# On linear-combinatorial problems associated with subspaces spanned by $\{ \pm 1\}$-vectors 

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#### Abstract

A complete answer to the question about subspaces generated by $\{ \pm 1\}$-vectors, which arose in the work of I. Kanter and H. Sompolinsky on associative memories, is given. More precisely, let vectors $v_{1}, \ldots, v_{p}$, $p \leq n-1$, be chosen at random uniformly and independently from $\{ \pm 1\}^{n} \subset \mathbf{R}^{n}$. Then the probability $\mathbb{P}(p, n)$ that $$
\operatorname{span}\left\langle v_{1}, \ldots, v_{p}\right\rangle \cap\left\{\{ \pm 1\}^{n} \backslash\left\{ \pm v_{1}, \ldots, \pm v_{p}\right\}\right\} \neq \emptyset
$$


is shown to be

$$
4\binom{p}{3}\left(\frac{3}{4}\right)^{n}+O\left(\left(\frac{5}{8}+o_{n}(1)\right)^{n}\right) \quad \text { as } \quad n \rightarrow \infty
$$

where the constant implied by the $O$-notation does not depend on $p$. The main term in this estimate is the probability that some 3 vectors $v_{j_{1}}, v_{j_{2}}, v_{j_{3}}$ of $v_{j}, j=1, \ldots, p$, have a linear combination that is a $\{ \pm 1\}$ vector different from $\pm v_{j_{1}}, \pm v_{j_{2}}, \pm v_{j_{3}}$.

Keywords. $\{ \pm 1\}$-vector, Threshold function, singular Bernoulli matrices, $\eta^{\star}$-function.

## 1 Introduction.

In the paper [11], A.M.Odlyzko gave a partial answer to the following question that arose in the paper [8] on associative memories. Let vectors $v_{1}, \ldots, v_{p}$ be chosen at random uniformly and independently from $\{ \pm 1\}^{n} \subset \mathbf{R}^{n}$. What is the probability $\mathbb{P}(p, n)$ that the subspace spanned by $v_{1}, \ldots, v_{p}$ over reals contains a $\{ \pm 1\}$-vector different from $\pm v_{1}, \ldots, \pm v_{p}$, i.e.

$$
\operatorname{span}\left\langle v_{1}, \ldots, v_{p}\right\rangle \cap\left\{\{ \pm 1\}^{n} \backslash\left\{ \pm v_{1}, \ldots, \pm v_{p}\right\}\right\} \neq \emptyset ?
$$

G.Kalai and N.Linial conjectured (see [11]) that this probability is dominated by the probability $\mathbb{P}_{3}(p, n)$ that some 3 vectors $v_{j_{1}}, v_{j_{2}}, v_{j_{3}}$ of $v_{j}, j=1, \ldots, p$,

[^0]have a linear combination that is a $\{ \pm 1\}$-vector different from $\pm v_{j_{1}}, \pm v_{j_{2}}, \pm v_{j_{3}}$, where
\[

$$
\begin{equation*}
\mathbb{P}_{3}(p, n)=4\binom{p}{3}\left(\frac{3}{4}\right)^{n}+O\left(p^{4}\left(\frac{5}{8}\right)^{n}\right) \quad \text { as } \quad n \rightarrow \infty \tag{1.1}
\end{equation*}
$$

\]

for $3 \leq p \leq n$. In the paper [11] this conjecture was proven for

$$
\begin{equation*}
p \leq n-\frac{10 n}{\ln n} \tag{1.2}
\end{equation*}
$$

Theorem 1.1 (A.M.Odlyzko [11]) If $p \leq n-\frac{10 n}{\ln n}$ and vectors $v_{1}, \ldots, v_{p}$ are chosen at random uniformly and independently from $\{ \pm 1\}^{n} \subset \mathbf{R}^{n}$, then the probability $\mathbb{P}(p, n)$ that the subspace spanned by $v_{1}, \ldots, v_{p}$ over reals contains $a$ $\{ \pm 1\}$-vector different from $\pm v_{1}, \ldots, \pm v_{p}$ equals

$$
\begin{equation*}
\mathbb{P}(p, n)=\mathbb{P}_{3}(p, n)+O\left(\left(\frac{7}{10}\right)^{n}\right) \quad \text { as } \quad n \rightarrow \infty . \tag{1.3}
\end{equation*}
$$

The constants implied by the O-notation in (1.1) and (1.3) are independent of $p$.

The paper 11 made a significant contribution to estimating the number of threshold functions $P(2, n)$. Namely, in the paper [17], as a corollary of Theorem 1, T.Zaslavsky's formula [16] and G.-C.Rota's theorem on the inequality of the Möbius function of a geometric lattice to zero (Theorem 4 [12, p.357), the asymptotics of the logarithm of $P(2, n)$ was obtained. In [3], the Theorem 1 was used together with the original $(A, B, C)$-construction to improve the lower bound of $P(2, n)$ obtained in [17] by a factor of $\sim P\left(2,\left\lfloor\frac{7 n \ln 2}{\ln n}\right\rfloor\right)$. In [4], the Theorem 1 was generalized to the case of $E_{K}$-vectors, where $E_{K}=\{0, \pm 1, \ldots, \pm Q\}$ if $K=2 Q+1$, and $E_{K}=\{ \pm 1, \pm 3, \ldots, \pm(2 Q-1)\}$ if $K=2 Q$, to obtain the asymptotics of the logarithm of the number of threshold functions of K-valued logic.

