\frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1}\Big|_{\alpha_1=1,\epsilon=\tau=0}} = 0, \qquad \frac{\partial \beta_1}{\partial \tau^2}\Big|_{\alpha_1=1,\epsilon=\tau=0} = -\frac{\frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \tau^2}\Big|_{\alpha_1=1,\epsilon=\tau=0}}{\frac{\partial \varphi_1^{(\boldsymbol{\mu})}}{\partial \beta_1}\Big|_{\alpha_1=1,\epsilon=\tau=0}} = 0. \tag{4.63}$$

Next, we further take the derivatives on equation (4.61) with respect to $\varphi_1^{(\boldsymbol{\mu})}$, β_1, ϵ, and τ^2 respectively, and combine with the estimates in Lemmas A.1–A.2 to obtain

$$\begin{aligned}
\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\varphi_1^{(\boldsymbol{\mu})2}} &= 2+(\delta+\varphi_1^{(\boldsymbol{\mu})})(1-\tau^2\mathcal{B}^{(\epsilon)})^{-\frac{3}{2}}\tau^2\delta-2\tau^2\delta^2\\
&\quad+2(1-\tau^2\delta^2)^{\frac{1}{2}}(1-\tau^2\mathcal{B}^{(\epsilon)})^{-\frac{3}{2}}\Big\{(\delta+\varphi_1^{(\boldsymbol{\mu})})\Big((\delta+\varphi_1^{(\boldsymbol{\mu})})\delta+\frac{2\beta_1\big(\beta_1^{\epsilon}-1\big)}{a_\infty^2\epsilon(\beta_1+1)}\Big)\tau^2\\
&\qquad\qquad+(1-\tau^2\mathcal{B}^{(\epsilon)})\Big(2(\delta+\varphi_1^{(\boldsymbol{\mu})})\delta+\frac{2\beta_1\big(\beta_1^{\epsilon}-1\big)}{a_\infty^2\epsilon(\beta_1+1)}\Big)\Big\}\\
&=2+O(1)(\delta+\epsilon+\tau^2)\delta, \qquad (4.64)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\beta_1\partial\varphi_1^{(\boldsymbol{\mu})}} &= \Big\{\frac{\big((\beta_1+1)\epsilon+1\big)\beta_1^{\epsilon}-1}{(\beta_1+1)^2\epsilon}+\frac{\beta_1^{\epsilon-1}\delta}{\delta+\varphi_1^{(\boldsymbol{\mu})}}+\frac{(\delta+\varphi_1^{(\boldsymbol{\mu})})\beta_1^{\epsilon-1}\tau^2}{2(1-\tau^2\mathcal{B}^{(\epsilon)})}\Big((\delta+\varphi_1^{(\boldsymbol{\mu})})\delta+\frac{2\beta_1\big(\beta_1^{\epsilon}-1\big)}{a_\infty^2(\beta_1+1)\epsilon}\Big)\Big\}\\
&\quad\times\Big(\frac{1-\tau^2\delta^2}{1-\tau^2\mathcal{B}^{(\epsilon)}}\Big)^{\frac{1}{2}}\frac{(\delta+\varphi_1^{(\boldsymbol{\mu})})\tau^2}{a_\infty^2}\\
&=O(1)\tau^2\delta, \qquad (4.65)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\varphi_1^{(\boldsymbol{\mu})}\partial\epsilon} &= \Big\{\delta+2(\delta+\varphi_1^{(\boldsymbol{\mu})})(1-\tau^2\delta)^{\frac{1}{2}}\Big((\delta+\varphi_1^{(\boldsymbol{\mu})})\delta+\frac{2\beta_1\big(\beta_1^{\epsilon}-1\big)}{a_\infty^2\epsilon(\beta_1+1)}\Big)\tau^2\\
&\qquad+\frac{4(\delta+\varphi_1^{(\boldsymbol{\mu})})(1-\tau^2\delta)^{\frac{1}{2}}(1-\tau^2\mathcal{B}^{(\epsilon)})}{\beta_1+1}\Big\}\frac{(1-\tau^2\mathcal{B}^{(\epsilon)})^{-\frac{3}{2}}\tau^2}{a_\infty^2}\frac{(\epsilon\ln\beta_1-1)\beta_1^{\epsilon}+1}{\epsilon^2}\\
&=O(1)\tau^2\delta^3, \qquad (4.66)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\varphi_1^{(\boldsymbol{\mu})}\partial\tau^2} &= \frac{1}{2}\Big((1-\tau^2\mathcal{B}^{(\epsilon)})^{-\frac{3}{2}}\mathcal{B}^{(\epsilon)}+(1-\tau^2\delta^2)^{-\frac{1}{2}}\delta^2\Big)\delta-2(\delta+\varphi_1^{(\boldsymbol{\mu})})\delta^2\\
&\quad+2(\delta+\varphi_1^{(\boldsymbol{\mu})})\Big((\delta+\varphi_1^{(\boldsymbol{\mu})})\delta+\frac{2\beta_1\big(\beta_1^{\epsilon}-1\big)}{a_\infty^2\epsilon(\beta_1+1)}\Big)\Big(\frac{1-\tau^2\delta^2}{1-\tau^2\mathcal{B}^{(\epsilon)}}\Big)^{\frac{1}{2}}\\
&\quad\times\Big(1+\frac{(1-\tau^2\delta^2)^{\frac{1}{2}}\mathcal{B}^{(\epsilon)}-(1-\tau^2\mathcal{B}^{(\epsilon)})^{\frac{1}{2}}\delta^2}{2(1-\tau^2\delta^2)(1-\tau^2\mathcal{B}^{(\epsilon)})}\Big)\tau^2\\
&=O(1)(\delta+\epsilon+\tau^2)\delta, \qquad (4.67)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\beta_1\partial\epsilon} =\;& \frac{2}{a_\infty^2(\beta_1+1)^2}\frac{\big(\epsilon(\epsilon+1)\ln\beta_1-1\big)\beta_1^\epsilon+\epsilon^2\beta^{\epsilon-1}\ln\beta_1+1}{\epsilon^2}\\
&-\frac{2\big(1-\tau^2\delta^2\big)^{\frac12}\big(1-\tau^2\mathcal{B}^{(\epsilon)}\big)^{\frac12}}{a_\infty^2(\beta_1+1)^2}\frac{\big((\beta_1+1)\epsilon+1\big)\epsilon\beta_1^\epsilon\ln\beta_1-\beta^\epsilon+1}{\epsilon^2}\\
&+\Big\{\frac{2}{a_\infty^2(\beta_1+1)^2}\frac{\big[(\beta_1+1)\epsilon+1\big]\beta_1^\epsilon-1}{\epsilon}+\frac{\beta_1^\epsilon}{\beta_1+1}+\Big((\delta+\varphi_1^{(\boldsymbol{\mu})})\delta+\frac{2\beta_1\big(\beta_1^\epsilon-1\big)}{a_\infty^2(\beta_1+1)\epsilon}\Big)\frac{\beta_1^\epsilon\tau^2}{2}\Big\}\\
&\times\frac{2\tau^2}{a_\infty^4}\Big(\frac{1-\tau^2\delta^2}{1-\tau^2\mathcal{B}^{(\epsilon)}}\Big)^{\frac12}\frac{\big(\epsilon\ln\beta_1-1\big)\beta_1^\epsilon+1}{\epsilon^2}\\
&+\Big\{\Big(\frac{1-\tau^2\delta^2}{1-\tau^2\mathcal{B}^{(\epsilon)}}\Big)^{\frac12}\Big((\delta+\varphi_1^{(\boldsymbol{\mu})})\delta+\frac{2\beta_1\big(\beta_1^\epsilon-1\big)}{a_\infty^2(\beta_1+1)\epsilon}\Big)-\delta^2\Big\}\frac{\beta_1^\epsilon\ln\beta_1}{a_\infty^2}\tau^2\\
=\;&\frac{1}{2a_\infty^2}\Big(\frac{3a_\infty}{2}\delta+O(1)(\delta+\epsilon+\tau^2)\delta\Big)-\frac{1}{2a_\infty^2}\Big(2a_\infty\delta+O(1)(\delta+\epsilon+\tau^2)\delta\Big)+O(1)(\epsilon+\tau^2)\delta\\
=\;&-\frac{1}{4a_\infty}\delta+O(1)(\delta+\epsilon+\tau^2)\delta,
\end{aligned}\tag{4.68}
$$

