Noncommutative pfaffians and classification of states of five-dimensional quasi-spin.

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Noncommutative pfaffians associated with an orthogonal algebra $\mathfrak{o}_N$ are some special elements of the universal enveloping algebra $U(\mathfrak{o}_N)$. Using pfaffians we construct the fourth quantum number which together with the naturally defined three quantum numbers allow to classify the states of a five-dimensional quasi-spin. The pfaffians are treated as creation operators for the new quantum number.

1 The quasi-spin algebra

In many problems of quantum numbers there appears naturally an algebra of observables whose elements are quadratic combinations of fermion creation and annihilation operators. Such an algebra is called a generalized fermion algebra. Some its particular cases are called the quasi-spin algebra [1].

For example, the three-dimensional quasi-spin algebra appears naturally when one considers particles of the same type. The three dimensional quasi-spin is constructed as follows. One takes the system of noninteracting fermions with the fixed angular momentum j. Denote by a^+_m, a_m, $m=-j,...,j$, the creation and annihilation operators of particles with the projection m of the angular momentum to the axes z.

Consider the operators

$$s^+_k = a^+_k a^+_{-k},$$

$$s^-_k = a_{-k} a_k,$$

$$s_k^0 = \frac{1}{2}(a_k^+ a_k - a_{-k} a_{-k}^+).$$

Let us construct a three-dimensional algebra of quasi-spin, isomorphic to $\mathfrak{o}_3 = \mathfrak{sl}_2$, see [2],[3]. It is spanned by elements

$$S_+ = \sum_k s_k^+,$$

$$S_- = \sum_k s_k^-,$$

$$S_0 = \sum_k s_k^0.$$

For the classification of states of the three-dimensional quasi-spin the quantum numbers seniority and the number of particles are used. Note that although the three-dimensional quasi-spin algebra is isomorphic to the spin algebra the two quantum numbers mentioned above are different from the quantum numbers used for the classification of the states of the spin algebra.

The three-dimensional quasi-spin algebra appears in the theory superconductivity, in description of pair correlations between nucleons, [3].

Suppose we have particles of two types, protons and neutrons. Then the five-dimensional quasi-spin algebra naturally appears. Each creation and annihilation operators has two indices. Let a_{pm}^+, a_{nm}^+ be creation operators of protons and neutrons with the projection m of angular momentum to the z-axes, and a_{pm}, a_{nm} the corresponding annihilation operators. The commutation relations between these operators are the following (the index τ is either p or n):

$$a_{\tau m} a_{\tau' m'} + a_{\tau' m'} a_{\tau m} = 0$$

$$a_{\tau m}^+ a_{\tau' m'}^+ + a_{\tau' m'}^+ a_{\tau m}^+ = 0$$

$$a_{\tau m} a_{\tau' m'}^+ + a_{\tau' m'}^+ a_{\tau m} = \delta_{\tau,\tau'} \delta_{m,m'}$$

In this case one can naturally define the five-dimensional quasi-spin algebra [4] (see also [2]), spanned by elements

$$\tau_+ = \sum_m a^+_{pm} a_{nm}, \quad \tau_0 = \frac{1}{2}\sum_m (a^+_{pm} a_{pm} - a^+_{pm} a_{nm}), \quad \tau_- = \sum_m a^+_{nm} a_{pm}$$

$$N = \frac{1}{2}\sum_m (a^+_{pm} a_{pm} + a^+_{nm} a_{nm}) - \frac{2j+1}{2}$$

$$A(1) = \sum_{m>0} (-1)^{j-m} a^+_{pm} a^+_{p-m}, \quad A(0) = \frac{1}{\sqrt{2}} \sum_{m>0} (-1)^{j-m} (a^+_{pm} a^+_{n-m} + a^+_{n-m} a^+_{p-m}),$$

$$A(-1) = \sum_{m>0} (-1)^{j-m} a^+_{nm} a^+_{n-m}$$

$$B(1) = \sum_{m>0} (-1)^{j-m} a_{n-m} a_{pm}, \quad B(0) = \frac{1}{\sqrt{2}} \sum_{m>0} (-1)^{j-m} (a_{n-m} a^+_{pm} + a^+_{p-m} a^+_{nm}),$$

$$B(-1) = \sum_{m>0} (-1)^{j-m} a_{n-m} a_{nm}$$

These operators form a Lie algebra isomorphic to $\mathfrak{o}_5$. Note that the subalgebra $\mathfrak{o}_3$ spanned by τ_+, τ_0, τ_- is the isospin subalgebra and N is the number of particles plus a constant. But below for simplicity of terminology we call N itself the number of particles. The reasons for the introduction of this five-dimensional quasi-spin are discussed in [5].

One can obtain an explicit isomorphism of the quasi-spin algebra with the algebra $\mathfrak{o}_5$ in the split realization (see Sec. 2) in the following way. First one must establish a relation between the written above elements $\tau_i, A(j), B(l)$ and the Chevalley base of $\mathfrak{o}_5$. Second one must establish a relation between canonical generators of $\mathfrak{o}_5$ and the Chevalley base of $\mathfrak{o}_5$. The commutation relation between elements $\tau_i, A(j), B(l)$ that are necessary for the above procedure are given in [1]. As the result we have

$$F_{0-1} = \frac{\tau_+}{\sqrt{2}} \;\; F_{-1-2} = A(-1), \;\; F_{0-2} = A(0), \;\; F_{2-1} = -A(1), \;\; F_{-1-1} = -\tau_0,$$

$$F_{-10} = \frac{\tau_-}{\sqrt{2}} \;\; F_{-2-1} = B(-1), \;\; F_{-20} = B(0), \;\; F_{-12} = -B(1), \;\; F_{-2-2} = -N,$$

where F_{ij} are defined in Sec. 2.

Now let us proceed to the problem of classification of states of a finite-dimensional representation of the five-dimensional the quasi-spin algebra.

There are three natural quantum numbers indexing the base vectors of a representation. They are the number of particles N, the isospin T and the projection of the isospin τ_0.

These numbers are not enough to classify the states, there are linear independent states with the same collections of these three numbers.

Thus, in order to classify the states it is necessary to construct at least one new quantum number k.

First construction of such a number in [6],[7] were of the following type.

At first one fixes the set of states, to which the zero value of the fourth quantum number is assigned. Then the creation operator for the new quantum number is introduced. Then one defines the fourth quantum number inductively in the following manner. If a state is obtained from a state with the fourth quantum number k by application of the creation operator for the fourth quantum number, then we assign to it the fourth quantum number $k+1$.

However, it turns out that the creation operator of the new quantum number changes other quantum numbers in a not clear way.

Later other constructions of the fourth quantum number were given in [5],[8],[9],[10],[11]. There exist other solutions of the same classification problem that use the technique of the vector coherent states [12],[13],[14].

Note that the problem of classification of states of the five-dimensional quasi-spin appears not only in the shell model of nuclear structure described above but also in of classification of states of two-dimensional oscillator [6],[5],[8], in the description of states of the Bohr-Mottelson model [15] in the model of interacting bosons [16] and in others (see references in [14]).

In the present paper another solution of the problem of construction of the fourth quantum number is given. The quantum number constructed in the paper has several advantages over quantum numbers constructed earlier. As in [6],[7] the fourth quantum number is introduces using the creation operator for this quantum number. This allows to give a simple physical interpretation for the new quantum number. But in contrast to [6],[7] our creation operator changes other quantum numbers in a very simple way.

We use as creation operator the noncommutative pfafians associated with the algebra $\mathfrak{o}_5$.

The paper consists of two parts. In the first part we give a construction of the fourth quantum number using noncommutative pfaffians. In this construction and in the proof that our quantum number does solve the problem of classification of states a technical theorem is used. The second part of the paper, named the appendix, is devoted to the proof of this theorem.