In (11, A.M.Odlyzko noted that removing constraint (1.2) is an open and hard problem. The hardness of this problem is manifested in the fact (see [7], p. 238), that if $\mathbb{P}(n-1, n)$ tends to zero as $n \rightarrow \infty$, then one could obtain the asymptotics of $P(2, n)$ :

$$
\begin{equation*}
P(2, n) \sim 2\binom{2^{n}-1}{n}, \quad n \rightarrow \infty \tag{1.4}
\end{equation*}
$$

We will show this fact (see Corollary 2.3) using the properties of $\eta_{n}^{\star}$ function from the paper [5].

Also, in [14], one can see an implicit connection between estimates of the probability $\mathbb{P}(n-1, n)$ and the probability $\mathbb{P}_{n}$ that a random Bernoulli $n \times n$ $\{ \pm 1\}$-matrix $M_{n}$ is singular:

$$
\mathbb{P}_{n} \stackrel{\text { def }}{=} \operatorname{Pr}\left(\operatorname{det} M_{n}=0\right) .
$$

In this work, T. Tao and V. Vu investigate the properties of the combinatorial Grassmannian $G r$, consisting of hyperplanes $V$ in an n-dimensional space over a finite field $\mathbb{F},|\mathbb{F}|=p>n^{\frac{n}{2}}$, where $p$ is a prime number, such that $V=$ span $\left\langle V \cap\{ \pm 1\}^{n}\right\rangle$, and estimate $\mathbb{P}_{n}$ based on the formula

$$
\mathbb{P}_{n}=2^{o(n)} \sum_{V \in G r, V \text { is a non-trival hyperplane in } \mathbb{F}^{n}} \mathbb{P}\left(A_{V}\right) .
$$

Here $A_{V}$ denotes the event that vectors $v_{1}, \ldots, v_{n}$, chosen at random uniformly and independently from $\{ \pm 1\}^{n}$, span $V$, and $\mathbb{P}\left(A_{V}\right)$ denotes the probability of this event. We can essentially improve the upper bound of $\mathbb{P}\left(A_{V}\right)$ in the Small combinatorial dimension estimate Lemma from [7] (see inequality (6) in Lemma 2.3. from [14]) if we substitute the probability $\mathbb{P}\left(v_{1}, \ldots, v_{n-1}\right.$ span $\left.V\right)$ by the probability

$$
\mathbb{P}\left(\left\{v_{1}, \ldots, v_{n-1} \text { span } V\right\} \wedge\left\{V \cap\left\{\{ \pm 1\}^{n} \backslash\left\{ \pm v_{1}, \ldots, \pm v_{n-1}\right\}\right\} \neq \emptyset\right\}\right)
$$

On the other hand, J. Kahn, J. Komlós, and E. Szemerédi in 7], relying on a technique developed to estimate $\mathbb{P}_{n}$, showed (corollary 4, [7) that there is a constant $C$ such that the Theorem 1 is true for $p \leq n-C$.

In this paper, using the asymptotics of the probability $\mathbb{P}_{n}$ (Theorem 6, [5])

$$
\begin{equation*}
\mathbb{P}_{n} \sim(n-1)^{2} 2^{1-n}, \quad n \rightarrow \infty \tag{1.5}
\end{equation*}
$$

and the lower bound for $P(2, n)$ (inequality (139) of the Theorem 7 [5])

$$
\begin{equation*}
P(2, n) \geq 2\left[1-\frac{n^{2}}{2^{n}}\left(1+o\left(\frac{n^{3}}{2^{n}}\right)\right)\right]\binom{2^{n}-1}{n} \tag{1.6}
\end{equation*}
$$

we remove in the Theorem 1 the restriction (1.2) and prove the conjecture of G.Kalai, N.Linial, and A.M.Odlysko for $p \leq n-1$.

Theorem 1.2 (Main theorem) If $p \leq n-1$ and vectors $v_{1}, \ldots, v_{p}$ are chosen at random uniformly and independently from $\{ \pm 1\}^{n} \subset \mathbf{R}^{n}$, then the probability $\mathbb{P}(p, n)$ that the subspace spanned by $v_{1}, \ldots, v_{p}$ over reals contains a $\{ \pm 1\}$-vector different from $\pm v_{1}, \ldots, \pm v_{p}$ equals

$$
\begin{equation*}
\mathbb{P}(p, n)=\mathbb{P}_{3}(p, n)+O\left(\left(\frac{5}{8}+o_{n}(1)\right)^{n}\right) \quad \text { as } \quad n \rightarrow \infty \tag{1.7}
\end{equation*}
$$

The constants implied by the O-notation in (1.1) and 1.7) are independent of p.

## 2 Interdependence between $\eta_{n}^{\star}$-function, $\mathbb{P}(n, n+$

 1 ), and the fraction of singular $(n+1) \times(n+1)$ $\{ \pm 1\}$-matrices with different rows.Let $\langle H\rangle=\left\{\left\langle w_{1}\right\rangle, \ldots,\left\langle w_{T}\right\rangle\right\} \subset \mathbf{R P}^{n}$ be a subset of the $n$-dimensional projective space, where points $\left\langle w_{i}\right\rangle, i=1, \ldots, T$, are represented by lines $t w_{i} \subset \mathbf{R}^{n+1}$ and $w_{i} \in \mathbf{R}^{n+1}, i=1, \ldots, T$. Let $K^{H}$ be a simplicial compex defined as follows. The set of vertices of $K^{H}$ coincides with the set $\langle H\rangle$. A subset $\left\{\left\langle w_{i_{1}}\right\rangle, \ldots,\left\langle w_{i_{s}}\right\rangle\right\}$ of $\langle H\rangle$ forms a simplex of $K^{H}$ iff

$$
\operatorname{span}\left\langle w_{i_{1}}, \ldots, w_{i_{s}}\right\rangle \neq \operatorname{span}\left\langle w_{1}, \ldots, w_{T}\right\rangle .
$$