and

$$
\begin{aligned}
\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\beta_1\partial\tau^2} =\;& \frac{(1-\tau^2\mathcal{B}^{(\epsilon)})\delta^2+(1-\tau^2\delta^2)\mathcal{B}^{(\epsilon)}}{(1-\tau^2\delta^2)^{\frac12}(1-\tau^2\mathcal{B}^{(\epsilon)})^{\frac12}}\frac{\big((\beta_1+1)\epsilon+1\big)\beta_1^\epsilon-1}{a_\infty^2(\beta_1+1)^2\epsilon}\\
&+\frac{\beta_1^{\epsilon-1}}{a_\infty^2}\Big(\frac{1-\tau^2\delta^2}{1-\tau^2\mathcal{B}^{(\epsilon)}}\Big)^{\frac12}\Big(1+\frac{\big(\mathcal{B}^{(\epsilon)}-\delta^2\big)\tau^2}{2(1-\tau^2\delta^2)(1-\tau^2\mathcal{B}^{(\epsilon)})}\Big)\Big((\delta+\varphi_1^{(\boldsymbol{\mu})})\delta+\frac{2\beta_1\big(\beta_1^\epsilon-1\big)}{a_\infty^2(\beta_1+1)\epsilon}\Big)\\
=\;&O(1)(\delta+\epsilon+\tau^2)\delta.
\end{aligned}\tag{4.69}
$$

Then we obtain from estimates (4.64)–(4.69) that

$$
\begin{aligned}
\frac{\partial^2\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1\partial\epsilon} =\;& -\frac{\Big(\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\varphi_1^{(\boldsymbol{\mu})2}}\frac{\partial\varphi_1^{(\boldsymbol{\mu})}}{\partial\epsilon}+\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\varphi_1^{(\boldsymbol{\mu})}\partial\epsilon}\Big)\frac{\partial\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1}+\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\beta_1\partial\varphi_1^{(\boldsymbol{\mu})}}\frac{\partial\varphi_1^{(\boldsymbol{\mu})}}{\partial\epsilon}+\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\beta_1\partial\epsilon}}{\frac{\partial\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\varphi_1^{(\boldsymbol{\mu})}}}\\
=\;&\Big(\frac{1}{a_\infty}+O(1)(\delta+\epsilon+\tau^2)\Big)\\
&\quad\times\frac{\Big(\big(2+O(1)(\delta+\epsilon+\tau^2)\delta\big)\big(-\frac{a_\infty}{2}\delta^2+O(1)(\delta+\epsilon+\tau^2)\delta^2\big)+O(1)\tau^2\delta^3\Big)}{-2\delta+O(1)(\delta+\epsilon+\tau^2)\delta}\\
&\quad-\frac{\Big(\frac{a_\infty}{2}\delta^2+O(1)(\delta+\epsilon+\tau^2)\delta^2\Big)O(1)\tau^2\delta}{2\delta+O(1)(\delta+\epsilon+\tau^2)\delta}-\frac{\frac{1}{4a_\infty}\delta+O(1)(\delta+\epsilon+\tau^2)\delta}{2\delta+O(1)(\delta+\epsilon+\tau^2)\delta}\\
=\;&-\frac{1}{8a_\infty}+O(1)\big(\delta+\epsilon+\tau^2\big),
\end{aligned}
$$