2 The orthogonal algebra and nocommutative pfaffians

To present our construction of the fourth quantum number we must introduce noncommutative pfaffians and explain, what is the Gelfand-Tsetlin-Molev base of a $\mathfrak{o}_5$-representation.

Let $\Phi = (\Phi_{ij})$, $i, j = 1, ..., 2n$ be a skew-symmetric $2n \times 2n$-matrix, whose matrix entries belong to a noncommutative ring.

Definition 1. The noncommutative pfaffian of Φ is defined by the formula

$$Pf\Phi = \frac{1}{n!2^n} \sum_{\sigma \in S_{2n}} (-1)^{\sigma} \Phi_{\sigma(1)\sigma(2)} ... \Phi_{\sigma(2n-1)\sigma(2n)},$$

were σ is a permutation of the set $\{1, ..., 2n\}$.

In the paper a split realization of $\mathfrak{o}_N$ is used. To formulate it we use the following indexation of rows and columns of matrices from $\mathfrak{o}_N$. When N is odd the indices i, j of rows and columns belong to the set $\{-n, ..., -1, 0, 1, ..., n\}$, where $n = \frac{N-1}{2}$. When N is even the indices i, j belong to the set $\{-n, ..., -1, 1, ..., n\}$, where $n = \frac{N}{2}$. Shortly in both cases this set of indices is denoted in the paper as $\{-n, ..., n\}$.

Then the algebra $\mathfrak{o}_N$ is defined as the Lie algebra spanned by matrices $F_{ij} = E_{ij} - E_{-j-i}$, where E_{ij} are matrix units.

One can prove that elements $F_{-n-n}, ..., F_{-1-1}$ form a base in the Cartan subalgebra, and elements $F_{ij}, j < -i$ are root elements.

The precise correspondence is the following. Let e_i be the element F_{ii}^* in the dual space to the Cartan subalgebra. Put $e_{-r} := -e_r$ and $e_0 = 0$. Then the element F_{ij} corresponds to the root $e_i - e_j$.

The commutation relations between the generators are the following

$$[F_{ij}, F_{kl}] = \delta_{kj}F_{il} - \delta_{il}F_{kj} - \delta_{-ki}F_{-jl} + \delta_{-lj}F_{k-i}.$$

In the paper the following noncommutative pfaffians are considered.

Definition 2. Let F be the matrix $F = (F_{ij})$. For every subset $I \subset \{-n, ..., n\}$ which consists of an even number k of elements define a submatrix F_I by the formulae $F_I = (F_{ij})_{-i,j\in I}$. Put

$$PfF_I = Pf(F_{-ij})_{-i,j\in I}.$$

In [18] the author in terms of these pfaffians defines some special elements of $U(\mathfrak{o}_N)$ called the Capelli elements $C_k = \sum_{I\subset\{-n,...,n\},|I|=k} PfF_I PfF_{-I}$, $k = 2, 4, ..., [\frac{N}{2}]$. It is proved that the elements C_k belong to the center of $U(\mathfrak{o}_N)$.

In the case $N = 2n + 1$ put

$$PfF_{\widehat{-n}} := PfF_{\{-n+1,...,n\}} \text{ and } PfF_{\widehat{n}} := PfF_{\{-n,...,n-1\}}$$

If one identifies $\mathfrak{o}_5$ with the quasi-spin algebra then the pfaffians $PfF_{\widehat{2}}$, $PfF_{\widehat{-2}}$ are written as follows

$$PfF_{\widehat{2}} = A(-1) \star \frac{\tau_+}{\sqrt{2}} + A(0) \star \tau_0 + A(1) \star \frac{\tau_-}{\sqrt{2}},$$

$$PfF_{\widehat{-2}} = B(-1) \star \frac{\tau_-}{\sqrt{2}} + B(0) \star \tau_0 + B(1) \star \frac{\tau_+}{\sqrt{2}},$$

where $a \star b = \frac{1}{2}(ab + ba)$.

3 The Gelfand-Tsetlin-Molev base of a $\mathfrak{o}_5$-representation

For the construction of the fourth quantum number we use the Gelfand-Tsetlin-Molev base of a $\mathfrak{o}_5$-representation. In this section we give its definition and describe the action of the pfaffian $PfF_{\widehat{2}}$ in this base.

The Gelfand-Tsetlin-Molev base of a $\mathfrak{o}_{2n+1}$-representation is a base of a Gelfand-Tsetlin type, whose construction is based on restrictions $\mathfrak{o}_{2n+1} \downarrow \mathfrak{o}_{2n-1}$, in contrast to the classical Gelfand-Tsetlin base, whose construction is based on restrictions $\mathfrak{o}_N \downarrow \mathfrak{o}_{N-1}$.

One can give the following formal definition. The Gelfand-Tsetlin-Molev base is a weight base of a $\mathfrak{o}_{2n+1}$-representation and for different n the procedures of constructions of such bases must be coherent in the following sense. A base of a $\mathfrak{o}_{2n+1}$-representation must be a union of bases in $\mathfrak{o}_{2n-1}$-representation, into which a $\mathfrak{o}_{2n+1}$-representation splits when one restricts $\mathfrak{o}_{2n+1} \downarrow \mathfrak{o}_{2n-1}$.

Note that in the case $\mathfrak{sp}_{2n}$ an analogous base was firstly constructed by Zhelobenko [17].

Describe a base of a $\mathfrak{o}_5$-representation. All notations are taken from [18]. In construction of such a base only one restriction $\mathfrak{o}_5 \downarrow \mathfrak{o}_3$ appears. Hence base vectors are weight vectors for the algebra $\mathfrak{o}_5$ and every base vector is contained in a $\mathfrak{o}_3$-representation that appears, when one restricts $\mathfrak{o}_5 \downarrow \mathfrak{o}_3$.

The base vectors of a $\mathfrak{o}_5$-representation with the highest weight (λ_1, λ_2),

$$0 \geq \lambda_1 \geq \lambda_2$$

are indexed by tableaus Λ of type

λ_1, λ_2
$\sigma, \lambda'_{21}, \lambda'_{22}$
λ_{11}
σ_1, λ'_{11}

Let us give an interpretation of these numbers and give inequalities for them.

1. The number λ_{11} is a weight of a $\mathfrak{o}_3$-representation λ_{11} that contains the base vector. This number must satisfy the inequalities

$$0 \geq \lambda_1 \geq \lambda_{11} \geq \lambda_2.$$

2. The numbers σ_1 and λ'_{21} give an index of a base vector in a $\mathfrak{o}_3$-representation with the highest weight λ_{11} that contains the considered base vector. The constraints on these numbers are

$$0 \geq \lambda'_{21} \geq \lambda_{21}$$

and $\sigma_1 = 0, 1$.

3. The numbers σ, $\lambda'_{21}, \lambda'_{22}$ define an element of a base in the space of $\mathfrak{o}_3$-highest vectors with the $\mathfrak{o}_3$-weight λ^+_{11}. These numbers satisfy the constraints

$$0 \geq \lambda'_{21} \geq \lambda_1 \geq \lambda'_{22} \geq \lambda_2,$$

$$0 \geq \lambda'_{21} \geq \lambda_{11} \geq \lambda'_{22},$$

$$\sigma = 0, 1; \text{ if } \lambda'_{21} = 0, \text{ then } \sigma = 0.$$

All numbers λ are simultaneously integer or half integer.

Let us establish a relation between quantum numbers N, T, τ_0 and elements of the tableau Λ.

