# On linear-combinatorial problems associated with subspaces spanned by $\{\pm 1\}$ -vectors

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#### Abstract

A complete answer to the question about subspaces generated by  $\{\pm 1\}$ -vectors, which arose in the work of I. Kanter and H. Sompolinsky on associative memories, is given. More precisely, let vectors  $v_1, \ldots, v_p$ ,  $p \leq n-1$ , be chosen at random uniformly and independently from  $\{\pm 1\}^n \subset \mathbf{R}^n$ . Then the probability  $\mathbb{P}(p, n)$  that

$$span \langle v_1, \ldots, v_p \rangle \cap \{ \{ \pm 1 \}^n \setminus \{ \pm v_1, \ldots, \pm v_p \} \} \neq \emptyset$$

is shown to be

$$4\binom{p}{3}\left(\frac{3}{4}\right)^n + O\left(\left(\frac{5}{8} + o_n(1)\right)^n\right) \quad \text{as} \quad n \to \infty,$$

where the constant implied by the *O*-notation does not depend on *p*. The main term in this estimate is the probability that some 3 vectors  $v_{j_1}, v_{j_2}, v_{j_3}$  of  $v_j, j = 1, \ldots, p$ , have a linear combination that is a  $\{\pm 1\}$ -vector different from  $\pm v_{j_1}, \pm v_{j_2}, \pm v_{j_3}$ .

**Keywords.**  $\{\pm 1\}$ -vector, Threshold function, singular Bernoulli matrices,  $\eta^*$ -function.

### 1 Introduction.

In the paper [11], A.M.Odlyzko gave a partial answer to the following question that arose in the paper [8] on associative memories. Let vectors  $v_1, \ldots, v_p$  be chosen at random uniformly and independently from  $\{\pm 1\}^n \subset \mathbb{R}^n$ . What is the probability  $\mathbb{P}(p, n)$  that the subspace spanned by  $v_1, \ldots, v_p$  over reals contains a  $\{\pm 1\}$ -vector different from  $\pm v_1, \ldots, \pm v_p$ , i.e.

$$span \langle v_1, \ldots, v_p \rangle \cap \{ \{ \pm 1 \}^n \setminus \{ \pm v_1, \ldots, \pm v_p \} \} \neq \emptyset ?$$

G.Kalai and N.Linial conjectured (see [11]) that this probability is dominated by the probability  $\mathbb{P}_3(p, n)$  that some 3 vectors  $v_{j_1}, v_{j_2}, v_{j_3}$  of  $v_j, j = 1, \ldots, p$ ,

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have a linear combination that is a  $\{\pm 1\}$ -vector different from  $\pm v_{j_1}, \pm v_{j_2}, \pm v_{j_3}$ , where

(1.1) 
$$\mathbb{P}_3(p,n) = 4\binom{p}{3}\left(\frac{3}{4}\right)^n + O\left(p^4\left(\frac{5}{8}\right)^n\right) \quad \text{as} \quad n \to \infty,$$

for  $3 \le p \le n$ . In the paper [11] this conjecture was proven for

$$(1.2) p \le n - \frac{10n}{\ln n}.$$

**Theorem 1.1** (A.M.Odlyzko [11]) If  $p \leq n - \frac{10n}{\ln n}$  and vectors  $v_1, \ldots, v_p$  are chosen at random uniformly and independently from  $\{\pm 1\}^n \subset \mathbf{R}^n$ , then the probability  $\mathbb{P}(p, n)$  that the subspace spanned by  $v_1, \ldots, v_p$  over reals contains a  $\{\pm 1\}$ -vector different from  $\pm v_1, \ldots, \pm v_p$  equals

(1.3) 
$$\mathbb{P}(p,n) = \mathbb{P}_3(p,n) + O\left(\left(\frac{7}{10}\right)^n\right) \quad as \quad n \to \infty$$

The constants implied by the O-notation in (1.1) and (1.3) are independent of p.

The paper [11] made a significant contribution to estimating the number of threshold functions P(2, n). Namely, in the paper [17], as a corollary of Theorem 1, T.Zaslavsky's formula [16] and G.-C.Rota's theorem on the inequality of the Möbius function of a geometric lattice to zero (Theorem 4 [12], p.357), the asymptotics of the logarithm of P(2, n) was obtained. In [3], the Theorem 1 was used together with the original (A, B, C)-construction to improve the lower bound of P(2, n) obtained in [17] by a factor of  $\sim P(2, \lfloor \frac{7n \ln 2}{\ln n} \rfloor)$ . In [4], the Theorem 1 was generalized to the case of  $E_K$ -vectors, where  $E_K = \{0, \pm 1, \ldots, \pm Q\}$  if K = 2Q + 1, and  $E_K = \{\pm 1, \pm 3, \ldots, \pm (2Q - 1)\}$  if K = 2Q, to obtain the asymptotics of the logarithm of the number of threshold functions of K-valued logic.