and

$$\begin{aligned}\frac{\partial^2\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1\partial\tau^2} &= -\frac{\Big(\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\varphi_1^{(\boldsymbol{\mu})2}}\frac{\partial\varphi_1^{(\boldsymbol{\mu})}}{\partial\tau^2}+\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\varphi_1^{(\boldsymbol{\mu})}\partial\tau^2}\Big)\frac{\partial\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1}+\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\beta_1\partial\varphi_1^{(\boldsymbol{\mu})}}\frac{\partial\varphi_1^{(\boldsymbol{\mu})}}{\partial\tau^2}+\frac{\partial^2\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\beta_1\partial\tau^2}}{\frac{\partial\mathcal{H}_S^{(\boldsymbol{\mu})}}{\partial\varphi_1^{(\boldsymbol{\mu})}}}\\
&= -\Big(\frac{1}{a_\infty}+O(1)(\delta+\epsilon+\tau^2)\Big)\\
&\quad\times\frac{\Big(\big(2+O(1)(\delta+\epsilon+\tau^2)\delta\big)\big(\frac{1}{2a_\infty^2}+O(1)(\delta+\epsilon+\tau^2)\delta\big)+O(1)(\delta+\epsilon+\tau^2)\delta\Big)}{2\delta+O(1)(\delta+\epsilon+\tau^2)\delta}\\
&\quad+\frac{O(1)(\delta+\epsilon+\tau^2)\delta}{2\delta+O(1)(\delta+\epsilon+\tau^2)\delta}\\
&= -\frac{1}{2a_\infty^3}+O(1)(\delta+\epsilon+\tau^2),\end{aligned}$$

which show that

$$\frac{\partial^2\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1\partial\epsilon}\Big|_{\alpha_1=1,\epsilon=\tau=0}=-\frac{1}{8a_\infty},\qquad\frac{\partial^2\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1\partial\tau^2}\Big|_{\alpha_1=1,\epsilon=\tau=0}=-\frac{1}{2a_\infty^3}. \tag{4.70}$$

Using Lemma 2.6, (4.63), and (4.70), we thus obtain

$$\begin{aligned}\frac{\partial^2\beta_1}{\partial\alpha_1\partial\epsilon}\Big|_{\alpha_1=1,\epsilon=\tau=0}&=-\frac{\Big(\frac{\partial^2\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1^2}\frac{\partial\beta_1}{\partial\epsilon}+\frac{\partial^2\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1\partial\epsilon}\Big)\frac{\partial\beta_1}{\partial\alpha_1}\Big|_{\alpha_1=1,\epsilon=\tau=0}}{\frac{\partial\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1}\Big|_{\alpha_1=1,\epsilon=\tau=0}}=-\frac{1}{8},\\
\frac{\partial^2\beta_1}{\partial\alpha_1\partial\tau^2}\Big|_{\alpha_1=1,\epsilon=\tau=0}&=-\frac{\Big(\frac{\partial^2\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1^2}\frac{\partial\beta_1}{\partial\tau^2}+\frac{\partial^2\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1\partial\tau^2}\Big)\frac{\partial\beta_1}{\partial\alpha_1}\Big|_{\alpha_1=1,\epsilon=\tau=0}}{\frac{\partial\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1}\Big|_{\alpha_1=1,\epsilon=\tau=0}}=-\frac{1}{2a_\infty^2}.\end{aligned} \tag{4.71}$$

Finally, combining the Taylor formula again with (4.71), we arrive at

$$
\begin{aligned}
\beta_1(\alpha_1,\epsilon,\tau^2) &= \beta_1(\alpha_1,0,\tau^2)+\beta_1(1,\epsilon,\tau^2)-\beta_1(1,0,\tau^2)\\
&\quad+(\alpha_1-1)\epsilon\int_0^1\int_0^1\frac{\partial^2\beta_1}{\partial\alpha_1\partial\epsilon}(\xi_1(\alpha_1-1)+1,\xi_2\epsilon,\tau^2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\\
&= \beta_1(\alpha_1,0,0)+\beta_1(1,0,\tau^2)-\beta_1(1,0,0)\\
&\quad+(\alpha_1-1)\tau^2\int_0^1\int_0^1\frac{\partial^2\beta_1}{\partial\alpha_1\partial\epsilon}(\xi_1(\alpha_1-1)+1,0,\xi_3\tau^2)\,\mathrm{d}\xi_1\mathrm{d}\xi_3\\
&\quad+\beta_1(1,\epsilon,\tau^2)-\beta_1(1,0,\tau^2)\\
&\quad+(\alpha_1-1)\epsilon\int_0^1\int_0^1\frac{\partial^2\beta_1}{\partial\alpha_1\partial\epsilon}(\xi_1(\alpha_1-1)+1,\xi_2\epsilon,\tau^2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\\
&= \alpha_1+\frac{\partial^2\beta_1}{\partial\alpha_1\partial\epsilon}\Big|_{\alpha_1=1,\epsilon=\tau^2=0}(\alpha_1-1)\epsilon+\frac{\partial^2\beta_1}{\partial\alpha_1\partial\tau^2}\Big|_{\alpha_1=1,\epsilon=\tau^2=0}(\alpha_1-1)\tau^2\\
&\quad+O(1)\big(|\alpha_1-1|+\epsilon+\tau^2\big)|\alpha_1-1|(\epsilon+\tau^2)\\
&= \alpha_1-\frac{1}{8}(\alpha_1-1)\epsilon-\frac{1}{2a_\infty^2}(\alpha_1-1)\tau^2+O(1)\big(|\alpha_1-1|+\epsilon+\tau^2\big)|\alpha_1-1|(\epsilon+\tau^2),
\end{aligned}
\tag{4.72}
$$

where the bounds of $O(1)$ depend only on $\underline{U}$.