The number T is a highest weight of a $\mathfrak{o}_3$-representation that contains the base vector. Thus

$$T = \lambda_{11}.$$

The number τ_0 is the index of the base vector in it $\mathfrak{o}_3$-representation that contains it. In the Gelfand-Tsetlin-Molev base an element of a $\mathfrak{o}_3$-representation with the highest weight $0 \geq \lambda_{11}$ is indexed by numbers $\sigma_1 = 0, 1$ and λ'_{11}, such that $0 \geq \lambda'_{11} \geq \lambda_{11}$. Traditionally the elements of a weight base are indexed by one number τ_0, such that $\lambda_{11} \leq \tau_0 \leq -\lambda_{11}$. The relation between these approaches is the following

$$\tau_0 = (-1)^{\sigma_1} \lambda'_{11}.$$

The number $N = F_{22}$ is expressed as follows [18]

$$N = \sigma + 2(\lambda'_{2,1} + \lambda'_{2,2}) - (\lambda_1 + \lambda_2) - \lambda_{1,1}$$

From the mathematical point of view the problem of construction of the fourth quantum number is a problem of construction of an indexation of the vectors of a base space of $\mathfrak{o}_3$-highest vectors with the $\mathfrak{o}_3$-weight T (this space is called the multiplicity space) using the number N and some new quantum number.

Using $\sigma, \lambda'_{2,1}, \lambda'_{2,2}$ one can solve the problem of construction of the fourth quantum number. But then the physical interpretation of the obtained quantum number is difficult.

Introduce notation $\rho_i = i - \frac{1}{2}$ for $i > 0$, and $\rho_{-i} = -\rho_i$.

Define

$$\gamma_i = \lambda'_{2i} + \rho_i + \frac{1}{2}.$$

Also put

$$\overline{\sigma} = \sigma + 1 \, mod \, 2.$$

Theorem 1. *On the vector ξ_Λ the pfaffian $PfF_{\widehat{2}}$ acts as follows.*

Let $\Lambda = (\lambda, \sigma, \lambda', \Lambda')$, where λ is the first row of Λ, σ, λ' is the second row of Λ and Λ' is the remaining part of Λ.

Let $\xi_{\sigma,\lambda',\Lambda'}$ be the corresponding base vector.

If $\sigma = 0$, then

$$PfF_{\widehat{2}}\xi_{\sigma,\lambda',\Lambda'} = \xi_{\bar{\sigma},\lambda,\Lambda'}.$$

If $\sigma = 1$, then

$$PfF_{\widehat{2}}\xi_{\sigma,\lambda',\Lambda'} = (-1)^n \sum_{j=1}^{2} \Pi_{t=1,t\neq j}^{2} \frac{-\gamma_t^2}{\gamma_j^2 - \gamma_t^2} \xi_{\bar{\sigma},\lambda'+\delta_j,\Lambda'},$$

Where $\lambda' + \delta_j$ is the row λ' with 1 added to the j-th component.

The general form of the theorem 1 (the theorem 4) is proved in the appendix.

4 Construction of an additional quantum number using pfaffian

Let us show how to construct the fourth quantum number using pfaffians.

Denote as $|l_1, ..., l_p >$ a vector corresponding to quantum numbers $l_1, ..., l_p$.

The main result of this section is the following.

Theorem 2. *There exists an indexation of base weight vectors of a representation of $\mathfrak{o}_5$ by numbers T, τ_0, N, k, where the additional number k is a nonnegative integer. States which have different collections of indices are independent.*

If $N < 0$, then $PfF_{\widehat{2}}$ maps $|T, \tau_0, N, k >$ to $|T, \tau_0, N + 1, k + 1 >$.

If $N > 0$, then $PfF_{\widehat{-2}}$ maps $|T, \tau_0, N, k >$ to $|T, \tau_0, N - 1, k + 1 >$.

All the rest part of the section is devoted to the construction of the quantum number k with the prescribed properties and to the proof that the numbers T, τ_0, N, k are sufficient for the classification the base vectors of a $\mathfrak{o}_5$-representation.

Proof. First of all we construct the additional number for the state for with $N \leq 0$ and prove that the new quantum number solves the problem of

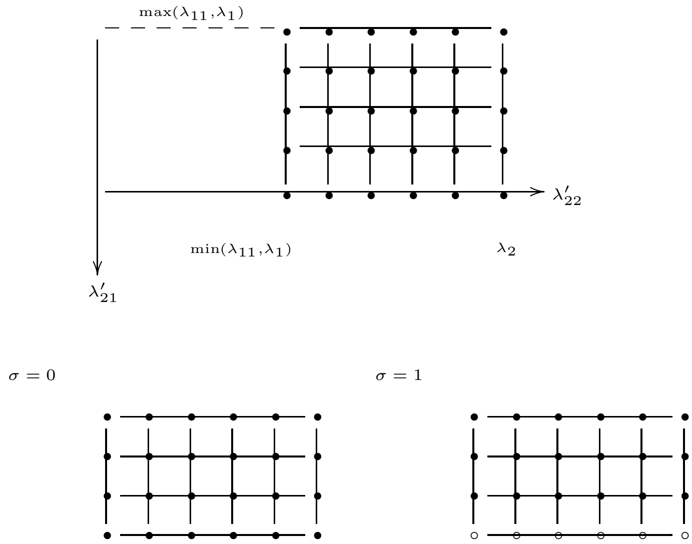


Figure 1: The cases $\sigma = 0$ and $\sigma = 1$.

classification. Then for states with $N > 0$ the quantum number is constructed using the reflection of the algebra $\mathfrak{o}_5$.

The construction of the quantum number and the proof that it solves the classification problem is done by induction by N, the numbers T, τ_0 are suggested to be fixed.

The base vectors of the Gelfand-Tsetlin-Molev base of a $\mathfrak{o}_5$-representation are encoded by tableaus, which include actually the numbers λ''_{11}, σ_1 related to T, τ_0, and also the numbers $\sigma, \lambda'_{2,1}, \lambda'_{2,2}$. The restriction on $\sigma, \lambda'_{2,1}, \lambda'_{2,2}$ are the following

$$0 \geq \lambda'_{2,1} \geq \lambda_1 \geq \lambda'_{2,2} \geq \lambda_2,$$

$$\lambda'_{2,1} \geq \lambda_{1,1} \geq \lambda'_{2,2},$$

$$\sigma = 0, 1.$$

In other words the point $(\lambda'_{2,1}, \lambda'_{2,2})$ belongs to the rectangle on the figure 4.

If $\sigma = 0$ then the points on the lower edge are included, and if $\sigma = 1$ then they are not included, see figure 1.

The rectangle can have one of two forms, see figure 2.

In the first case we say that the rectangle is narrow and in the second case we say that the rectangle is wide.

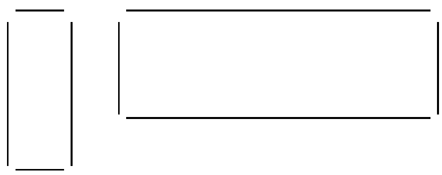


Figure 2: A narrow and a wide rectangle.

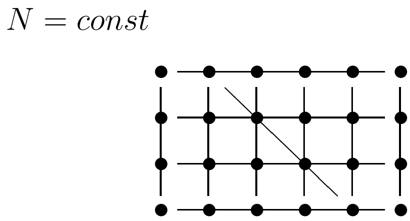


Figure 3: Points corresponding to the fixed N

Write the formula for $N = F_{22}$ in the Gelfant-Tsetlin-Molev base [18]

$$N = \sigma + 2(\lambda'_{2,1} + \lambda'_{2,2}) - (\lambda_1 + \lambda_2) - \lambda_{21}$$

The point with fixed N belong to the line $\lambda'_{21} + \lambda'_{22} = const$, see figure 3.

The line $N = 0$ cuts the rectangle into two equal parts, see figure 4.

Now we can begin the construction of the quantum number k for the vectors with $N \leq 0$.

Remind that the construction is inductive by N with fixed numbers T, τ_0.