In [11], A.M.Odlyzko noted that removing constraint (1.2) is an open and hard problem. The hardness of this problem is manifested in the fact (see [7], p. 238), that if  $\mathbb{P}(n-1,n)$  tends to zero as  $n \to \infty$ , then one could obtain the asymptotics of P(2, n):

(1.4) 
$$P(2,n) \sim 2\binom{2^n-1}{n}, \quad n \to \infty.$$

We will show this fact (see Corollary 2.3) using the properties of  $\eta_n^{\bigstar}$  function from the paper [5].

Also, in [14], one can see an implicit connection between estimates of the probability  $\mathbb{P}(n-1,n)$  and the probability  $\mathbb{P}_n$  that a random Bernoulli  $n \times n$   $\{\pm 1\}$ -matrix  $M_n$  is singular:

$$\mathbb{P}_n \stackrel{\text{def}}{=} \mathbf{Pr}(\det M_n = 0).$$

In this work, T. Tao and V. Vu investigate the properties of the combinatorial Grassmannian Gr, consisting of hyperplanes V in an n-dimensional space over a finite field  $\mathbb{F}$ ,  $|\mathbb{F}| = p > n^{\frac{n}{2}}$ , where p is a prime number, such that  $V = span \langle V \cap \{\pm 1\}^n \rangle$ , and estimate  $\mathbb{P}_n$  based on the formula

$$\mathbb{P}_n = 2^{o(n)} \sum_{V \in Gr, \ V \ is \ a \ non-trivial \ hyperplane \ in \ \mathbb{F}^n} \mathbb{P}(A_V).$$

Here  $A_V$  denotes the event that vectors  $v_1, \ldots, v_n$ , chosen at random uniformly and independently from  $\{\pm 1\}^n$ , span V, and  $\mathbb{P}(A_V)$  denotes the probability of this event. We can essentially improve the upper bound of  $\mathbb{P}(A_V)$  in the Small combinatorial dimension estimate Lemma from [7] (see inequality (6) in Lemma 2.3. from [14]) if we substitute the probability  $\mathbb{P}(v_1, \ldots, v_{n-1} \text{ span } V)$  by the probability

$$\mathbb{P}(\{v_1,\ldots,v_{n-1} \text{ span } V\} \land \{V \cap \{\{\pm 1\}^n \setminus \{\pm v_1,\ldots,\pm v_{n-1}\}\} \neq \emptyset\}).$$

On the other hand, J. Kahn, J. Komlós, and E. Szemerédi in [7], relying on a technique developed to estimate  $\mathbb{P}_n$ , showed (corollary 4, [7]) that there is a constant C such that the Theorem 1 is true for  $p \leq n - C$ .

In this paper, using the asymptotics of the probability  $\mathbb{P}_n$  (Theorem 6, [5])

(1.5) 
$$\mathbb{P}_n \sim (n-1)^2 2^{1-n}, \quad n \to \infty$$

and the lower bound for P(2, n) (inequality (139) of the Theorem 7 [5])

(1.6) 
$$P(2,n) \ge 2 \left[ 1 - \frac{n^2}{2^n} \left( 1 + o\left(\frac{n^3}{2^n}\right) \right) \right] \binom{2^n - 1}{n},$$

we remove in the Theorem 1 the restriction (1.2) and prove the conjecture of G.Kalai, N.Linial, and A.M.Odlysko for  $p \leq n-1$ .

**Theorem 1.2 (Main theorem)** If  $p \leq n-1$  and vectors  $v_1, \ldots, v_p$  are chosen at random uniformly and independently from  $\{\pm 1\}^n \subset \mathbf{R}^n$ , then the probability  $\mathbb{P}(p,n)$  that the subspace spanned by  $v_1, \ldots, v_p$  over reals contains a  $\{\pm 1\}$ -vector different from  $\pm v_1, \ldots, \pm v_p$  equals

(1.7) 
$$\mathbb{P}(p,n) = \mathbb{P}_3(p,n) + O\left(\left(\frac{5}{8} + o_n(1)\right)^n\right) \quad as \quad n \to \infty.$$

The constants implied by the O-notation in (1.1) and (1.7) are independent of p.

# 2 Interdependence between $\eta_n^{\bigstar}$ -function, $\mathbb{P}(n, n+1)$ , and the fraction of singular $(n+1) \times (n+1)$ - $\{\pm 1\}$ -matrices with different rows.