We define on the set $2{\underset{f i n}{ }{ }^{\mathbf{R}}{ }^{n} \text { of finite subsets of } \mathbf{R} \mathbf{P}^{n} \text { the function } \eta_{n}^{\star}: 2_{f \text { in }}{ }^{\mathbf{R P}^{n}} \rightarrow}$ $\mathbb{Z}_{\geq 0}$ by the following formula (see [5]):

$$
\begin{equation*}
\eta_{n}^{\star}(\langle H\rangle)=\operatorname{rank} \tilde{H}_{n-1}\left(K^{H} ; \mathbf{F}\right), \quad\langle H\rangle \subset \mathbf{R} \mathbf{P}^{n} \tag{2.1}
\end{equation*}
$$

if

$$
\operatorname{span}\left\langle w_{1}, \ldots, w_{T}\right\rangle=\mathbf{R}^{n+1}
$$

and

$$
\begin{equation*}
\eta_{n}^{\star}(\langle H\rangle)=0 \tag{2.2}
\end{equation*}
$$

if

$$
\operatorname{span}\left\langle w_{1}, \ldots, w_{T}\right\rangle \neq \mathbf{R}^{n+1}
$$

Here $\tilde{H}_{n-1}\left(K^{H} ; \mathbf{F}\right)$ denotes the reduced homology group of the complex $K^{H}$ with coefficients in an arbitrary field $\mathbf{F}$.

Let us denote by $\langle H\rangle^{\times s}, s=1, \ldots, T$, the set of ordered collections $\left(\left\langle w_{i_{1}}\right\rangle, \ldots,\left\langle w_{i_{s}}\right\rangle\right)$ of different $s$ elements from $\langle H\rangle$ and $\langle H\rangle_{\neq 0}^{\times s} \subset\langle H\rangle^{\times s},\langle H\rangle_{=0}^{\times s} \subset$ $\langle H\rangle^{\times s}$ be the subsets

$$
\begin{align*}
& \langle H\rangle \neq 0 \times \text { def }=\left\{\left(\left\langle w_{i_{1}}\right\rangle, \ldots,\left\langle w_{i_{s}}\right\rangle\right) \in\langle H\rangle^{\times s} \mid \text { dim } \operatorname{span}\left\langle w_{i_{1}}, \ldots, w_{i_{s}}\right\rangle=s\right\} .  \tag{2.3}\\
& \left.\langle H\rangle=0 \times \text { def }=\left(\left\langle w_{i_{1}}\right\rangle, \ldots,\left\langle w_{i_{s}}\right\rangle\right) \in\langle H\rangle^{\times s} \mid \operatorname{dim} \operatorname{span}\left\langle w_{i_{1}}, \ldots, w_{i_{s}}\right\rangle<s\right\} . \tag{2.4}
\end{align*}
$$

For any $W=\left(\left\langle w_{i_{1}}\right\rangle, \ldots,\left\langle w_{i_{n}}\right\rangle\right) \in\langle H\rangle^{\times n}$ and $l=1, \ldots, n$, let

$$
\begin{align*}
& L_{l}(W) \stackrel{\text { def }}{=} \operatorname{span}\left\langle w_{i_{n-l+1}}, \ldots, w_{i_{n}}\right\rangle \subset \mathbf{R}^{n+1} ; \\
& q_{l}^{W}(H) \stackrel{\text { def }}{=}\left|L_{l}(W) \cap\langle H\rangle\right| . \tag{2.5}
\end{align*}
$$

Definition 2.1 For any $W \in\langle H\rangle^{\times n}$, the ordered set of numbers

$$
\begin{equation*}
W(\langle H\rangle) \stackrel{\text { def }}{=}\left(q_{n}^{W}(H), q_{n-1}^{W}(H), \ldots, q_{1}^{W}(H)\right) \tag{2.6}
\end{equation*}
$$

is called a combinatorial flag on $\langle H\rangle \subset \mathbf{R P}^{n}$ of the ordered set $W$.

For the sake of simplicity, we will use the following notation:

$$
\begin{equation*}
W[H] \stackrel{\text { def }}{=} q_{n}^{W}(H) \cdot q_{n-1}^{W}(H) \cdots q_{1}^{W}(H) \tag{2.7}
\end{equation*}
$$

In [5], it was proven the following theorem.
Theorem 2.2 ([5], [6) For any $p=\left(p_{1}, \ldots, p_{T}\right), p_{i} \in \mathbf{R}, i=1, \ldots, T$, such that $\sum_{i=1}^{T} p_{i}=1$, and subset $\langle H\rangle=\left\{\left\langle w_{1}\right\rangle, \ldots,\left\langle w_{T}\right\rangle\right\} \subset \mathbf{R P}^{n}$, such that span $\langle H\rangle=\mathbf{R}^{n+1}$, the following equality is true:

$$
\begin{equation*}
\eta_{n}^{\star}(\langle H\rangle)=\sum_{W \in\langle H\rangle \ngtr \neq 0} \frac{1-p_{i_{1}}-p_{i_{2}}-\cdots-p_{i_{q_{n}^{W}}}}{W[H]} . \tag{2.8}
\end{equation*}
$$