Base of induction. Note that the state with $N = N_{min}$ and fixed T, τ_0

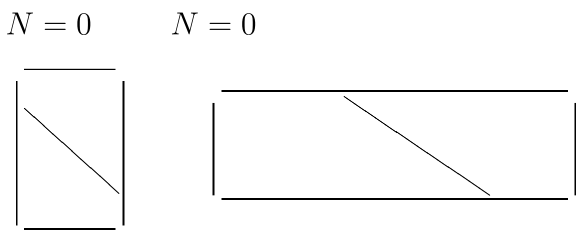


Figure 4: Line $N = 0$

is unique. This follows from the fact that to the minimal value of N there corresponds the upper right angle of the rectangle. In other words for the minimal N one has $\lambda'_{2,1} = max\{\lambda_1, \lambda_{1,1}\}$, $\lambda'_{2,2} = \lambda_2$, $\sigma = 0$.

Thus all numbers in the tableau Λ are uniquely defined, hence the states with $N = N_{min}$, and fixed T, τ_0 is unique.

Assign to this state the zero number k. Since for $N = N_{min}$ there exist only one independent state for $N = N_{min}$ all states are classified by four quantum numbers by obvious reasons.

Preliminary discussion of the induction process. Suppose that for $N = N^*$ the states are classified by the number k. Consider the case $N = N^* + 1$.

A base in the space of states with fixed T, τ_0, N is formed by those vectors of the Gelfand-Tsetlin-Molev base, for which the second row of the tableau Λ has the form

$$(\sigma, K - t, t),$$

where K and N^* are related by $N^* = \sigma + 2K - (\lambda_1 + \lambda_2) - T$.

Denote such vectors $\xi_t^{N^*}$.

Note that for fixed N one has for all states either $\sigma = 0$, or $\sigma = 1$.

The index t runs only the values for which the point $(\lambda'_{2,1}, \lambda'_{2,2}) = (K-t, t)$ belongs to the rectangle.

The action of the pfaffian $PfF_{\widehat{2}}$ on the second row of the tableau Λ is given (see Theorem 1) by formulae:

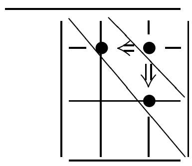
$$(0, \lambda'_{2,1}, \lambda'_{2,2}) \mapsto (1, \lambda'_{2,1}, \lambda'_{2,2}) \tag{1}$$

$$(1, \lambda'_{2,1}, \lambda'_{2,2}) \mapsto c_1(0, \lambda'_{2,1} + 1, \lambda'_{2,2}) + c_2(0, \lambda'_{2,1}, \lambda'_{2,2} + 1), \tag{2}$$

where $c_1 = \frac{-\gamma_2^2}{(\gamma_1^2 - \gamma_2^2)}$, $c_2 = \frac{-\gamma_1^2}{(\gamma_2^2 - \gamma_2^2)}$, $\gamma_1 = \lambda'_{2,1}$, $\gamma_2 = \lambda'_{2,2} - 1$.

If a row contained in these formulas does not satisfy the constraints on the second row of a Gelfand-Tsetlin-Molev tableau (that is the corresponding point does not belong to the rectangle), then it must be replaced to zero.

Since $\lambda'_{2,2}$ is non-positive, then $c_2 \neq 0$, and $c_1 = 0$ if and only if $\lambda'_{2,1} = 0$. But then since $\lambda'_{2,1} + 1 > 0$ the vector $(0, \lambda'_{2,1} + 1, \lambda'_{2,2})$ which is multiplied by c_1 is zero. That is why we can put in this case $c_1 = 1$ in the formula 2 and suggest that always $c_1 \neq 0$.



$N = N^*,\ \sigma = 1$

$N = N^* + 1,\ \sigma = 0$

Figure 5: The action of the pfaffian $PfF_{\hat{2}}$ in the case $\sigma = 1$

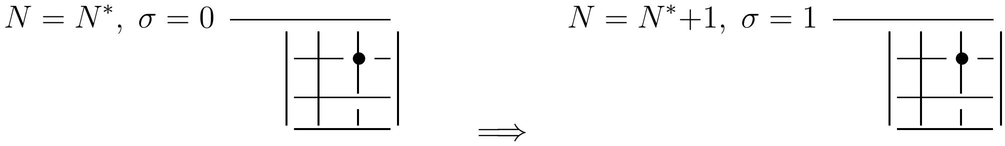


Figure 6: The action of the pfaffian $PfF_{\hat{2}}$ in the case $\sigma = 0$

To the formulas 1 and 2 for the action of the pfaffian $PfF_{\hat{2}}$ the following figures correspond. To the formula 1 there corresponds the figure 5.

To the formula 2 there corresponds the figure 6

Thus, if one considers the vectors $\xi_t^{N^*}$ for all t, then one gets that the matrix of mapping from the span $<\xi_t^{N^*}>_{t\in\mathbb{Z}}$ to the span $<\xi_t^{N^*+1}>_{t\in\mathbb{Z}}$, defined by formula 1,2 is the following. If $N = N^*$ is such that $\sigma = 0$, then the matrix is unit. If $N = N^*$ is such that $\sigma = 1$, then the matrix has nonzero (i,i) and $(i+1,i)$. elements.

At last we can do a step of induction. Consider the cases i)-iii) in dependence of geometry of the figure which the line $N = N^*$ cuts from the rectangle on the upper right corner.

For the a narrow rectangle there two different case and for a wide rectangle there also two case. They are presented on the figure 7

The considerations in the cases A and C are similar, thus we actually have three cases.

For a fixed $N = N^*$ we have that for all N either $\sigma = 0$ or $\sigma = 1$. Thus in each of three cases above we have to consider two subcases.

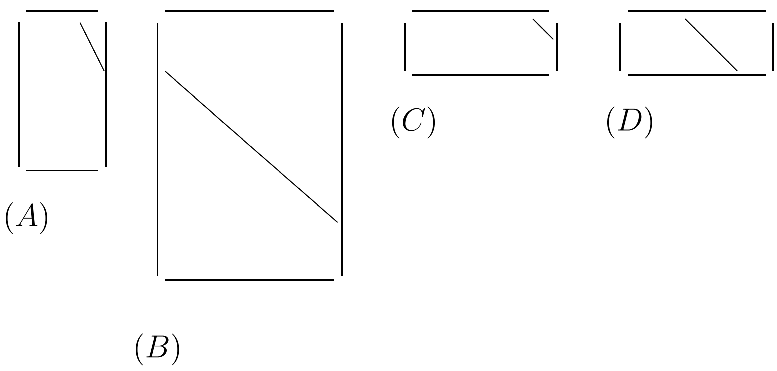


Figure 7: The intersection of the rectangle and the line $N = const$.

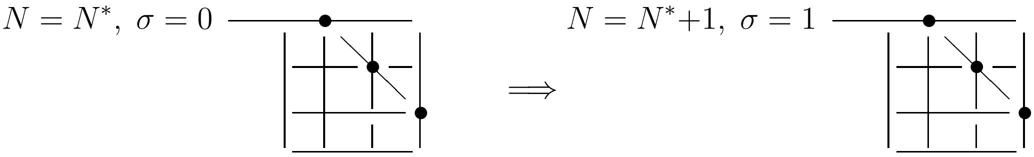


Figure 8: The action of the pfaffian $PfF_{\hat{2}}$ in the case $\sigma = 0$

Cases A and C.

Firstly consider the case $\sigma = 0$. The pfaffian $PfF_{\hat{2}}$ acts as it is shown on the figure 8, i.e. the pfaffian changes $\sigma = 0$ to $\sigma = 1$.

Obviously in this case the space with $N = N^*$ is mapped isomorphically to the space of states with $N = N^* + 1$.

Using this isomorphism we assign to the states with $N = N^* + 1$ the fourth quantum number by the following rule. If a state with $N = N^*$ has the fourth quantum number k then state's image has the fourth quantum number equal to $k + 1$.

By induction one immediately gets that the one can classify all states with $N = N^* + 1$ with four quantum numbers.