Let  $\langle H \rangle = \{ \langle w_1 \rangle, \dots, \langle w_T \rangle \} \subset \mathbf{RP}^n$  be a subset of the *n*-dimensional projective space, where points  $\langle w_i \rangle$ ,  $i = 1, \dots, T$ , are represented by lines  $tw_i \subset \mathbf{R}^{n+1}$  and  $w_i \in \mathbf{R}^{n+1}$ ,  $i = 1, \dots, T$ . Let  $K^H$  be a simplicial compex defined as follows. The set of vertices of  $K^H$  coincides with the set  $\langle H \rangle$ . A subset  $\{ \langle w_{i_1} \rangle, \dots, \langle w_{i_s} \rangle \}$  of  $\langle H \rangle$  forms a simplex of  $K^H$  iff

span 
$$\langle w_{i_1}, \ldots, w_{i_s} \rangle \neq span \langle w_1, \ldots, w_T \rangle$$
.

We define on the set  $2_{fin}^{\mathbf{RP}^n}$  of finite subsets of  $\mathbf{RP}^n$  the function  $\eta_n^{\bigstar} : 2_{fin}^{\mathbf{RP}^n} \to \mathbb{Z}_{\geq 0}$  by the following formula (see [5]):

(2.1) 
$$\eta_n^{\bigstar}(\langle H \rangle) = rank \; \tilde{H}_{n-1}(K^H; \mathbf{F}), \quad \langle H \rangle \subset \mathbf{RP}^n,$$

 $\mathbf{i}\mathbf{f}$ 

span 
$$\langle w_1, \ldots, w_T \rangle = \mathbf{R}^{n+1}$$

and

(2.2) 
$$\eta_n^{\bigstar}(\langle H \rangle) = 0$$

if

span 
$$\langle w_1, \ldots, w_T \rangle \neq \mathbf{R}^{n+1}$$

Here  $\tilde{H}_{n-1}(K^H; \mathbf{F})$  denotes the reduced homology group of the complex  $K^H$  with coefficients in an arbitrary field  $\mathbf{F}$ .

Let us denote by  $\langle H \rangle^{\times s}$ ,  $s = 1, \ldots, T$ , the set of ordered collections  $(\langle w_{i_1} \rangle, \ldots, \langle w_{i_s} \rangle)$  of different *s* elements from  $\langle H \rangle$  and  $\langle H \rangle_{\neq 0}^{\times s} \subset \langle H \rangle^{\times s}$ ,  $\langle H \rangle_{=0}^{\times s} \subset \langle H \rangle^{\times s}$  be the subsets

$$(2.3) \quad \langle H \rangle_{\neq 0}^{\times s} \stackrel{\text{def}}{=} \{ (\langle w_{i_1} \rangle, \dots, \langle w_{i_s} \rangle) \in \langle H \rangle^{\times s} \mid \dim span \langle w_{i_1}, \dots, w_{i_s} \rangle = s \}.$$

$$(2.4) \quad \langle H \rangle_{=0}^{\times s} \stackrel{\text{def}}{=} \{ (\langle w_{i_1} \rangle, \dots, \langle w_{i_s} \rangle) \in \langle H \rangle^{\times s} \mid \dim span \ \langle w_{i_1}, \dots, w_{i_s} \rangle < s \}.$$

For any  $W = (\langle w_{i_1} \rangle, \dots, \langle w_{i_n} \rangle) \in \langle H \rangle^{\times n}$  and  $l = 1, \dots, n$ , let

(2.5) 
$$L_l(W) \stackrel{\text{def}}{=} span \ \langle w_{i_{n-l+1}}, \dots, w_{i_n} \rangle \subset \mathbf{R}^{n+1};$$
$$q_l^W(H) \stackrel{\text{def}}{=} |L_l(W) \cap \langle H \rangle|.$$

**Definition 2.1** For any  $W \in \langle H \rangle^{\times n}$ , the ordered set of numbers

(2.6) 
$$W(\langle H \rangle) \stackrel{\text{def}}{=} (q_n^W(H), q_{n-1}^W(H), \dots, q_1^W(H))$$

is called a combinatorial flag on  $\langle H \rangle \subset \mathbf{RP}^n$  of the ordered set W.

For the sake of simplicity, we will use the following notation:

(2.7) 
$$W[H] \stackrel{\text{def}}{=} q_n^W(H) \cdot q_{n-1}^W(H) \cdots q_1^W(H).$$

In [5], it was proven the following theorem.

**Theorem 2.2** ([5], [6]) For any  $p = (p_1, \ldots, p_T)$ ,  $p_i \in \mathbf{R}$ ,  $i = 1, \ldots, T$ , such that  $\sum_{i=1}^{T} p_i = 1$ , and subset  $\langle H \rangle = \{\langle w_1 \rangle, \ldots, \langle w_T \rangle\} \subset \mathbf{RP}^n$ , such that span  $\langle H \rangle = \mathbf{R}^{n+1}$ , the following equality is true:

(2.8) 
$$\eta_n^{\bigstar}(\langle H \rangle) = \sum_{W \in \langle H \rangle_{\neq 0}^{\times n}} \frac{1 - p_{i_1} - p_{i_2} - \dots - p_{i_{q_n^W}}}{W[H]}.$$