Denoted by $\sigma_1(\alpha_1)$ and $\sigma_1^{(\boldsymbol{\mu})}(\beta_1)$ the speeds of α_1 and β_1, respectively. Then, for δ, ϵ, and τ are sufficiently small, by (4.48), we have

$$
\sigma_1(\alpha_1)=\frac{\rho_b v_b-\rho_l v_l}{\rho_b-\rho_l}=-\frac{\delta}{\alpha_1-1}=-\frac{1}{a_\infty}+O(1)\delta, \tag{4.73}
$$

$$
\begin{aligned}
\sigma_1^{(\tau)}(\beta_1)&=\frac{\rho_b^{(\boldsymbol{\mu})}v_b^{(\boldsymbol{\mu})}-\rho_l v_l}{\rho_b^{(\boldsymbol{\mu})}\sqrt{1-\tau^2\mathcal{B}^{(\epsilon)}}-\rho_l\sqrt{1-\tau^2\delta}}=-\frac{\delta}{\beta_1\sqrt{1-\tau^2\mathcal{B}^{(\epsilon)}}-\sqrt{1-\tau^2\delta}}\\
&=-\frac{\delta}{\beta_1-1}+O(1)\tau^2\delta.
\end{aligned}
\tag{4.74}
$$

Notice from (4.72) that

$$
\beta_1-1 \quad =(\alpha_1-1)\Big(1-\tfrac{1}{8}\epsilon-\tfrac{1}{2a_\infty^2}\tau^2+O(1)\big(|\alpha_1-1|+\epsilon+\tau^2\big)(\epsilon+\tau^2)\Big).
$$

Therefore, we can further deduce from (4.74) that

$$
\begin{aligned}
\sigma_1^{(\tau)}(\beta_1)&=-\frac{\delta}{(\alpha_1-1)\Big(1-\tfrac{1}{8}\epsilon-\tfrac{1}{2a_\infty^2}\tau^2+O(1)\big(|\alpha_1-1|+\epsilon+\tau^2\big)(\epsilon+\tau^2)\Big)}+O(1)\tau^2\delta\\
&=-\frac{\delta}{\alpha_1-1}\Big(1+\frac{1}{8}\epsilon+\frac{1}{2a_\infty^2}\tau^2+O(1)\big(\delta+\epsilon+\tau^2\big)(\epsilon+\tau^2)\Big)+O(1)\tau^2\delta.
\end{aligned}
\tag{4.75}
$$

Thus, by (4.73) and (4.75), we have

$$\begin{aligned}\sigma_1(\alpha_1)-\sigma_1^{(\tau)}(\beta_1)&=\frac{\delta}{\alpha_1-1}\Big(1+\frac{1}{8}\epsilon+\frac{1}{2a_\infty^2}\tau^2+O(1)\big(\delta+\epsilon+\tau^2\big)(\epsilon+\tau^2)\Big)-\frac{\delta}{\alpha_1-1}+O(1)\tau^2\delta\\&=\frac{\delta}{\alpha_1-1}\Big(\frac{3}{8}\epsilon+\frac{1}{2a_\infty^2}\tau^2\Big)+O(1)\big(\delta+\epsilon+\tau^2\big)(\epsilon+\tau^2)+O(1)\delta\tau^2\\&=\frac{1}{8a_\infty}\epsilon+\frac{1}{2a_\infty^3}\tau^2+O(1)\big(\delta+\epsilon+\tau^2\big)(\epsilon+\tau^2).\end{aligned}\tag{4.76}$$

On the other hand, using estimates (4.48) and (4.54), we have

$$\begin{aligned}|\rho_b^{(\boldsymbol{\mu})}-\rho_l|&=|\beta_1-1|\\&=\Big|(\alpha_1-1)\big(1-\frac{1}{8}\epsilon-\frac{1}{2a_\infty^2}\tau^2+O(1)(|\alpha_1-1|+\epsilon+\tau^2)(\epsilon+\tau^2)\big)\Big|\\&=a_\infty\delta+O(1)\big(\delta+\epsilon+\tau^2\big)\delta,\end{aligned}\tag{4.77}$$

$$\begin{aligned}|\rho_b^{(\boldsymbol{\mu})}-\rho_b|&=|\beta_1-\alpha_1|\\&=\Big|(\alpha_1-1)\big(-\frac{3}{8}\epsilon-\frac{1}{2a_\infty^2}\tau^2+O(1)(|\alpha_1-1|+\epsilon+\tau^2)(\epsilon+\tau^2)\big)\Big|\\&=\Big(\frac{1}{8}\epsilon+\frac{1}{2a_\infty^2}\tau^2\Big)a_\infty\delta+O(1)\big(\delta+\epsilon+\tau^2\big)(\epsilon+\tau^2)\delta.\end{aligned}\tag{4.78}$$

With estimates (4.73) and (4.76)–(4.78) in hand, we thus have

$$\begin{aligned}&\big(|\rho_b^{(\boldsymbol{\mu})}-\rho_l|+|v_b^{(\boldsymbol{\mu})}-v_l|\big)\,\big(\sigma_1(\alpha_1)-\sigma_1^{(\boldsymbol{\mu})}(\beta_1)\big)\\&=\Big(a_\infty\delta+O(1)\big(\delta+\epsilon+\tau^2\big)\delta+\delta\Big)\Big(\frac{3}{8a_\infty}\epsilon+\frac{1}{2a_\infty^3}\tau^2+O(1)\big(\delta+\epsilon+\tau^2\big)(\epsilon+\tau^2)\Big)\\&=\frac{a_\infty+1}{8a_\infty}\epsilon\delta+\frac{a_\infty+1}{2a_\infty^3}\tau^2\delta+O(1)\big(\delta+\epsilon+\tau^2\big)(\epsilon+\tau^2)\delta,\end{aligned}\tag{4.79}$$