Consider now the case $\sigma = 1$. The pfaffian $PfF_{\hat{2}}$ acts as it is shown on the figure 9.

Using the formulae of the pfaffian's and information about the matrix of this action we see that the pfaffian defines an injective mapping and the

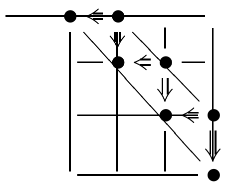


Figure 9: The action of the pfaffian $PfF_{\hat{2}}$ in the case $\sigma = 1$

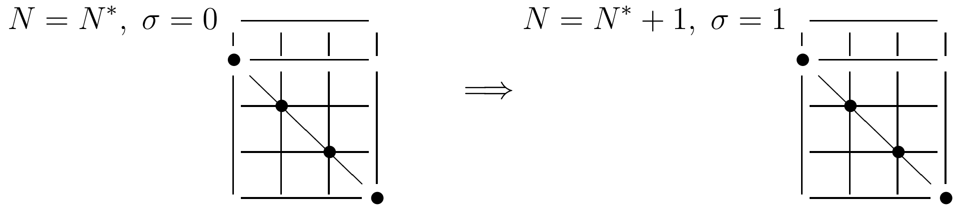


Figure 10: The action of the pfaffian $PfF_{\hat{2}}$ in the case $\sigma = 0$

codimension of the image equals 1.

Choose as a complementary vector to the image a vector corresponding to the lower point on the line $N = N^* + 1$. To the corresponding states we assign zero value if the fourth quantum number. For the states from the image of the pfaffian we use the following rule. If a state with $N = N^*$ has the fourth quantum number k, then its image has the fourth quantum number $k + 1$.

One gets by induction that one can classify all states with $N = N^* + 1$ with four quantum numbers.

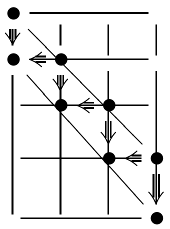
Case B.

Consider the case when for $N = N^*$ we have $\sigma = 0$, then the pfaffian acts from the space of states with $N = N^*$ to the space of states with $N = N^* + 1$ as it is shown on the figure 10.

One sees that the pfaffian is an isomorphism of these spaces.

If a state with $N = N^*$ has the fourth quantum number k, then we assign to its image the fourth quantum number $k + 1$.

One gets by construction that one can classify all states with $N = N^* + 1$ with four quantum numbers.

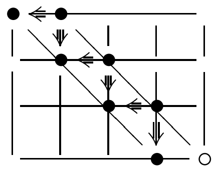


$N = N^*, \ \ \sigma = 1$

$N = N^* + 1, \ \ \sigma = 0$

Figure 11: The action of the pfaffian $PfF_{\hat{2}}$ in the case $\sigma = 1$

$N = N^* + 1, \sigma = 0$



$N = N^*, \sigma = 1$

Figure 12: The action of the pfaffian $PfF_{\hat{2}}$ in the case $\sigma = 1$

If for $N = N^*$ we have $\sigma = 1$, then the pfaffian acts from the space of states with $N = N^*$ to the spaces of states with $N = N^* + 1$ as it is shown on the figure 11.

Again using information about the action of the pfaffian one sees that the pfaffian is an isomorphism of these spaces.

If a state with $N = N^*$ has the fourth quantum number k, then we assign to its image has the fourth quantum number $k + 1$.

One gets by construction that one can classify all states with $N = N^* + 1$ with four quantum numbers.

Since we consider only the states with $N < 0$, then the case of a wide rectangle is considered completely.

Case D

Let $N = N^*$ be such that one has $\sigma = 1$. Then the pfaffian act as it shown on the figure 12.

One sees that the pfaffian is an injective mapping and the codimension of the image equals 1. As a complimentary vector to the image we take a vector corresponding to the lower point on the line $N = N^* + 1$.

Assign to this state the zero fourth number, and to the vectors from the

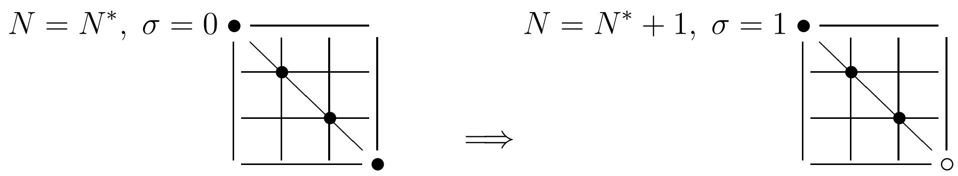


Figure 13: The action of the pfaffian $PfF_{\hat{2}}$ in the case $\sigma = 1$

image of the pfaffian assign the fourth number according to the following rule. If a state with $N = N^*$ has the fourth quantum number k, then its image has the fourth quantum number $k + 1$.

One gets by induction that one can classify all states with $N = N^* + 1$ with four quantum numbers.

Now let $N = N^*$ be such that one has $\sigma = 0$. Then the pfaffian acts as it is shown on the figure 13.

The pfaffian acts as a non-injective mapping, it has a one-dimensional kernel. This kernel is generated by the state corresponding to the lower point on the line $N = N^*$. This is just the vector to which we have assigned a definite quantum number on the previous step. The image is factor space by the kernel, but since the vectors in the kernel have definite fourth quantum number one can define correctly the fourth quantum number for the vectors in the factor space. As usual if a state with $N = N^*$ has the fourth quantum number k, then its image has the fourth quantum number $k + 1$.

One gets by construction that one can classify all states with $N = N^* + 1$ with four quantum numbers.

Thus we have assigned the fourth quantum number for all states with $N \leq 0$. The induction is completed.

Now let us construct the fourth quantum number for the states with $N > 0$.

There exists a reflection ω in the Weil group of the root system B_2, that transforms x into $-x$. This element acts on the algebra $U(\mathfrak{o}_5)$, and maps F_{ij} into F_{ji}, one has

$$\omega(PfF_{\widehat{-2}}) = -PfF_{\hat{2}}.$$

Also ω acts in every representation V of the algebra $\mathfrak{o}_5$. Under the action of ω in a representation a weight space V_λ is mapped into $V_{-\lambda}$.

The action of ω on the algebra and in the representation are related by the formula

$$\omega(g)\omega(v) = \omega(gv),$$

where $g \in U(\mathfrak{o}_5)$, $v \in V$.

Now let us construct for the states with $N \geq 0$ the fourth quantum number by the formula

$$k(v) = k(\omega v).$$

Note that for the states with $N = 0$ one actually obtains two ways for definition of the fourth quantum number. Fist time we have assigned the quantum number when we have considered states with $N \geq 0$ and second time we have assigned the quantum number when we have considered states with $N \leq 0$. We are going to prove that they are equivalent.

This follows from the fact the the fourth quantum number together with N give an indexation of $\mathfrak{o}_3$-highest vectors with $\mathfrak{o}_3$-weight T. If one considers such $\mathfrak{o}_3$-highest vectors with $N = 0$, then the mapping ω acts on the space of such $\mathfrak{o}_3$-highest as multiplication by scalar. Indeed, since being restricted on this space ω turns into analogous $\mathfrak{o}_3$-reflection, then it preserves $\mathfrak{o}_3$ representations. Thus each $\mathfrak{o}_3$-highest vector is mapped into itself multiplied by a scalar. Since this is true for all $\mathfrak{o}_3$-highest vectors these scalars are the same for all $\mathfrak{o}_3$-highest vectors.

That is why two way of construction of the fourth quantum number for the states with $N = 0$ are equivalent.

$\square$

Thus for $N < 0$ the pfaffian $PfF_{\widehat{-2}}$ acts as a raising operator for the quantum number k, and for $N > 0$ the pfaffian $PfF_{\widehat{2}}$ acts as a raising operator for the the quantum number k

The theorem 2 is proved.