Here, the indices used in the numerator correspond to elements from

$$L_n(W) \cap \langle H \rangle = \left\{ \langle w_{i_1} \rangle, \dots, \langle w_{i_n} \rangle, \dots, \langle w_{i_{q_n^W}} \rangle \right\}.$$

Let

(2.9) 
$$E_n = \{(1, b_1, \dots, b_n) \mid b_i \in \{\pm 1\}, i = 1, \dots, n\}$$

and

(2.10) 
$$\{E_n\}^p = \underbrace{E_n \times \cdots \times E_n}_p.$$

We say that an ordered collection  $W = (w_1, \ldots, w_p), w_i \in \{\pm 1\}^{n+1}, i = 1, \ldots, p$ , satisfies to KSO-condition, and we write  $W \in KSO(p, n+1)$ , iff

(2.11) 
$$span \langle w_1, \dots, w_p \rangle \cap \{\{\pm 1\}^{n+1} \setminus \{\pm w_1, \dots, \pm w_p\}\} \neq \emptyset.$$

Then

$$\mathbb{P}(n, n+1) = \frac{|KSO(n, n+1)|}{2^{n(n+1)}} = \frac{|\{W \in \{E_n\}^n \mid W \in KSO(n, n+1)\}|}{2^{n^2}},$$

and

(2.12) 
$$|KSO(n, n + 1) \cap \langle E_n \rangle_{\neq 0}^{\times n}| = |\{W \in \langle E_n \rangle_{\neq 0}^{\times n} | W \in KSO(n, n + 1)\}| < 2^{n^2} \mathbb{P}(n, n + 1).$$

**Corollary 2.3** If  $\lim_{n\to\infty} \mathbb{P}(n, n+1) = 0$ , then  $P(2, n) \sim 2\binom{2^n-1}{n}$ ,  $n \to \infty$ .

**Proof.** Let us apply Theorem 2.2 to the set  $H = E_n \subset \mathbb{R}^{n+1}$  and the collection of weights  $p = (1, 0, \dots, 0)$ , where  $w_1 = (1, \dots, 1) \in \mathbb{R}^{n+1}$ . Then

(2.13) 
$$\eta_n^{\bigstar}(\langle E_n \rangle) = \Sigma_1 + \Sigma_2 - \Sigma_3 - \Sigma_4,$$

where

$$\Sigma_{1} = \sum_{W \in \langle E_{n} \rangle_{\neq 0}^{\times n}, W \notin KSO(n, n+1)} \frac{1}{W[E_{n}]},$$

$$\Sigma_{2} = \sum_{W \in \langle E_{n} \rangle_{\neq 0}^{\times n}, W \in KSO(n, n+1)} \frac{1}{W[E_{n}]},$$

$$\Sigma_{3} = \sum_{W \in \langle E_{n} \rangle_{\neq 0}^{\times n}, w_{1} \notin \{W\}, w_{1} \in span\langle W \rangle} \frac{1}{W[E_{n}]},$$

$$\Sigma_{4} = \sum_{W \in \langle E_{n} \rangle_{\neq 0}^{\times n}, w_{1} \in \{W\}} \frac{1}{W[E_{n}]}.$$

From (2.12), we have

(2.14) 
$$\Sigma_{1} = \frac{|\langle E_{n} \rangle_{\neq 0}^{\times n}| - |KSO(n, n+1) \cap \langle E_{n} \rangle_{\neq 0}^{\times n}|}{n!} > \frac{2^{n} \cdots (2^{n} - n+1) - 2^{n^{2}} \mathbb{P}_{n} - |KSO(n, n+1) \cap \langle E_{n} \rangle_{\neq 0}^{\times n}|}{n!} > \frac{2^{n^{2}}}{n!} > \frac{2^{n^{2}}}{2^{n}(2^{n} - 1) \cdots (2^{n} - n+1)} (\mathbb{P}_{n} + \mathbb{P}(n, n+1)),$$

(2.15) 
$$\Sigma_3 < \mathbb{P}_n(\{0,1\}) \frac{2^{n^2}}{2^n(2^n-1)\cdots(2^n-n+1)} \binom{2^n}{n},$$

(2.16) 
$$\Sigma_4 < n \frac{(2^n - 1)(2^n - 2) \cdots (2^n - n + 1)}{n!} = \binom{2^n - 1}{n - 1}.$$

Here  $\mathbb{P}_n(\{0,1\})$  denotes the probability that a (0,1)- $n \times n$ -matrix, with entries chosen at random, uniformly, and independently from  $\{0,1\}$ , is singular. It follows from (2.13), (2.14), (2.15), and (2.16) that

(2.17) 
$$\eta_n^{\bigstar}(\langle E_n \rangle) > \binom{2^n - 1}{n} (1 - c_n)),$$

where

$$c_n = \frac{2^{n^2}}{(2^n - 1)\cdots(2^n - n)} (\mathbb{P}_n + \mathbb{P}(n, n+1) + \mathbb{P}_n(\{0, 1\})).$$