Here, the indices used in the numerator correspond to elements from

$$
L_{n}(W) \cap\langle H\rangle=\left\{\left\langle w_{i_{1}}\right\rangle, \ldots,\left\langle w_{i_{n}}\right\rangle, \ldots,\left\langle w_{i_{q_{n}^{W}}}\right\rangle\right\} .
$$

Let

$$
\begin{equation*}
E_{n}=\left\{\left(1, b_{1}, \ldots, b_{n}\right) \mid b_{i} \in\{ \pm 1\}, i=1, \ldots, n\right\} \tag{2.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\{E_{n}\right\}^{p}=\underbrace{E_{n} \times \cdots \times E_{n}}_{p} \tag{2.10}
\end{equation*}
$$

We say that an ordered collection $W=\left(w_{1}, \ldots, w_{p}\right), w_{i} \in\{ \pm 1\}^{n+1}, i=$ $1, \ldots, p$, satisfies to KSO-condition, and we write $W \in K S O(p, n+1)$, iff

$$
\begin{equation*}
\operatorname{span}\left\langle w_{1}, \ldots, w_{p}\right\rangle \cap\left\{\{ \pm 1\}^{n+1} \backslash\left\{ \pm w_{1}, \ldots, \pm w_{p}\right\}\right\} \neq \emptyset \tag{2.11}
\end{equation*}
$$

Then

$$
\mathbb{P}(n, n+1)=\frac{|K S O(n, n+1)|}{2^{n(n+1)}}=\frac{\left|\left\{W \in\left\{E_{n}\right\}^{n} \mid W \in K S O(n, n+1)\right\}\right|}{2^{n^{2}}}
$$

and

$$
\begin{align*}
& \left|K S O(n, n+1) \cap\left\langle E_{n}\right\rangle_{\neq 0}^{\times n}\right|= \\
& =\left|\left\{W \in\left\langle E_{n}\right\rangle_{\neq 0}^{\times n} \mid W \in K S O(n, n+1)\right\}\right|<2^{n^{2}} \mathbb{P}(n, n+1) . \tag{2.12}
\end{align*}
$$

Corollary 2.3 If $\lim _{n \rightarrow \infty} \mathbb{P}(n, n+1)=0$, then $P(2, n) \sim 2\binom{2^{n}-1}{n}, n \rightarrow \infty$.
Proof. Let us apply Theorem 2.2 to the set $H=E_{n} \subset \mathbf{R}^{n+1}$ and the collection of weights $p=(1,0, \ldots, 0)$, where $w_{1}=(1, \ldots, 1) \in \mathbf{R}^{n+1}$. Then

$$
\begin{equation*}
\eta_{n}^{\star}\left(\left\langle E_{n}\right\rangle\right)=\Sigma_{1}+\Sigma_{2}-\Sigma_{3}-\Sigma_{4} \tag{2.13}
\end{equation*}
$$

where

From (2.12), we have

$$
\begin{align*}
\Sigma_{1} & =\frac{\left|\left\langle E_{n}\right\rangle_{\neq 0}^{\times n}\right|-\left|K S O(n, n+1) \cap\left\langle E_{n}\right\rangle_{\neq 0}^{\times n}\right|}{n!}> \\
& >\frac{2^{n} \cdots\left(2^{n}-n+1\right)-2^{n^{2}} \mathbb{P}_{n}-\left|K S O(n, n+1) \cap\left\langle E_{n}\right\rangle_{\neq 0}^{\times n}\right|}{n!}>  \tag{2.14}\\
& >\binom{2^{n}}{n}-\binom{2^{n}}{n} \frac{2^{n^{2}}}{2^{n}\left(2^{n}-1\right) \cdots\left(2^{n}-n+1\right)}\left(\mathbb{P}_{n}+\mathbb{P}(n, n+1)\right),
\end{align*}
$$

$$
\begin{equation*}
\Sigma_{3}<\mathbb{P}_{n}(\{0,1\}) \frac{2^{n^{2}}}{2^{n}\left(2^{n}-1\right) \cdots\left(2^{n}-n+1\right)}\binom{2^{n}}{n} \tag{2.15}
\end{equation*}
$$

$$
\begin{equation*}
\Sigma_{4}<n \frac{\left(2^{n}-1\right)\left(2^{n}-2\right) \cdots\left(2^{n}-n+1\right)}{n!}=\binom{2^{n}-1}{n-1} \tag{2.16}
\end{equation*}
$$

Here $\mathbb{P}_{n}(\{0,1\})$ denotes the probability that a $(0,1)-n \times n$-matrix, with entries chosen at random, uniformly, and independenly from $\{0,1\}$, is singular. It follows from (2.13), (2.14), (2.15), and (2.16) that

$$
\begin{equation*}
\left.\eta_{n}^{\star}\left(\left\langle E_{n}\right\rangle\right)>\binom{2^{n}-1}{n}\left(1-c_{n}\right)\right) \tag{2.17}
\end{equation*}
$$

where

$$
c_{n}=\frac{2^{n^{2}}}{\left(2^{n}-1\right) \cdots\left(2^{n}-n\right)}\left(\mathbb{P}_{n}+\mathbb{P}(n, n+1)+\mathbb{P}_{n}(\{0,1\})\right)
$$