Note that the numbers N and T give an indexation of base vectors in the space of $\mathfrak{o}_3$-highest vectors with the $\mathfrak{o}_3$-weight T.

5 Appendix. Proof of the theorem 4

To prove the theorem we need first to investigate the action of pfaffians in representations.

5.1 Action of a pfaffian on a weight vector.

Remind that e_i denotes the standard base vectors F_{ii}^* of dual space to the Cartan subalgebra.

Proposion 1. *Let V be a representation of $\mathfrak{o}_N$. Under the action of the pfaffian PfF_I a weight vector with the weight μ is mapped to a weight vector with the weight $\mu - \sum_{i\in I} e_i$.*

Proof. If v is a weight vector in a representation of $\mathfrak{o}_N$ with the weight μ, g_α is a root element in $\mathfrak{o}_N$ corresponding to the root α, then $g_\alpha v_\mu$ is a weight vector of to the weight $\alpha + \mu$.

Consider the vector $PfF_I v$. By definition one has

$$PfF_I = \frac{1}{\frac{k}{2}!2^{\frac{k}{2}}} \sum_{\sigma\in S^k} (-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} ... F_{-\sigma(i_{k-1})\sigma(i_k)}.$$

To prove the proposition it suffices to show that every summand changes the weight by subtracting of the same expression $-\sum_{i\in I} e_i$. Using the correspondence between roots and elements F_{ij} from the sec. 2 one gets the following. When one acts by $F_{-\sigma(i_1)\sigma(i_2)} ... F_{-\sigma(i_{k-1})\sigma(i_k)}$ on v then to the weight the vector

$$e_{-\sigma(i_1)} - e_{\sigma(i_2)} - ... + e_{-\sigma(i_{k-1})} - e_{\sigma(i_k)} = -\sum_{i\in I} e_i$$

is added. This proves the proposition.

$\square$

Consider the most interesting case $\mathfrak{o}_N = \mathfrak{o}_{2n+1}$ and $|I| = 2n$.

Corrolary 1. *Let $\mathfrak{o}_N = \mathfrak{o}_{2n+1}$.*

The action of $PfF_{\widehat{-n}}$ adds the vector $-\sum_{i\in I} e_i = -e_n$ to the weight.

The action of $PfF_{\widehat{n}}$ adds the vector $-\sum_{i\in I} e_i = -e_{-n} = e_n$ to the weight.

5.2 Commutators of pfaffians and F_{ij}.

Lemma 1. *Let $I = \{i_1, ..., i_k\}$, where k is even. Then the commutator $[PfF_I, F_{j_1-j_2}]$ is calculated according to the following rule.*

1. *If $j_1, j_2 \notin I$, then $[PfF_I, F_{j_1-j_2}] = 0$.*

2. *If* $j_1 \in I, j_2 \notin I$, *then* $[PfF_I, F_{j_1 -j_2}] = PfF_{I_{j_1 \to -j_2}}$.

3. *If* $j_1 \notin I, j_2 \in I$, *then* $[PfF_I, F_{j_1 -j_2}] = -PfF_{I_{j_2 \to -j_1}}$.

4. *If* $j_1 \in I, j_2 \in I$, *then* $[PfF_I, F_{j_1 -j_2}] = PfF_{I_{j_1 \to -j_2}} - PfF_{I_{j_2 \to -j_1}}$

Proof. The quadratic form is $G = (\delta_{i,-j})$. Hence one can identify E_{ij} with $e_i \otimes e_{-j}$. Then F_{ij} is identified with $e_i \wedge e_{-j}$. Remind that

$$PfF_I = \frac{1}{2^{\frac{k}{2}}\frac{k}{2}!}\sum_{\sigma \in S_k}(-1)^{\sigma}F_{-\sigma(i_1)\sigma(i_2)}...F_{-\sigma(i_{k-1})\sigma(i_k)}.$$

Thus PfF_I with indexing set $I = \{i_1, ..., i_k\}$ is identified with the polyvector $e_{-i_1} \wedge ... \wedge e_{-i_k}$.

This identification is compatible with the action of $\mathfrak{o}_N$. Thus

$$[PfF_I, F_{j_1 -j_2}] = -[F_{j_1 -j_2}, PfF_I] =$$
$$F_{j_1 -j_2}e_{i_1} \wedge e_{i_2} ... \wedge e_{i_k} + e_{i_1} \wedge F_{j_1 -j_2}e_{i_2} ... \wedge e_{i_k} + e_{i_1} \wedge e_{i_2} ... \wedge F_{j_1 -j_2}e_{i_k}.$$

One has $F_{j_1 -j_2}e_{-i_1} = e_{j_2}$ if $j_2 = i_1$ and $F_{j_1 -j_2}e_{-i_1} = -e_{j_2}$ if $j_1 = i_1$.

Suggest that $\{j_1, j_2\} \cap I = \emptyset$. Then $[PfF_I, F_{j_1 -j_2}] = 0$.

Suggest that $j_1 \in I, j_2 \notin I$. Then $j_1 = i_t$ and the only nonzero summand is that containing $F_{j_1 j_2}e_{-i_t}$. Thus we have $[PfF_I, F_{j_1 -j_2}] = -PF_{I_{j_1 \mapsto -j_2}}$. Here $I_{j_1 \mapsto -j_2}$ is obtained from I by replacing the index j_1 to j_2.

The case $j_1 \notin I, j_2 \in I$ is considered in the same manner.

Suggest that $\{j_1, j_2\} \subset I$. That is $j_1 = e_{i_t}$, $j_2 = e_{i_s}$. Then in the sum there are two nonzero summands one contains $F_{j_1 j_2}e_{i_t}$, the other contains $F_{j_1 j_2}e_{i_s}$. Each of them is a wedge product with a new indexing set. So one gets that $[PfF_I, F_{j_1 j_2}] = PfF_{I_{j_1 \to -j_2}} - PfF_{I_{j_2 \to -j_1}}$.

□

Corrolary 2. *In the case* $\mathfrak{o}_{2n+1}$ *the pfaffians* $PfF_{\widehat{n}}$, $PfF_{\widehat{-n}}$ *commute with elements* F_{ij}, $-n < i, j < n$, *that span the subalgebra* $\mathfrak{o}_{2n-1}$

5.3 Some formulas involving pfaffians.

In this subsection some summation formulas are proved.

Lemma 2. *$PfF_I = \frac{(\frac{p}{2})!(\frac{q}{2})!}{(\frac{k}{2})!}\sum_{I=I'\sqcup I'',|I'|=p,|I''|=q}(-1)^{(I'I'')}PfF_{I'}PfF_{I''}$.*

Here $(-1)^{(I'I'')}$ is a sign of a permutation of the set $I=\{i_1,...,i_k\}$ that places first the subset $I'\subset I$ and then the subset $I''\subset I$.

The numbers p, q are even fixed numbers, they satisfy $p+q=k=|I|$.

Proof. By definition one has

$$PfF_I = \frac{1}{2^{\frac{k}{2}}(\frac{k}{2})!}\sum_{\sigma\in S_k}(-1)^{\sigma}F_{-\sigma(i_1)\sigma(i_2)}...F_{-\sigma(i_{k-1}),\sigma(i_k)}.$$

The summand $(-1)^{\sigma}F_{-\sigma(i_1)\sigma(i_2)}...F_{-\sigma(i_{k-1}),\sigma(i_k)}$ can be written as

$$(-1)^{(I'I'')}(-1)^{\sigma'}F_{-\sigma'(i'_1)\sigma'(i'_2)}...F_{-\sigma'(i'_{p-1}),\sigma'(i'_p)}(-1)^{\sigma''}F_{-\sigma''(i''_1)\sigma''(i''_2)}...F_{-\sigma''(i''_{q-1}),\sigma''(i''_q)}.$$

Here $I'=\{i'_1,...,i'_p\}$ is the set of indices $\{\sigma(i_1),...,\sigma(i_p)\}$ placed in a natural order, $I''=\{i''_1,...,i''_q\}$ is set of indices $\{\sigma(i_{p+1}),...,\sigma(i_k)\}$ placed in a natural order, σ' is a permutation $\{\sigma(i_1),...,\sigma(i_p)\}$ of the set I' and σ'' is a permutation of the set I'' defined in a similar way. Note that $(-1)^{(I'I'')}(-1)^{\sigma'}(-1)^{\sigma''}=(-1)^{\sigma}$.