Taking into account the inequality (see the formulas (19) and (25) of [5])

$$P(2,n) \ge 2\eta_n^{\bigstar}(\langle E_n \rangle),$$

L. Schläfli's upper bound (see the formula 2 in [5] and [13])

$$P(2,n) \le 2\sum_{i=0}^{n} \binom{2^{n}-1}{i},$$

the given fact that  $\mathbb{P}(n, n+1) \to 0$ , and the well known results  $\mathbb{P}_n(\{0, 1\}) \to 0$ , and  $\mathbb{P}_n \to 0$  as  $n \to \infty$  (see [9], [10], [7], [14], [1], [15], [5]), we can conclude that

$$c_n \to 0 \ as \ n \to \infty,$$

and

$$P(2,n) \sim 2 \binom{2^n - 1}{n}, \ n \to \infty.$$
 Q.E.D.

We define  $\delta_{n,k}$ ,  $k = 1, \ldots, n+1$ , as

$$\delta_{n,k} \stackrel{\text{def}}{=} \frac{|\langle E_n \rangle_{=0}^{\times k}|}{|\langle E_n \rangle^{\times k}|}, \quad k = 1, \dots, n+1.$$

For  $W = (\langle w_{i_1} \rangle, \dots, \langle w_{i_n} \rangle) \in \langle E_n \rangle_{\neq 0}^{\times n}$ , we use the following notations:

$$L_n(W) \stackrel{\text{def}}{=} span \langle w_{i_1}, \dots, w_{i_n} \rangle \subset \mathbf{R}^{n+1};$$
  

$$q_n^W \stackrel{\text{def}}{=} |L_n(W) \cap E_n|;$$
  

$$E_n^m \stackrel{\text{def}}{=} \left\{ W \in \langle E_n \rangle_{\neq 0}^{\times n} \mid q_n^W = n + m \right\}, \quad m = 0, 1, \dots, 2^{n-1} - n$$

**Theorem 2.4** For sufficiently large n, we have

(2.18) 
$$\frac{|KSO(n, n+1) \cap \langle E_n \rangle_{\neq 0}^{\times n}|}{2^{n^2}} \le \frac{n^2}{2^{n-1}}$$

**Proof.** Let us take a vector  $w \in \mathbf{R}^{n+1}$  in general position to the set  $E_n$ , i.e. for any vectors  $w_{i_1}, \ldots, w_{i_n} \in E_n \subset \mathbf{R}^{n+1}$ ,

 $w \notin span\langle w_{i_1}, \ldots, w_{i_n} \rangle.$ 

From the Theorem 2.2 applyed to the set  $H = \langle E_n \rangle \cup \langle w \rangle \subset \mathbf{RP}^n$  and the collection of weights p(w) = 1 and  $p(w_i) = 0$  for  $w_i \in E_n$ ,  $i = 1, ..., 2^n$ , we get

$$\begin{split} \eta_n^{\bigstar}(\langle E_n \rangle \cup \langle w \rangle) &= \sum_{W \in \langle E_n \rangle_{\neq 0}^{\times n}} \frac{1}{W[E_n]} = \\ &= \sum_{m=0}^{2^{n-1}-n} \sum_{W \in E_n^m} \frac{1}{W[E_n]} = \frac{1}{n!} \sum_{m=0}^{2^{n-1}-n} |E_n^m| - \sum_{m=1}^{2^{n-1}-n} \sum_{W \in E_n^m} \left(\frac{1}{n!} - \frac{1}{W[E_n]}\right) \leq \\ &\left(\frac{1}{n!} - \frac{1}{W[E_n]} \ge \frac{1}{n!} - \frac{1}{(n-1)!(n+m)} = \frac{m}{n!(n+m)}\right) \\ &\leq \frac{1}{n!} \left( |\langle E_n \rangle^{\times n} | (1 - \delta_{n,n}) - \frac{1}{n+1} \left| \cup_{m=1}^{2^{n-1}-n} E_n^m \right| \right), \end{split}$$

or

(2.19)  
$$\eta_n^{\bigstar}(\langle E_n \rangle \cup \langle w \rangle) \le {\binom{2^n}{n}}(1 - \delta_{n,n}) - \frac{1}{n!(n+1)} \left| KSO(n, n+1) \cap \langle E_n \rangle_{\neq 0}^{\times n} \right|.$$

From the Theorem 5 of the paper [5], we have (see the formula (135) in [5]):

(2.20) 
$$\eta_n^{\bigstar}(\langle E_n \rangle \cup \{\langle w \rangle\}) \ge {\binom{2^n}{n}} \left[1 - \delta_{n,n} - \frac{n-1}{2^{n-1}} \left(1 + o\left(\frac{n^3}{2^n}\right)\right)\right].$$

Combining inequalities (2.19) and (2.20), we get

$$(2.21) \left| KSO(n, n+1) \cap \langle E_n \rangle_{\neq 0}^{\times n} \right| \le 2^n \cdots (2^n - n + 1) \frac{n^2 - 1}{2^{n-1}} \left( 1 + o\left(\frac{n^3}{2^n}\right) \right).$$

The Theorem 2.4 follows from the inequality (2.21).