Taking into account the inequality (see the formulas (19) and (25) of [5])

$$
P(2, n) \geq 2 \eta_{n}^{\star}\left(\left\langle E_{n}\right\rangle\right)
$$

$$
\begin{aligned}
& \Sigma_{1}=\sum_{W \in\left\langle E_{n}\right\rangle_{\substack{\times n \\
\neq 0}} \sum_{W \notin K S O(n, n+1)} \frac{1}{W\left[E_{n}\right]}, ~}^{\substack{W \neq}} \\
& \Sigma_{2}=\sum_{\substack{W \in\left\langle E_{n}\right\rangle \times n \\
\neq 0}} \sum_{W \in K S O(n, n+1)} \frac{1}{W\left[E_{n}\right]}, \\
& \Sigma_{3}=\sum_{W \in\left\langle E_{n}\right\rangle \neq 0}^{\substack{\times n \\
\neq 0}} \sum_{w_{1} \notin\{W\}, w_{1} \in \operatorname{span}\langle W\rangle} \frac{1}{W\left[E_{n}\right]}, \\
& \Sigma_{4}=\sum_{W \in\left\langle E_{n}\right\rangle \times n}^{\substack{\times n \\
\neq w_{1} \in\{W\}}}{ }^{W\left[E_{n}\right]} .
\end{aligned}
$$

L. Schläfli's upper bound (see the formula 2 in [5] and [13])

$$
P(2, n) \leq 2 \sum_{i=0}^{n}\binom{2^{n}-1}{i}
$$

the given fact that $\mathbb{P}(n, n+1) \rightarrow 0$, and the well known results $\mathbb{P}_{n}(\{0,1\}) \rightarrow 0$, and $\mathbb{P}_{n} \rightarrow 0$ as $n \rightarrow \infty$ (see [9], [10], [7], [14], [1], [15], [5]), we can conclude that

$$
c_{n} \rightarrow 0 \quad \text { as } n \rightarrow \infty
$$

and

$$
P(2, n) \sim 2\binom{2^{n}-1}{n}, n \rightarrow \infty
$$

Q.E.D.

We define $\delta_{n, k}, k=1, \ldots, n+1$, as

$$
\delta_{n, k} \stackrel{\text { def }}{=} \frac{\left|\left\langle E_{n}\right\rangle_{=0}^{\times k}\right|}{\left|\left\langle E_{n}\right\rangle^{\times k}\right|}, \quad k=1, \ldots, n+1 .
$$

For $W=\left(\left\langle w_{i_{1}}\right\rangle, \ldots,\left\langle w_{i_{n}}\right\rangle\right) \in\left\langle E_{n}\right\rangle_{\neq 0}^{\times n}$, we use the following notations:

$$
\begin{aligned}
& L_{n}(W) \stackrel{\text { def }}{=} \operatorname{span}\left\langle w_{i_{1}}, \ldots, w_{i_{n}}\right\rangle \subset \mathbf{R}^{n+1} ; \\
& q_{n}^{W} \stackrel{\text { def }}{=}\left|L_{n}(W) \cap E_{n}\right| ; \\
& E_{n}^{m} \stackrel{\text { def }}{=}\left\{W \in\left\langle E_{n}\right\rangle_{\neq 0}^{\times n} \mid q_{n}^{W}=n+m\right\}, \quad m=0,1, \ldots, 2^{n-1}-n \text {. }
\end{aligned}
$$

Theorem 2.4 For sufficiently large $n$, we have

$$
\begin{equation*}
\frac{\left|K S O(n, n+1) \cap\left\langle E_{n}\right\rangle_{\neq 0}^{\times n}\right|}{2^{n^{2}}} \leq \frac{n^{2}}{2^{n-1}} \tag{2.18}
\end{equation*}
$$

Proof. Let us take a vector $w \in \mathbf{R}^{n+1}$ in general position to the set $E_{n}$, i.e. for any vectors $w_{i_{1}}, \ldots, w_{i_{n}} \in E_{n} \subset \mathbf{R}^{n+1}$,

$$
w \notin \operatorname{span}\left\langle w_{i_{1}}, \ldots, w_{i_{n}}\right\rangle
$$

From the Theorem 2.2 applyed to the set $H=\left\langle E_{n}\right\rangle \cup\langle w\rangle \subset \mathbf{R P}^{n}$ and the collection of weights $p(w)=1$ and $p\left(w_{i}\right)=0$ for $w_{i} \in E_{n}, i=1, \ldots 2^{n}$, we get

$$
\begin{aligned}
& \eta_{n}^{\star}\left(\left\langle E_{n}\right\rangle \cup\langle w\rangle\right)=\sum_{W \in\left\langle E_{n}\right\rangle_{\neq 0}^{\times n}} \frac{1}{W\left[E_{n}\right]}= \\
& =\sum_{m=0}^{2^{n-1}-n} \sum_{W \in E_{n}^{m}} \frac{1}{W\left[E_{n}\right]}=\frac{1}{n!} \sum_{m=0}^{2^{n-1}-n}\left|E_{n}^{m}\right|-\sum_{m=1}^{2^{n-1}-n} \sum_{W \in E_{n}^{m}}\left(\frac{1}{n!}-\frac{1}{W\left[E_{n}\right]}\right) \leq \\
& \left(\frac{1}{n!}-\frac{1}{W\left[E_{n}\right]} \geq \frac{1}{n!}-\frac{1}{(n-1)!(n+m)}=\frac{m}{n!(n+m)}\right) \\
& \leq \frac{1}{n!}\left(\left|\left\langle E_{n}\right\rangle^{\times n}\right|\left(1-\delta_{n, n}\right)-\frac{1}{n+1}\left|\cup_{m=1}^{2^{n-1}-n} E_{n}^{m}\right|\right)
\end{aligned}
$$

$$
\begin{align*}
\eta_{n}^{\star}\left(\left\langle E_{n}\right\rangle \cup\langle w\rangle\right) & \leq\binom{ 2^{n}}{n}\left(1-\delta_{n, n}\right)-  \tag{2.19}\\
& -\frac{1}{n!(n+1)}\left|K S O(n, n+1) \cap\left\langle E_{n}\right\rangle \not{ }_{\neq 0}\right| .
\end{align*}
$$