$$\begin{aligned}&\big(|\rho_b-\rho_b^{(\boldsymbol{\mu})}|+|v_b-v_b^{(\boldsymbol{\mu})}|\big)\,\big(-\sigma_1(\alpha_1)\big)\\&=\Big(\big(\frac{3}{8}\epsilon+\frac{1}{2a_\infty^2}\tau^2\big)a_\infty\delta+O(1)\big(\delta+\epsilon+\tau^2\big)(\epsilon+\tau^2)\delta\Big)\Big(a_\infty\delta+O(1)\big(\delta+\epsilon+\tau^2\big)\delta\Big)\\&=\frac{1}{8}\epsilon\delta+\frac{1}{2a_\infty^2}\tau^2\delta+O(1)\big(\delta+\epsilon+\tau^2\big)(\epsilon+\tau^2)\delta.\end{aligned}\tag{4.80}$$

Finally, combining estimate (4.79) with estimate (4.80), we conclude

$$\begin{aligned}\big\|U^{(\boldsymbol{\mu})}-U\big\|_{L^1}&=\int_{\sigma_1^{(\boldsymbol{\mu})}(\beta_1)x}^{\sigma_1(\alpha_1)x}\big|U^{(\boldsymbol{\mu})}-U\big|\,\mathrm{d}y+\int_{\sigma_1(\alpha_1)x}^{0}\big|U^{(\boldsymbol{\mu})}-U\big|\,\mathrm{d}y\\&=\big(|\rho_b^{(\boldsymbol{\mu})}-\rho_l|+|v_b^{(\boldsymbol{\mu})}-v_l|\big)\,\big(\sigma_1(\alpha_1)-\sigma_1^{(\boldsymbol{\mu})}(\beta_1)\big)x\\&\quad+\big(|\rho_b-\rho_b^{(\boldsymbol{\mu})}|+|v_b-v_b^{(\boldsymbol{\mu})}|\big)\,\big(-\sigma_1(\alpha_1)\big)x\\&=\frac{2a_\infty+1}{8a_\infty}\epsilon\delta x+\frac{2a_\infty+1}{2a_\infty^3}\tau^2\delta x+O(1)\big(\delta+\epsilon+\tau^2\big)(\epsilon+\tau^2)\delta x.\end{aligned}\tag{4.81}$$

This implies that the convergence rate we have obtained in Theorem 1.1 is optimal.

5. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2 to establish the convergence rate between the entropy solutions $(\rho^{(\boldsymbol{\mu})}, u^{(\boldsymbol{\mu})}, v^{(\boldsymbol{\mu})})$ of problems (1.6)–(1.9) and the entropy solution (ρ, u, v) of problem (1.10)–(1.12).

For $\rho > 0$, define

$$\Psi(U,\boldsymbol{\mu}) = \frac{B^{(\epsilon)}(\rho,v,\epsilon)}{\sqrt{1-\tau^2 B^{(\epsilon)}(\rho,v,\epsilon)}+1}, \qquad \Psi(U) = \frac{1}{2}v^2 + \frac{\ln\rho}{a_\infty^2}, \tag{5.1}$$

where $U = (\rho, v)^\tau$ and $B^{(\epsilon)}$ are given by (1.15). Clearly, for fixed U, we have

$$\Psi(U,\mathbf{0}) = \Psi(U). \tag{5.2}$$

We first introduce two lemmas which are useful to the proof of Theorem 1.2.

Lemma 5.1. *For $\rho > 0$,*

$$\partial_\rho B^{(\epsilon)} = \frac{2\rho^{\epsilon-1}}{a_\infty^2}, \qquad \partial_v B^{(\epsilon)} = 2v, \qquad \partial_\epsilon B^{(\epsilon)} = \frac{2(\epsilon\rho^\epsilon \ln\rho - \rho^\epsilon + 1)}{a_\infty^2 \epsilon^2}. \tag{5.3}$$

Lemma 5.1 is obtained by direct calculation, so we omit the details.

Lemma 5.2. *For $\rho > 0$,*

$$\partial_\rho \Psi(U,\boldsymbol{\mu}) = \frac{\rho^{\epsilon-1}}{a_\infty^2\sqrt{1-\tau^2 B^{(\epsilon)}}}, \qquad \partial_v \Psi(U,\boldsymbol{\mu}) = \frac{v}{a_\infty^2\sqrt{1-\tau^2 B^{(\epsilon)}}}, \tag{5.4}$$

and

$$\partial_\epsilon \Psi(U,\boldsymbol{\mu}) = \frac{\epsilon\rho^\epsilon\ln\rho - \rho^\epsilon + 1}{a_\infty^2\sqrt{1-\tau^2 B^{(\epsilon)}}}, \qquad \partial_{\tau^2}\Psi(U,\boldsymbol{\mu}) = \frac{B^{(\epsilon)^2}}{2\sqrt{1-\tau^2 B^{(\epsilon)}}\big(\sqrt{1-\tau^2 B^{(\epsilon)}}+1\big)^2}. \tag{5.5}$$

Proof. By direct computation, we have

$$\begin{aligned}
\partial_\rho \Psi(U,\boldsymbol{\mu}) &= \frac{\big(\sqrt{1-\tau^2 B^{(\epsilon)}}+1\big)\partial_\rho B^{(\epsilon)} + \frac{1}{2}\tau^2 B^{(\epsilon)}(1-\tau^2 B^\epsilon)^{-\frac{1}{2}}\partial_\rho B^{(\epsilon)}}{\big(\sqrt{1-\tau^2 B^{(\epsilon)}}+1\big)^2} \\
&= \frac{\big(2+2\sqrt{1-\tau^2 B^{(\epsilon)}} - \tau^2 B^{(\epsilon)}\big)\partial_\rho B^{(\epsilon)}}{2\sqrt{1-\tau^2 B^{(\epsilon)}}\big(\sqrt{1-\tau^2 B^{(\epsilon)}}+1\big)^2} \\
&= \frac{\partial_\rho B^{(\epsilon)}}{2\sqrt{1-\tau^2 B^{(\epsilon)}}}.
\end{aligned}$$

Then the expression of $\partial_\rho\Psi(U,\boldsymbol{\mu})$ follows from Lemma 5.1. The expressions of $\partial_v\Psi(U,\boldsymbol{\mu})$ and $\partial_\epsilon\Psi(U,\boldsymbol{\mu})$ can be obtained by similar arguments from Lemma 5.1.