Q.E.D.

## **3** Proof of the Main Theorem.

Let  $\mathbb{P}_m(p,n)$ ,  $m \leq p \leq n-1$ , denote the probability that in the set of p vectors  $v_1, \ldots, v_p \in \{\pm 1\}^n \subset \mathbf{R}^n$ , chosen at random uniformly and independently, there are some m vectors  $v_{j_1}, \ldots, v_{j_m}$  such that

$$\alpha_1 v_{j_1} + \dots + \alpha_m v_{j_m} \in \{\pm 1\}^n$$
 for some  $\alpha_1, \dots, \alpha_m \in \mathbb{R} \setminus \{0\}$ 

Let  $\mathcal{M}_m(p,n)$  denote the set of  $(p \times n)$ - $\{\pm 1\}$ -matrices M with linear independent rows  $w_1, \ldots, w_p \in \{\pm 1\}^n$  satisfying the following property. There are a subset of m rows  $w_{i_1}, \ldots, w_{i_m}$  and some nonzero coefficients  $\alpha_1, \ldots, \alpha_m \in \mathbb{R} \setminus \{0\}$  such that

$$\alpha_1 w_{i_1} + \dots + \alpha_m w_{i_m} \in \{\pm 1\}^n.$$

Let  $\mathcal{Q}(p,n)$  be the set of  $(p \times n)$ - $\{\pm 1\}$ -matrices M with rank less than p (< p). Denote by  $R_m(p,n)$  the probability that a  $(p \times n)$ - $\{\pm 1\}$ -matrix M chosen at random belongs to  $\mathcal{M}_m(p,n)$  and by  $\mathbb{P}_{p,n}$  the probability that a  $(p \times n)$ - $\{\pm 1\}$ -matrix M chosen at random has rank less than p (< p). Then

$$KSO(p,n) \subset \bigcup_{m=3}^{p} \mathcal{M}_{m}(p,n) \bigsqcup \mathcal{Q}(p,n),$$

(3.1) 
$$R_m(p,n) \le \binom{p}{m} R_m(m,n),$$

(3.2) 
$$\mathbb{P}_m(p,n) \le R_m(p,n) + \mathbb{P}_{p,n} \le \binom{p}{m} R_m(m,n) + \mathbb{P}_{p,n},$$

and

(3.3) 
$$\mathbb{P}(p,n) \le \sum_{m=3}^{p} R_m(p,n) + \mathbb{P}_{p,n} \le \sum_{m=3}^{p} \binom{p}{m} R_m(m,n) + \mathbb{P}_{p,n},$$

It follows from [5] (see Lemma 5 and the proof of the Theorem 6) that for  $p = 1, \ldots, n$ 

(3.4) 
$$\mathbb{P}_{p,n} \le \frac{(p-1)^2}{2^{n-1}} (1+o_n(1)).$$

The proof of the Theorem 1.2 is divided into 3 cases of evaluation  $R_m(p,n)$ :

$$\begin{array}{ll} Case \ 1. & 5 \leq m \leq \frac{n}{a(\epsilon)}, & a(\epsilon) = \frac{1}{\epsilon^2}, & 0 < \epsilon < \frac{1}{100}, & m \leq p \leq n-1; \\ Case \ 2. & \frac{n}{a(\epsilon)} < m \leq n - \frac{cn}{\log_2 n}, & c \geq 7.36, & m \leq p \leq n-1; \\ Case \ 3. & n - \frac{cn}{\log_2 n} < m \leq n-1, & c \geq 7.36, & m \leq p \leq n-1. \end{array}$$

It was shown in the paper [11] that

(3.5) 
$$\mathbb{P}_2(p,n) = O(p^2 2^{-n}) \text{ as } n \to \infty, \text{ for } 2 \le p \le n-1;$$

(3.6) 
$$\mathbb{P}_3(p,n) = 4\binom{p}{3}\left(\frac{3}{4}\right)^n + O\left(p^4\left(\frac{5}{8}\right)^n\right) \text{ as } n \to \infty, \text{ for } 3 \le p \le n-1;$$

(3.7) 
$$\mathbb{P}_4(p,n) = O\left(p^4 2^{-n}\right) \text{ as } n \to \infty, \text{ for } 4 \le p \le n-1.$$

The proofs of cases 1 and 2 repeat some arguments of the papers [11] and [4]. Here we present the proofs of cases 1 and 2 for completeness of presentation and clarification of some constants. The proof of case 3 is based on Theorem 2.4.