From the Theorem 5 of the paper [5], we have (see the formula (135) in [5]):

$$
\begin{equation*}
\eta_{n}^{\star}\left(\left\langle E_{n}\right\rangle \cup\{\langle w\rangle\}\right) \geq\binom{ 2^{n}}{n}\left[1-\delta_{n, n}-\frac{n-1}{2^{n-1}}\left(1+o\left(\frac{n^{3}}{2^{n}}\right)\right)\right] . \tag{2.20}
\end{equation*}
$$

Combining inequalities (2.19) and (2.20), we get

$$
\begin{equation*}
\left|K S O(n, n+1) \cap\left\langle E_{n}\right\rangle_{\neq 0}^{\times n}\right| \leq 2^{n} \cdots\left(2^{n}-n+1\right) \frac{n^{2}-1}{2^{n-1}}\left(1+o\left(\frac{n^{3}}{2^{n}}\right)\right) . \tag{2.21}
\end{equation*}
$$

The Theorem 2.4 follows from the inequality (2.21).
Q.E.D.

## 3 Proof of the Main Theorem.

Let $\mathbb{P}_{m}(p, n), m \leq p \leq n-1$, denote the probability that in the set of $p$ vectors $v_{1}, \ldots, v_{p} \in\{ \pm 1\}^{n} \subset \mathbf{R}^{n}$, chosen at random uniformly and independently, there are some $m$ vectors $v_{j_{1}}, \ldots, v_{j_{m}}$ such that

$$
\alpha_{1} v_{j_{1}}+\cdots+\alpha_{m} v_{j_{m}} \in\{ \pm 1\}^{n} \text { for some } \alpha_{1}, \ldots, \alpha_{m} \in \mathbb{R} \backslash\{0\} .
$$

Let $\mathcal{M}_{m}(p, n)$ denote the set of $(p \times n)$ - $\{ \pm 1\}$-matrices $M$ with linear independent rows $w_{1}, \ldots, w_{p} \in\{ \pm 1\}^{n}$ satisfying the following property. There are a subset of $m$ rows $w_{i_{1}}, \ldots w_{i_{m}}$ and some nonzero coefficients $\alpha_{1}, \ldots, \alpha_{m} \in \mathbb{R} \backslash\{0\}$ such that

$$
\alpha_{1} w_{i_{1}}+\cdots+\alpha_{m} w_{i_{m}} \in\{ \pm 1\}^{n} .
$$

Let $\mathcal{Q}(p, n)$ be the the set of $(p \times n)$ - $\{ \pm 1\}$-matrices $M$ with rank less than $p$ $(<p)$. Denote by $R_{m}(p, n)$ the probability that a $(p \times n)-\{ \pm 1\}$-matrix $M$ chosen at random belongs to $\mathcal{M}_{m}(p, n)$ and by $\mathbb{P}_{p, n}$ the probability that a $(p \times n)$ - $\{ \pm 1\}-$ matrix $M$ chosen at random has rank less than $p(<p)$. Then

$$
\begin{gathered}
K S O(p, n) \subset \bigcup_{m=3}^{p} \mathcal{M}_{m}(p, n) \bigsqcup \mathcal{Q}(p, n), \\
R_{m}(p, n) \leq\binom{ p}{m} R_{m}(m, n),
\end{gathered}
$$

$$
\begin{equation*}
\mathbb{P}_{m}(p, n) \leq R_{m}(p, n)+\mathbb{P}_{p, n} \leq\binom{ p}{m} R_{m}(m, n)+\mathbb{P}_{p, n} \tag{3.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{P}(p, n) \leq \sum_{m=3}^{p} R_{m}(p, n)+\mathbb{P}_{p, n} \leq \sum_{m=3}^{p}\binom{p}{m} R_{m}(m, n)+\mathbb{P}_{p, n} \tag{3.3}
\end{equation*}
$$

It follows from [5] (see Lemma 5 and the proof of the Theorem 6) that for $p=1, \ldots, n$

$$
\begin{equation*}
\mathbb{P}_{p, n} \leq \frac{(p-1)^{2}}{2^{n-1}}\left(1+o_{n}(1)\right) \tag{3.4}
\end{equation*}
$$

The proof of the Theorem 1.2 is divided into 3 cases of evaluation $R_{m}(p, n)$ :
Case 1. $5 \leq m \leq \frac{n}{a(\epsilon)}, \quad a(\epsilon)=\frac{1}{\epsilon^{2}}, \quad 0<\epsilon<\frac{1}{100}, \quad m \leq p \leq n-1 ;$
Case 2. $\frac{n}{a(\epsilon)}<m \leq n-\frac{c n}{\log _{2} n}, \quad c \geq 7.36, \quad m \leq p \leq n-1$;
Case 3. $n-\frac{c n}{\log _{2} n}<m \leq n-1, \quad c \geq 7.36, \quad m \leq p \leq n-1$.
It was shown in the paper [11] that

$$
\begin{equation*}
\mathbb{P}_{4}(p, n)=O\left(p^{4} 2^{-n}\right) \quad \text { as } n \rightarrow \infty, \text { for } \quad 4 \leq p \leq n-1 \tag{3.7}
\end{equation*}
$$

The proofs of cases 1 and 2 repeat some arguments of the papers [11] and 4]. Here we present the proofs of cases 1 and 2 for completeness of presentaton and clarification of some constants. The proof of case 3 is based on Theorem 2.4.
3.1 Case $1: 5 \leq m \leq \frac{n}{a}, \quad m \leq p \leq n-1$.