Finally, for $\partial_{\tau^2}\Psi(U,\boldsymbol{\mu})$, by (5.1) and direct calculations, we have

$$\begin{aligned}
\partial_{\tau^2}\Psi(U,\boldsymbol{\mu}) &= \frac{-B^{(\epsilon)}}{\big(\sqrt{1-\tau^2 B^{(\epsilon)}}+1\big)^2}\frac{-B^{(\epsilon)}}{2}\big(1-\tau^2 B^{(\epsilon)}\big)^{-\frac{1}{2}} \\
&= \frac{B^{(\epsilon)^2}}{2\sqrt{1-\tau^2 B^{(\epsilon)}}\big(\sqrt{1-\tau^2 B^{(\epsilon)}}+1\big)^2}.
\end{aligned}$$

This completes the proof of the lemma. □

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $(\rho^{(\boldsymbol{\mu})}, v^{(\boldsymbol{\mu})})$ be the entropy solution of problem (1.16)–(1.18) obtained by Proposition 3.1, and (ρ, v) be the entropy solution of problem (1.20)–(1.22) as given by Proposition 3.2. Then, by relations (1.14) and (1.19), we obtain $u^{(\boldsymbol{\mu})}$ and u from the solutions of problem (1.6)–(1.9) and problem (1.10)–(1.12), respectively, as

$$u^{(\boldsymbol{\mu})} = \frac{1}{\tau^2}\Big(\sqrt{1-\tau^2 B^{(\epsilon)}(\rho^{(\boldsymbol{\mu})}, v^{(\boldsymbol{\mu})}, \epsilon)} - 1\Big), \qquad u = -\frac{1}{2}v^2 - \frac{\ln\rho}{a_\infty^2}. \tag{5.6}$$

Then

$$\begin{aligned}
u^{(\boldsymbol{\mu})} - u &= \frac{1}{\tau^2}\Big(\sqrt{1-\tau^2 B^{(\epsilon)}(\rho^{(\boldsymbol{\mu})}, v^{(\boldsymbol{\mu})}, \epsilon)} - 1\Big) - \Big(-\frac{1}{2}v^2 - \frac{\ln\rho}{a_\infty^2}\Big)\\
&= -\Big(\frac{B^{(\epsilon)}(\rho^{(\boldsymbol{\mu})}, v^{(\boldsymbol{\mu})}, \epsilon)}{\sqrt{1-\tau^2 B^{(\epsilon)}(\rho^{(\boldsymbol{\mu})}, v^{(\boldsymbol{\mu})}, \epsilon)} + 1} - \frac{1}{2}v^2 - \frac{\ln\rho}{a_\infty^2}\Big)\\
&= -\big(\Psi(U^{(\boldsymbol{\mu})}, \boldsymbol{\mu}) - \Psi(U)\big)\\
&= -\big(\Psi(U^{(\boldsymbol{\mu})}, \boldsymbol{\mu}) - \Psi(U, \boldsymbol{\mu})\big) - \big(\Psi(U, \boldsymbol{\mu}) - \Psi(U)\big),
\end{aligned} \tag{5.7}$$

where $U^{(\boldsymbol{\mu})} = (\rho^{(\boldsymbol{\mu})}, v^{(\boldsymbol{\mu})})^\top$ and $U = (\rho, v)^\top$.

Next, we estimate the two terms $\Psi(U^{(\boldsymbol{\mu})}, \boldsymbol{\mu}) - \Psi(U, \boldsymbol{\mu})$ and $\Psi(U, \boldsymbol{\mu}) - \Psi(U)$ one by one. By Lemma 5.2, we have

$$\begin{aligned}
&\big\|\Psi(U^{(\boldsymbol{\mu})}, \boldsymbol{\mu}) - \Psi(U, \boldsymbol{\mu})\big\|_{L^1((-\infty, b_0 x))}\\
&= \Big\|\int_0^1 \nabla_U \Psi(U + t(U^{(\boldsymbol{\mu})} - U), \boldsymbol{\mu})\,\mathrm{d}t \cdot \big(U^{(\boldsymbol{\mu})} - U\big)\Big\|_{L^1((-\infty, b_0 x))}\\
&\le \Big\|\int_0^1 \frac{\big(\rho + t(\rho^{(\boldsymbol{\mu})} - \rho)\big)^{\epsilon+1}\,\mathrm{d}t}{a_\infty^2\sqrt{1-\tau^2 B^{(\epsilon)}(\rho + t(\rho^{(\boldsymbol{\mu})} - \rho), v + t(v^{(\boldsymbol{\mu})} - v), \epsilon)} + 1}\Big\|_{L^\infty(\Omega_{\rm w})} \big\|\rho^{(\boldsymbol{\mu})} - \rho\big\|_{L^1((-\infty, b_0 x))}\\
&\quad + \Big\|\int_0^1 \frac{\big(v + t(v^{(\boldsymbol{\mu})} - v)\big)^{\epsilon+1}\,\mathrm{d}t}{a_\infty^2\sqrt{1-\tau^2 B^{(\epsilon)}(\rho + t(\rho^{(\boldsymbol{\mu})} - \rho), v + t(v^{(\boldsymbol{\mu})} - v), \epsilon)} + 1}\Big\|_{L^\infty(\Omega_{\rm w})} \big\|v^{(\boldsymbol{\mu})} - v\big\|_{L^1((-\infty, b_0 x))}.
\end{aligned} \tag{5.8}$$