**3.1** Case 1:  $5 \le m \le \frac{n}{a}$ ,  $m \le p \le n - 1$ .

**Lemma 3.1** For any  $\epsilon$ , m, p, such that  $0 < \epsilon < \frac{1}{100}$ ,  $5 \leq m \leq \frac{n}{a}$ , where  $a = a(\epsilon) = \frac{1}{\epsilon^2}$ , and  $m \leq p \leq n-1$ , we have

$$R_m(p,n) < \left(\frac{5}{8}\right)^n (1+\epsilon)^n \text{ as } n \to \infty.$$

**Proof.** Let  $M \in \mathcal{M}_m(m, n)$ . Denote by  $w_1, \ldots, w_m$  the rows of M. If columns  $1 \leq j_1 < \cdots < j_m \leq n$  of the matrix M are linearly independent, then for each choice of  $\beta_1, \ldots, \beta_m \in \{\pm 1\}$ , there will be a unique set of coefficients  $\alpha_1, \ldots, \alpha_m$  with  $j_s$ th coordinate of the vector  $\alpha M = ((\alpha M)_1, \ldots, (\alpha M)_n) \stackrel{\text{def}}{=} \alpha_1 w_1 + \cdots + \alpha_m w_m$  equals to  $\beta_s, s = 1, \ldots, m$ . Hence, there are at most  $2^m$  sets  $\alpha_1, \ldots, \alpha_m \in \mathbb{R} \setminus \{0\}$  such that  $(\alpha M)_j = +1$  or -1 for  $j = j_1, \ldots, j_m$ . For each fixed vector  $\alpha = (\alpha_1, \ldots, \alpha_m), \alpha_i \in \mathbb{R} \setminus \{0\}, i = 1, \ldots, m$ , probability that  $(\alpha M)_j = +1$  or -1 for  $j \neq j_1, \ldots, j_m$ , is at most

$$2 \cdot 2^{-m} \binom{m}{\left\lfloor \frac{m}{2} \right\rfloor}$$

by Erdös-Littlewood-Offord lemma (see [2]). Since all columns  $j, j \neq j_1, \ldots, j_m$ , we choose independently of each other, we have

(3.8)  
$$R_m(m,n) \le 2^m \binom{n}{m} \left[ 2 \cdot 2^{-m} \binom{m}{\lfloor \frac{m}{2} \rfloor} \right]^{n-m} = 2^n \binom{n}{m} \left[ 2^{-m} \binom{m}{\lfloor \frac{m}{2} \rfloor} \right]^{n-m}.$$

Taking into account (3.1) and (3.8), we get

(3.9) 
$$R_m(p,n) \le 2^n \binom{p}{m} \binom{n}{m} \left[2^{-m} \binom{m}{\lfloor \frac{m}{2} \rfloor}\right]^{n-m}.$$

For  $5 \le m \le \frac{n}{a}$ ,  $m \le p \le n - 1$ , we have

$$2^{n} \binom{p}{m} \binom{n}{m} \leq 2^{n} \binom{n}{m}^{2} \leq 2^{n} \binom{n}{\frac{n}{a}}^{2} \leq 2^{n} (a \cdot e)^{\frac{2n}{a}} = 2^{n\left(1 + \frac{2}{a}\log_{2} a \cdot e\right)};$$
$$\left[2^{-m} \binom{m}{\lfloor \frac{m}{2} \rfloor}\right]^{n-m} \leq \left(\frac{5}{16}\right)^{n\left(1 - \frac{1}{a}\right)};$$

Thus, we have

$$R_m(p,n) \le 2^{n\left(1+\frac{2}{a}\log_2 a \cdot e\right)} \cdot \left(\frac{5}{16}\right)^{n\left(1-\frac{1}{a}\right)} = \left(\frac{5}{8}\right)^n \left(2^{\frac{2}{a}\log_2 a \cdot e} \left(\frac{16}{5}\right)^{\frac{1}{a}}\right)^n.$$

For any  $\epsilon$ ,  $0 < \epsilon < \frac{1}{100}$ , if we take  $a = a(\epsilon) = \frac{1}{\epsilon^2}$ , we get

$$\left(2^{\frac{2}{a}\log_2 a \cdot e} \left(\frac{16}{5}\right)^{\frac{1}{a}}\right) < 1 + \epsilon.$$

Hence, for any  $\epsilon$ , m, p, such that  $0 < \epsilon < \frac{1}{100}$ ,  $5 \le m \le \epsilon^2 n$ ,  $m \le p \le n-1$ , we have

(3.10) 
$$R_m(p,n) < \left(\frac{5}{8}\right)^n (1+\epsilon)^n \text{ as } n \to \infty.$$

Q.E.D.