Lemma 3.1 For any $\epsilon$, $m$, $p$, such that $0<\epsilon<\frac{1}{100}, 5 \leq m \leq \frac{n}{a}$, where $a=a(\epsilon)=\frac{1}{\epsilon^{2}}$, and $m \leq p \leq n-1$, we have

$$
R_{m}(p, n)<\left(\frac{5}{8}\right)^{n}(1+\epsilon)^{n} \quad \text { as } n \rightarrow \infty .
$$

Proof. Let $M \in \mathcal{M}_{m}(m, n)$. Denote by $w_{1}, \ldots, w_{m}$ the rows of $M$. If columns $1 \leq j_{1}<\cdots<j_{m} \leq n$ of the matrix $M$ are linearly independent, then for each choice of $\beta_{1}, \ldots, \beta_{m} \in\{ \pm 1\}$, there will be a unique set of coefficients $\alpha_{1}, \ldots, \alpha_{m}$ with $j_{s}$ th coordinate of the vector $\alpha M=\left((\alpha M)_{1}, \ldots,(\alpha M)_{n}\right) \stackrel{\text { def }}{=}$ $\alpha_{1} w_{1}+\cdots+\alpha_{m} w_{m}$ equals to $\beta_{s}, s=1, \ldots, m$. Hence, there are at most $2^{m}$ sets $\alpha_{1}, \ldots, \alpha_{m} \in \mathbb{R} \backslash\{0\}$ such that $(\alpha M)_{j}=+1$ or -1 for $j=j_{1}, \ldots, j_{m}$. For each fixed vector $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right), \alpha_{i} \in \mathbb{R} \backslash\{0\}, i=1, \ldots, m$, probability that $(\alpha M)_{j}=+1$ or -1 for $j \neq j_{1}, \ldots, j_{m}$, is at most

$$
2 \cdot 2^{-m}\binom{m}{\left\lfloor\frac{m}{2}\right\rfloor}
$$

by Erdös-Littlewood-Offord lemma (see [2]). Since all columns $j, j \neq j_{1}, \ldots, j_{m}$, we choose independently of each other, we have

$$
\begin{align*}
& R_{m}(m, n) \leq 2^{m}\binom{n}{m}\left[2 \cdot 2^{-m}\binom{m}{\left\lfloor\frac{m}{2}\right\rfloor}\right]^{n-m}= \\
& =2^{n}\binom{n}{m}\left[2^{-m}\binom{m}{\left\lfloor\frac{m}{2}\right\rfloor}\right]^{n-m} \tag{3.8}
\end{align*}
$$

Taking into account (3.1) and (3.8), we get

$$
\begin{equation*}
R_{m}(p, n) \leq 2^{n}\binom{p}{m}\binom{n}{m}\left[2^{-m}\binom{m}{\left\lfloor\frac{m}{2}\right\rfloor}\right]^{n-m} \tag{3.9}
\end{equation*}
$$

For $5 \leq m \leq \frac{n}{a}, \quad m \leq p \leq n-1$, we have

$$
\begin{aligned}
2^{n}\binom{p}{m}\binom{n}{m} \leq & 2^{n}\binom{n}{m}^{2} \leq 2^{n}\binom{n}{\frac{n}{a}}^{2} \leq 2^{n}(a \cdot e)^{\frac{2 n}{a}}=2^{n\left(1+\frac{2}{a} \log _{2} a \cdot e\right)} \\
& {\left[2^{-m}\binom{m}{\left.\frac{m}{2}\right\rfloor}\right]^{n-m} \leq\left(\frac{5}{16}\right)^{n\left(1-\frac{1}{a}\right)} }
\end{aligned}
$$

Thus, we have

$$
R_{m}(p, n) \leq 2^{n\left(1+\frac{2}{a} \log _{2} a \cdot e\right)} \cdot\left(\frac{5}{16}\right)^{n\left(1-\frac{1}{a}\right)}=\left(\frac{5}{8}\right)^{n}\left(2^{\frac{2}{a} \log _{2} a \cdot e}\left(\frac{16}{5}\right)^{\frac{1}{a}}\right)^{n}
$$

For any $\epsilon, 0<\epsilon<\frac{1}{100}$, if we take $a=a(\epsilon)=\frac{1}{\epsilon^{2}}$, we get

$$
\left(2^{\frac{2}{a} \log _{2} a \cdot e}\left(\frac{16}{5}\right)^{\frac{1}{a}}\right)<1+\epsilon
$$

Hence, for any $\epsilon, m, p$, such that $0<\epsilon<\frac{1}{100}, 5 \leq m \leq \epsilon^{2} n, \quad m \leq p \leq n-1$, we have

$$
\begin{equation*}
R_{m}(p, n)<\left(\frac{5}{8}\right)^{n}(1+\epsilon)^{n} \quad \text { as } n \rightarrow \infty \tag{3.10}
\end{equation*}
$$

Q.E.D.
3.2 Case 2: $\epsilon^{2} n<m \leq n-\frac{c n}{\log _{2} n}, 0<\epsilon<\frac{1}{100}, c \geq 7.36, m \leq$ $p \leq n-1$.