Note that, by Propositions 3.1–3.2, for $\tau > 0$ and $\epsilon > 0$ sufficiently small, we can choose a constant $C_2 > 0$ depending only on a_∞, ρ^*, and ρ_* such that

$$\begin{aligned}
&\Big\|\int_0^1 \frac{\big(\rho + t(\rho^{(\boldsymbol{\mu})} - \rho)\big)^{\epsilon+1}\,\mathrm{d}t}{a_\infty^2\sqrt{1-\tau^2 B^{(\epsilon)}(\rho + t(\rho^{(\boldsymbol{\mu})} - \rho), v + t(v^{(\boldsymbol{\mu})} - v), \epsilon)} + 1}\Big\|_{L^\infty(\Omega_{\rm w})}\\
&\quad + \Big\|\int_0^1 \frac{\big(v + t(v^{(\boldsymbol{\mu})} - v)\big)^{\epsilon+1}\,\mathrm{d}t}{a_\infty^2\sqrt{1-\tau^2 B^{(\epsilon)}(\rho + t(\rho^{(\boldsymbol{\mu})} - \rho), v + t(v^{(\boldsymbol{\mu})} - v), \epsilon)} + 1}\Big\|_{L^\infty(\Omega_{\rm w})} < C_2.
\end{aligned} \tag{5.9}$$

Thus, using Theorem 1.1 and estimates (5.8)–(5.9), we conclude

$$\big\|\Psi(U^{(\boldsymbol{\mu})}, \boldsymbol{\mu}) - \Psi(U, \boldsymbol{\mu})\big\|_{L^1((-\infty, b_0 x))} \le C_3 x \|\boldsymbol{\mu}\|, \tag{5.10}$$

where $C_3 > 0$ only depends on a_∞, ρ^*, and ρ_*.

Furthermore, by (5.2) and Lemma 5.2, we have

$$\begin{aligned}
&\big\|\Psi(U,\boldsymbol{\mu})-\Psi(U)\big\|_{L^1((-\infty,b_0x))}=\big\|\Psi(U,\boldsymbol{\mu})-\Psi(U,0)\big\|_{L^1((-\infty,b_0x))}\\
&=\Big\|\int_0^1\nabla_{\boldsymbol{\mu}}\Psi(U,\theta\boldsymbol{\mu})\,\mathrm{d}\theta\cdot\boldsymbol{\mu}\Big\|_{L^1((-\infty,b_0x))}\\
&\le\Big\|\int_0^1\frac{\big(\theta\epsilon\rho^{\theta\epsilon}\ln\rho-\rho^{\theta\epsilon}+1\big)\,\mathrm{d}\theta}{a_\infty^2\sqrt{1-\theta\tau^2B^{(\epsilon)}(\rho,v,\theta\epsilon)}+1}\Big\|_{L^1((-\infty,b_0x))}\epsilon\\
&\quad+\Big\|\int_0^1\frac{B^{(\epsilon)}(\rho,v,\theta\epsilon)\,\mathrm{d}\theta}{2\sqrt{1-\theta\tau^2B^{(\epsilon)}(\rho,v,\theta\epsilon)}\big(\sqrt{1-\theta\tau^2B^{(\epsilon)}(\rho,v,\theta\epsilon)}+1\big)^2}\Big\|_{L^1((-\infty,b_0x))}\tau^2.
\end{aligned}\tag{5.11}$$

For $\tau>0$ and $\epsilon>0$ sufficiently small, we can deduce from Proposition 3.2 that

$$\begin{aligned}
&\Big\|\int_0^1\frac{\big(\theta\epsilon\rho^{\theta\epsilon}\ln\rho-\rho^{\theta\epsilon}+1\big)\,\mathrm{d}\theta}{a_\infty^2\sqrt{1-\theta\tau^2B^{(\epsilon)}(\rho,v,\theta\epsilon)}+1}\Big\|_{L^1((-\infty,b_0x))}\\
&\quad+\Big\|\int_0^1\frac{B^{(\epsilon)}(\rho,v,\theta\epsilon)\,\mathrm{d}\theta}{2\sqrt{1-\theta\tau^2B^{(\epsilon)}(\rho,v,\theta\epsilon)}\big(\sqrt{1-\theta\tau^2B^{(\epsilon)}(\rho,v,\theta\epsilon)}+1\big)^2}\Big\|_{L^1((-\infty,b_0x))}\\
&\le C_4\|(\rho-1,v)\|_{L^1((-\infty,b_0x))},
\end{aligned}\tag{5.12}$$

where $C_4>0$ depends only on a_∞, ρ^*, and ρ_*.

It follows from (5.11)–(5.12) that a constant $C_5>0$ can be chosen, depending only on a_∞, ρ^*, and ρ_*, so that

$$\big\|\Psi(U,\boldsymbol{\mu})-\Psi(U)\big\|_{L^1((-\infty,b_0x))}\le C_5\|\boldsymbol{\mu}\|.\tag{5.13}$$

Then, combining estimates (5.10) and (5.13) altogether and employing equality (5.7), we obtain

$$\|u^{(\boldsymbol{\mu})}-u\|_{L^1((-\infty,b_0x))}\le C_6(1+x)\|\boldsymbol{\mu}\|,\tag{5.14}$$

where $C_6>0$ depends only on a_∞, ρ^*, and ρ_*.

Finally, combining estimate (5.14) with estimate (1.23) in Theorem 1.1, we conclude (1.24). This completes the proof of Theorem 1.2. □

Appendix A.

In this appendix, we give some basic estimates of the terms obtained from the derivatives of $\mathcal{H}^{(\boldsymbol{\mu})}$ which are used in proving the optimal convergence rate as stated in §4.3.