**3.2** Case 2:  $\epsilon^2 n < m \le n - \frac{cn}{\log_2 n}$ ,  $0 < \epsilon < \frac{1}{100}$ ,  $c \ge 7.36$ ,  $m \le p \le n - 1$ .

**Lemma 3.2** For any  $\epsilon$ , m, p, such that  $0 < \epsilon < \frac{1}{100}$ ,  $\epsilon^2 n < m \le n - \frac{cn}{\log_2 n}$ , where  $c \ge 7.36$ , and  $m \le p \le n - 1$ , we have

$$R_m(p,n) = o\left(\left(\frac{5}{8}\right)^n\right) \quad as \quad n \to \infty.$$

**Proof.** Using arguments from the first case, we have:

$$\begin{aligned} R_m(m,n) &\leq 2^m \binom{n}{m} \left[ 2 \cdot 2^{-m} \binom{m}{\lfloor \frac{m}{2} \rfloor} \right]^{n-m} \leq 2^{2n} \left[ 2^{-m} \binom{m}{\lfloor \frac{m}{2} \rfloor} \right]^{n-m} \leq \\ &\leq 2^{2n} \left[ \sqrt{\frac{2}{\pi\epsilon^2}} n^{-\frac{1}{2}} \right]^{\frac{cn}{\log_2 n}} = 2^{2n-\frac{cn}{2}} \left[ \left( \frac{2}{\pi\epsilon^2} \right)^{\frac{c}{2\log_2 n}} \right]^n. \end{aligned}$$

Then,

(3.11)  

$$R_{m}(p,m) \leq {\binom{p}{m}} R_{m}(m,n) \leq {\binom{p}{m}} 2^{2n - \frac{cn}{2}} \left[ \left(\frac{2}{\pi\epsilon^{2}}\right)^{\frac{c}{2\log_{2}n}} \right]^{n} \leq 2^{3n - \frac{cn}{2}} \left[ \left(\frac{2}{\pi\epsilon^{2}}\right)^{\frac{c}{2\log_{2}n}} \right]^{n} = o\left( \left(\frac{5}{8}\right)^{n} \right) \text{ for } c \geq 7, 36.$$
Q.E.D.

**3.3** Case 3: 
$$n - \frac{cn}{\log_2 n} < m \le n - 1$$
,  $c \ge 7.36$ ,  $m \le p \le n - 1$ .

**Lemma 3.3** For any m,  $n - \frac{cn}{\log_2 n} < m \leq n - 1$ , where  $c \geq 7.36$ , and p,  $m \leq p \leq n - 1$ , we have

$$R_m(p,n) = \left(\frac{1}{2} + o_n(1)\right)^n \quad as \quad n \to \infty.$$

**Proof.** Let  $M \in \mathcal{M}_m(m, n)$  and  $M(j_1, \ldots, j_{m+1})$  be its  $m \times (m+1)$ -submatrix with columns  $j_1 < \ldots < j_{m+1}$ . Denote by  $\mathcal{M}_m(m, n; j_1, \ldots, j_{m+1})$  the set

$$\mathcal{M}_m(m,n;j_1,\ldots,j_{m+1}) \stackrel{\text{def}}{=} \\ \stackrel{\text{def}}{=} \{ M \in \mathcal{M}_m(m,n) \mid M(j_1,\ldots,j_{m+1}) \in \mathcal{M}_m(m,m+1) \}.$$

Then

(3.12) 
$$\mathcal{M}_m(m,n) \subset \bigcup_{1 \le j_1 < \ldots < j_{m+1} \le n} \mathcal{M}_m(m,n;j_1,\ldots,j_{m+1}).$$

On the other hand, by Theorem 2.4 we have:

(3.13)  
$$R_m(m, m+1) = \frac{|\mathcal{M}_m(m, m+1)|}{2^{m(m+1)}} = \frac{|KSO(m, m+1) \cap \langle E_m \rangle_{\neq 0}^{\times m}|}{2^{m^2}} \le \frac{m^2}{2^{m-1}}.$$

From (3.12), (3.13), and (3.1), we get

$$R_{m}(p,n) \leq \binom{p}{m} \binom{n}{m+1} R_{m}(m,m+1) \leq \binom{n}{m}^{2} \frac{m^{2}}{2^{m-1}} \leq \\ \leq \binom{n}{\frac{cn}{\log_{2}n}}^{2} \frac{m^{2}}{2^{m-1}} \leq \left(\frac{e\log_{2}n}{c}\right)^{\frac{2cn}{\log_{2}n}} \cdot \frac{n^{2}}{2^{n-\frac{cn}{\log_{2}n}}} = \left(\frac{1}{2} + o_{n}(1)\right)^{n}.$$
Q.E.D

Now Theorem 1.2 follows from (3.3), (3.4), (3.5), (3.6), (3.7), Lemma 3.1, Lemma 3.2, and Lemma 3.3.

Q.E.D.

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