Lemma 3.2 For any $\epsilon, m, p$, such that $0<\epsilon<\frac{1}{100}, \epsilon^{2} n<m \leq n-\frac{c n}{\log _{2} n}$, where $c \geq 7.36$, and $m \leq p \leq n-1$, we have

$$
R_{m}(p, n)=o\left(\left(\frac{5}{8}\right)^{n}\right) \quad \text { as } n \rightarrow \infty
$$

Proof. Using arguments from the first case, we have:

$$
\begin{aligned}
R_{m}(m, n) & \leq 2^{m}\binom{n}{m}\left[2 \cdot 2^{-m}\binom{m}{\left\lfloor\frac{m}{2}\right\rfloor}\right]^{n-m} \leq 2^{2 n}\left[2^{-m}\binom{m}{\left\lfloor\frac{m}{2}\right\rfloor}\right]^{n-m} \leq \\
& \leq 2^{2 n}\left[\sqrt{\frac{2}{\pi \epsilon^{2}}} n^{-\frac{1}{2}}\right]^{\frac{c n}{\log _{2} n}}=2^{2 n-\frac{c n}{2}}\left[\left(\frac{2}{\pi \epsilon^{2}}\right)^{\frac{c}{2 \log _{2} n}}\right]^{n}
\end{aligned}
$$

Then,

$$
\begin{align*}
R_{m}(p, m) & \leq\binom{ p}{m} R_{m}(m, n) \leq\binom{ p}{m} 2^{2 n-\frac{c n}{2}}\left[\left(\frac{2}{\pi \epsilon^{2}}\right)^{\frac{c}{2 \log _{2} n}}\right]^{n} \leq  \tag{3.11}\\
& \leq 2^{3 n-\frac{c n}{2}}\left[\left(\frac{2}{\pi \epsilon^{2}}\right)^{\frac{c}{2 \log _{2} n}}\right]^{n}=o\left(\left(\frac{5}{8}\right)^{n}\right) \text { for } c \geq 7,36
\end{align*}
$$

Q.E.D.
3.3 Case 3: $n-\frac{c n}{\log _{2} n}<m \leq n-1, \quad c \geq 7.36, \quad m \leq p \leq n-1$.

Lemma 3.3 For any $m, n-\frac{c n}{\log _{2} n}<m \leq n-1$, where $c \geq 7.36$, and $p$, $m \leq p \leq n-1$, we have

$$
R_{m}(p, n)=\left(\frac{1}{2}+o_{n}(1)\right)^{n} \quad \text { as } n \rightarrow \infty
$$

Proof. Let $M \in \mathcal{M}_{m}(m, n)$ and $M\left(j_{1}, \ldots, j_{m+1}\right)$ be its $m \times(m+1)$-submatrix with columns $j_{1}<\ldots<j_{m+1}$. Denote by $\mathcal{M}_{m}\left(m, n ; j_{1}, \ldots, j_{m+1}\right)$ the set

$$
\begin{aligned}
& \mathcal{M}_{m}\left(m, n ; j_{1}, \ldots, j_{m+1}\right) \stackrel{\text { def }}{=} \\
& \stackrel{\text { def }}{=}\left\{M \in \mathcal{M}_{m}(m, n) \mid M\left(j_{1}, \ldots, j_{m+1}\right) \in \mathcal{M}_{m}(m, m+1)\right\}
\end{aligned}
$$

Then

$$
\begin{equation*}
\mathcal{M}_{m}(m, n) \subset \bigcup_{1 \leq j_{1}<\ldots<j_{m+1} \leq n} \mathcal{M}_{m}\left(m, n ; j_{1}, \ldots, j_{m+1}\right) \tag{3.12}
\end{equation*}
$$

On the other hand, by Theorem [2.4 we have:

$$
\begin{align*}
R_{m}(m, m+1) & =\frac{\left|\mathcal{M}_{m}(m, m+1)\right|}{2^{m(m+1)}}= \\
& =\frac{\left|K S O(m, m+1) \cap\left\langle E_{m}\right\rangle_{\neq 0}^{\times m}\right|}{2^{m^{2}}} \leq \frac{m^{2}}{2^{m-1}} \tag{3.13}
\end{align*}
$$

From (3.12), (3.13), and (3.1), we get

$$
\begin{aligned}
R_{m}(p, n) & \leq\binom{ p}{m}\binom{n}{m+1} R_{m}(m, m+1) \leq\binom{ n}{m}^{2} \frac{m^{2}}{2^{m-1}} \leq \\
& \leq\binom{ n}{\frac{c n}{\log _{2} n}}^{2} \frac{m^{2}}{2^{m-1}} \leq\left(\frac{e \log _{2} n}{c}\right)^{\frac{2 c n}{\log _{2} n}} \cdot \frac{n^{2}}{2^{n-\frac{c n}{\log _{2} n}}}=\left(\frac{1}{2}+o_{n}(1)\right)^{n}
\end{aligned}
$$

Q.E.D.

Now Theorem 1.2 follows from (3.3), (3.4), (3.5), (3.6), (3.7), Lemma 3.1, Lemma 3.2, and Lemma 3.3
Q.E.D.

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