Lemma A.1. *Let β_1 be given in* (4.50) *which satisfies* (4.54). *Then, for $\delta>0$, $\epsilon>0$, and $\tau>0$ sufficiently small, the following estimates hold*:

(i) $\frac{\beta_1^\epsilon-1}{\epsilon}=a_\infty\delta+O(1)(\delta+\epsilon+\tau^2)\delta$,

(ii) $\frac{((\beta_1+1)\epsilon+1)\beta_1^\epsilon-1}{\epsilon}=2(a_\infty+1)\delta+O(1)(\delta+\epsilon+\tau^2)\delta$,

(iii) $\frac{(\epsilon\ln\beta_1-1)\beta_1^\epsilon+1}{\epsilon^2}=\frac{a_\infty^2}{2}\delta^2+O(1)(\delta+\epsilon+\tau^2)\delta^2$,

(iv) $\frac{(\epsilon(\epsilon+1)\ln\beta_1-1)\beta_1^\epsilon+\epsilon^2\beta^{\epsilon-1}\ln\beta_1+1}{\epsilon^2}=\frac{3a_\infty}{2}\delta+O(1)(\delta+\epsilon+\tau^2)\delta$,

(v) $\frac{((\beta_1+1)\epsilon+1)\epsilon\beta_1^\epsilon\ln\beta_1-\beta^\epsilon+1}{\epsilon^2}=2a_\infty\delta+O(1)(\delta+\epsilon+\tau^2)\delta$,

where the bounds of $O(1)$ depend only on $\underline{U}$.

Proof. First, using estimate (4.54) and the Taylor formula, for $\delta > 0$ sufficiently small, we have

$$\ln\beta_1 = \beta_1 - 1 + O(1)(\beta_1 - 1)^2 = a_\infty\delta + O(1)(\delta + \epsilon + \tau^2)\delta. \tag{A.1}$$

Then, using estimate (A.1) and applying the Taylor formula again, we obtain

$$\beta_1^\epsilon = 1 + \epsilon\ln\beta_1 + \epsilon^2(\ln\beta_1)^2\int_0^1(1-t)\beta^{\epsilon t}\mathrm{d}t \tag{A.2}$$

$$= 1 + a_\infty\delta + O(1)(\delta + \epsilon + \tau^2)\delta\epsilon. \tag{A.3}$$

Therefore, estimate (i) can be obtained from (A.2). In the similar way, we can also show estimate (ii) with the help of (A.1).

Next, we turn to consider estimate (iii). To this end, we set

$$\psi(\epsilon) = (\epsilon\ln\beta_1 - 1)\beta_1^\epsilon + 1.$$

Then a direct calculation shows that $\psi(0) = 0$ and

$$\psi'(\epsilon) = (\ln\beta_1)^2\epsilon\beta_1^\epsilon, \qquad \psi''(\epsilon) = (\ln\beta_1)^2\big(1 + \epsilon\ln\beta_1\big)\beta_1^\epsilon, \qquad \psi'''(\epsilon) = (\ln\beta_1)^3\big(2 + \epsilon\ln\beta_1\big)\beta_1^\epsilon,$$

which satisfy

$$\psi'(0) = 0, \qquad \psi''(0) = (\ln\beta_1)^2.$$

Thus, by (A.2) and the Taylor formula, we obtain

$$\begin{aligned}\psi(\epsilon) &= \frac{1}{2}(\ln\beta_1)^2\epsilon^2 + \frac{1}{2}(\ln\beta_1)^3\epsilon^3\int_0^1(1-t)^2(2 + t\epsilon\ln\beta_1)\beta_1^{t\epsilon}\,\mathrm{d}t\\ &= \frac{1}{2}a_\infty^2\delta^2\epsilon^2 + O(1)(\delta + \epsilon + \tau^2)\delta^2\epsilon^2,\end{aligned}$$

which leads to estimate (iii). In the same way, we can also show estimates (iv)–(v). □

Lemma A.2. *Let $\mathcal{B}^{(\epsilon)}$ be defined by (4.58) with β_1 and $\varphi_1^{(\boldsymbol{\mu})}$ giving in (4.50) and satisfying (4.54) for β_1. Then, for $\delta > 0$, $\epsilon > 0$, and $\tau > 0$ sufficiently small,*

$$\mathcal{B}^{(\epsilon)} = \frac{2}{a_\infty}\delta + O(1)(\delta + \epsilon + \tau^2)\delta, \tag{A.4}$$

where the bounds of $O(1)$ depend only on $\underline{U}$.

Proof. Using the Taylor formula, Lemma 2.6, and estimate (4.54), for $\delta > 0$, $\epsilon > 0$, and $\tau > 0$ sufficiently small, we have

$$\begin{aligned}\varphi_1^{(\boldsymbol{\mu})} &= \varphi_1^{(\boldsymbol{\mu})}\Big|_{\beta_1=1} + \frac{\partial\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1}\Big|_{\beta_1=1}(\beta_1 - 1) + O(1)(\beta_1 - 1)^2\\ &= \Big(\frac{\partial\varphi_1^{(\boldsymbol{\mu})}}{\partial\beta_1}\Big|_{\beta_1=1,\epsilon=\tau=0} + O(1)(\epsilon + \tau^2)\Big)\big(a_\infty\delta + O(1)(\epsilon + \tau^2)\delta\big)\\ &\quad + O(1)\big(a_\infty\delta + O(1)(\epsilon + \tau^2)\delta\big)^2\\ &= -\delta + O(1)(\epsilon + \tau^2)\delta,\end{aligned} \tag{A.5}$$

which implies

$$\varphi_1^{(\boldsymbol{\mu})} + \delta = O(1)(\epsilon + \tau^2)\delta. \tag{A.6}$$

Then combining the (A.6) with estimate (i) in Lemma (A.1) leads to estimate (A.4). □

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