The mapping $\sigma\mapsto I',I'',\sigma',\sigma''$ is bijective.

Thus the pfaffian can be written as
$\frac{(\frac{p}{2})!(\frac{q}{2})!}{(\frac{k}{2})!}\sum_{I=I'\sqcup I'',|I'|=p,|I''|=q}(-1)^{(I'I'')}\frac{1}{2^{\frac{k}{2}}(\frac{p}{2})!(\frac{q}{2})!}\sum_{\sigma'}(-1)^{\sigma'}(-1)^{\sigma''}F_{-\sigma'(i'_1)\sigma'(i'_2)}...$
$...F_{-\sigma'(i'_{p-1}),\sigma'(i'_p)}F_{-\sigma''(i''_1)\sigma'(i''_2)}...F_{-\sigma''(i''_{q-1}),\sigma''(i''_q)}=$
$=\frac{(\frac{p}{2})!(\frac{q}{2})!}{(\frac{k}{2})!}\sum_{I=I'\sqcup I'',|I'|=p,|I''|=q}(-1)^{(I'I'')}PfF_{I'}PfF_{I''}$ □

Corrolary 3. *If $|I|=k$, then*
$PfF_I=\frac{1}{\frac{k}{2}+1}\sum_{I=I'\sqcup I''}(-1)^{(I'I'')}\frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!}PfF_{I'}PfF_{I''}$.

Lemma 3. *Let $-n\in I$. Then $PfF_I=$*
$=\sum_{i\in I\setminus\{-n\}}\sum_{I\setminus\{-n,i\}=I'\sqcup I''}\frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!}(-1)^{(I'-niI'')}PfF_{I'}F_{ni}PfF_{I''}$.

Here $(-1)^{(I'-niI'')}$ is a sign of the permutation $(I',-n,i,I'')$ of the set I.

Proof. By definition one has

$$PfF_I = \frac{1}{2^{\frac{k}{2}}(\frac{k}{2})!}\sum_{\sigma\in S_k}(-1)^{\sigma}F_{-\sigma(i_1)\sigma(i_2)}...F_{-\sigma(i_{k-1}),\sigma(i_k)}.$$

Since $F_{ij} = -F_{-j-i}$ the summation can be taken only over such permutation such that $\sigma(i_{2t-1}) < \sigma(i_{2t})$. But if the summation is done in such a way the multiple $\frac{1}{2^{\frac{k}{2}}}$ must be omitted.

Fix a such a permutation σ and find a place such that $(\sigma(i_{2t-1}), \sigma(i_{2t})) = (-n, i)$. The summand $(-1)^{\sigma} F_{-\sigma(i_1)\sigma(i_2)} ... F_{-\sigma(i_{k-1}),\sigma(i_k)}$ can be written as

$(-1)^{(I' in I'')}(-1)^{\sigma'} F_{-\sigma'(i'_1)\sigma'(i'_2)} ... F_{-\sigma'(i'_{p-1}),\sigma'(i_p)} F_{-ni}(-1)^{\sigma''} F_{-\sigma''(i''_1)\sigma''(i''_2)} ... F_{-\sigma''(i''_{q-1}),\sigma''(i''_q)}$.

Here $I' = \{i'_1, ..., i'_p\}$ is the set of indices $\{\sigma(i_1), ..., \sigma(i_{2t-2})\}$ placed in the natural order, $I'' = \{i''_1, ..., i''_q\}$ is set of indices $\{\sigma(i_{2t+1}), ..., \sigma(i_k)\}$ placed in the natural order, σ' is a permutation $\{\sigma(i_1), ..., \sigma(i_{2t-2})\}$ of the set I' and σ'' is a permutation of the set I'' defined in a similar way. Note that $(-1)^{(I'-niI'')}(-1)^{\sigma'}(-1)^{\sigma''} = (-1)^{\sigma}$. The permutation σ' satisfies the condition $\sigma'(i'_{2t-1}) < \sigma'(i'_{2t})$ as well as the permutation σ''.

The mapping $\sigma \mapsto I', I'', \sigma', \sigma''$ is bijective (since $\sigma(i_{2t-1}) < \sigma(i_{2t})$).

Thus the pfaffian can be written as

$$\sum_{i \in I \setminus \{n\}} \sum_{I \setminus \{i,n\} = I' \sqcup I''} \frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} \frac{1}{(\frac{|I'|}{2})!(\frac{|I''|}{2})!} (-1)^{(I'-niI'')}(-1)^{\sigma'} F_{-\sigma'(i'_1)\sigma'(i'_2)} \cdots$$

$$...F_{-\sigma'(i'_{p-1}),\sigma'(i_p)} F_{ni}(-1)^{\sigma''} F_{-\sigma''(i''_1)\sigma''(i''_2)} ... F_{-\sigma''(i''_{q-1}),\sigma''(i''_q)} =$$

$$= \sum_{i \in I \setminus \{n\}} \sum_{I \setminus \{i,n\} = I' \sqcup I''} \frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} (-1)^{(I'-niI'')} Pf F_{I'} F_{ni} Pf F_{I''}$$

□

5.4 The Mickelsson-Zhelobenko algebra and pfaffians.

In the construction of the Gelfand-Tsetlin-Molev base a key role is played by the Mickelsson-Zhelobenko algebra.

There exist a projection from the algebra $U(\mathfrak{o}_N)$ to the Mickelsson-Zhelobenko algebra $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$. To prove the theorem 4 we must calculate the image of pfaffians under this projection. The present section is devoted to the calculation of this image.

5.4.1 The Mickelsson-Zhelobenko algbera

Remind the idea of the construction of the Gelfand-Tsetlin-Molev base of a representation V of the algebra $\mathfrak{o}_{2n+1}$. An irreducible representation V of the algebra $\mathfrak{o}_{2n+1}$ becomes reducible as a representation $\mathfrak{o}_{2n-1}$. According to the idea of Gelfand and Tsetlin to construct a base in V, it is necessary to know all possible highest weights μ of $\mathfrak{o}_{2n-1}$-irreps, into which V splits. Also it is necessary to be able to construct a base in the multiplicity space, that is in the space of $\mathfrak{o}_{2n-1}$-highest vectors with the fixed $\mathfrak{o}_{2n-1}$-weight μ.

The first problem is solved quite easily, for the solution of the second one the Mickelsson-Zhelobenlo algebra is used.

Definition 3. Denote as V^+_μ the space of $\mathfrak{o}_{2n-1}$-highest vectors with the $\mathfrak{o}_{2n-1}$-weight μ in the $\mathfrak{o}_{2n+1}$-representation V.

To construct a base in V^+_μ Molev has used the Mickelsson-Zhelobenko algebra, acting on the space $\oplus_\mu V^+_\mu$.

Let us give the definition of the Mickelsson-Zhelobenko algebra, see [19],[20]. Al facts and notations are borrowed from and [18].

Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{k}$ be its reductive subalgebra. The main example is $\mathfrak{g} = \mathfrak{o}_{2n+1}$ and $\mathfrak{k} = \mathfrak{o}_{2n-1}$. Let $\mathfrak{k} = \mathfrak{k}^- + \mathfrak{h} + \mathfrak{k}^+$ be a triangular decomposition. Let $R(\mathfrak{h})$ be a field of fractions of the algebra $U(\mathfrak{h})$. Denote

$$U'(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} R(\mathfrak{h}).$$

Let

$$J' = U'(\mathfrak{g})\mathfrak{k}^+$$

be the left ideal in $U'(\mathfrak{g})$, generated by $\mathfrak{k}^+$. Put

$$M(\mathfrak{g}, \mathfrak{k}) = U'(\mathfrak{g})/J'.$$

For every positive root α of the algebra $\mathfrak{k}$ define a series

$$p_\alpha = 1 + \sum_{k=1}^{\infty} e^k_{-\alpha} e^k_\alpha \frac{(-1)^k}{k!(h_\alpha + \rho(h_\alpha) + 1)...(h_\alpha + \rho(h_\alpha) + k)},$$

where e_α is a root vector $\mathfrak{k}$, corresponding to α, h_α is a corresponding Cartan element, ρ is a half-sum of positive roots of $\mathfrak{k}$.

An order is normal if the following holds. Let a root be a sum of two roots, then it lies between them. Chose a normal ordering $\alpha_1 < ... < \alpha_m$ of positive roots of $\mathfrak{k}$.

Put

$$p = p_{\alpha_1} ... p_{\alpha_m}.$$

This element is called the extremal projector. It can be proved that nevertheless p is an infinite series it's action on $M(\mathfrak{g}, \mathfrak{k})$ by left multiplication is well defined [19].

The following equalities hold: $e_\alpha p = pe_{-\alpha} = 0$, where α is a positive root of $\mathfrak{k}$.

Put

$$Z(\mathfrak{g},\mathfrak{k}) = pM(\mathfrak{g},\mathfrak{k}).$$

This is the Mickelsson-Zhelobenko algebra. The multiplication in $Z(\mathfrak{g},\mathfrak{k})$ is defined using the isomorphism $Z(\mathfrak{g},\mathfrak{k}) = NormJ'/J'$, where $NormJ' = \{u \in U'(\mathfrak{g}) : J'u \subset J'\}$. Thus $Z(\mathfrak{g},\mathfrak{k})$ is an associative algebra and a bimodule over $R(\mathfrak{h})$ [19].

There exists a natural projection from $U(\mathfrak{o}_N)$ to $Z(\mathfrak{o}_{2n+1},\mathfrak{o}_{2n-1})$, which sends $x \in U(\mathfrak{o}_N)$ to $p\,x\,modJ'$.

Choose linear independent elements $v_1,...,v_n \in \mathfrak{g}$, such that $< v_1,...,v_n > \oplus\mathfrak{k} = \mathfrak{g}$ as linear spaces over $\mathbb{C}$. Put $z_i = pv_i modJ'$. It can be proved that monomials $\check{z}_1^{m_1}...\check{z}_n^{m_n}$, $m_i \in \mathbb{Z}^+$, form a bases of $Z(\mathfrak{g},\mathfrak{k})$ over $R(\mathfrak{h})$.

In the case $Z(\mathfrak{o}_{2n+1},\mathfrak{o}_{2n-1})$ put

$$\check{z}_{i\pm n} = pF_{i,\pm n}modJ', \quad i = -n,...,n.$$

Notations are taken from [18]. There exists an obvious symmetry $\check{z}_{ij} = \check{z}_{-j-i}$. From previous considerations it follows that $Z(\mathfrak{o}_{2n+1},\mathfrak{o}_{2n-1})$ is generated by elements $\check{z}_{ia}$, $i = 0,...,n$, $a = \pm n$ or $\check{z}_{ai}$, $i = 0,...,n$, $a = \pm n$.

Sometimes it is more useful to use the generators

$$z_{i\pm n} = \check{z}_{i\pm n}(f_i - f_{i-1})...(f_i - f_{-n+1}),$$

where

$$f_i = F_{ii} + \rho_i, \text{ for } i > 0 \ , \ f_0 = -\frac{1}{2}, \ f_{-i} = -f_i,$$

and

$$\rho_i = i - \frac{1}{2} \text{ for } i > 0 \text{ and } \rho_{-i} = -\rho_i.$$

In particular

$$z_{0n} = \check{z}_{0n}\prod_{i=1}^{n-1}(F_{ii} + i - \frac{1}{2}).$$

The Mickelsson-Zhelobenko algebra $Z(\mathfrak{o}_{2n+1},\mathfrak{o}_{2n-1})$ acts on the space $\oplus_\mu V_\mu^+$ (see [18]). A weight μ changes under this action according to the following rule. Let a be $\pm n$ and $\mu + \delta_i = (\mu_1,...,\mu_{i-1},\mu_i + 1,\mu_{i+1},...,\mu_{n-1})$. Then for $i = 1,...,n-1$ the following holds

$$z_{ia} : V_\mu^+ \to V_{\mu+\delta_i}^+, \quad z_{ai} : V_\mu^+ \to V_{\mu-\delta_i}^+$$

Elements z_{0a} do not change a $\mathfrak{o}_{2n-1}$-weight, that is they map V_μ^+ into itself.

5.4.2 Images of pfaffians in the Mickelsson-Zhelobenko alebra

Definition 4. A product of root and Cartan elements in the universal enveloping algebra is called normally ordered if in it at first (from the left side) the negative root elements occur, then Cartan elements occur and at the end positive root elements occur.

Every product of root and Cartan elements equals to a sum of normally ordered products.

Proposion 2. *Let $I \subset \{-n+1, ..., n-1\}$ be a subset which is not symmetric with respect to zero. Then $pPfF_I = 0$ in $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ or in $Z(\mathfrak{o}_{2n}, \mathfrak{o}_{2n-2})$.*

Proof. According to the definition a pfaffian a sum over permutations. The summands are products of root vectors and Cartan elements of $\mathfrak{o}_N$

The sum of root corresponding to element of each product equals $-\sum_{i\in I} e_i$. Since the set I is nonsymmetric one has $-\sum_{i\in I} e_i \neq 0$.

Impose a normal ordering in every summand. When one does the normal ordering new summands appear. But from the equality $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$ it follows that the sum of roots corresponding to the elements of these new products is again $-\sum_{i\in I} e_i$.

Since $-\sum_{i\in I} e_i \neq 0$ in every normally ordered summand in the pfaffian there is a root element. These elements either are zero modulo J', if there is a positive root element, or vanish after multiplication by p, if there is a negative root element.

□

Let us give a formula for the image of a pfaffian whose indexing set I is symmetric and is contained in $\{-n+1, ..., n-1\}$. In this case the calculation of the image in the Mickelsson-Zhelobenko algebra is equivalent to the calcualtion of the image of the pfaffian under the Harish-Candra homomorphism. This calculation was done in [21] (proposition 7.1), the result is the following.

Proposion 3. *[21]* $PfF_I = \frac{1}{(\frac{k}{2})!} D_{\frac{k}{2}}(F_{i_1 i_1}, ..., F_{i_{\frac{k}{2}} i_{\frac{k}{2}}})$,
where $D_r(h_1, ..., h_r) = \prod_{i=1}^{r}(h_i - \frac{r}{2} + i)$

Now let us find an image in $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ of the pfaffian $PfF_{\widehat{n}}$.

To formulate the next theorem define a polynomial C_n.

Definition 5. Let $C_{n-1}(h_1, ..., h_{n-1}) = (-1)^{n-1} D_{n-1}(h_1, ..., h_{n-1}) -$
$- 4\sum_{i=1}^{n-1}(-1)^{t+1} D_{n-2}(h_1, ..., \widehat{h_i}, ..., h_{n-1})$

Theorem 3. *The image of* $PfF_{\widehat{n}}$ *in* $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ *equals* $\check{z}_{n0} C_{n-1}(F_{11}, ..., F_{(n-1)(n-1)})$.

Proof. Take a set of indices of type $I = \{-n, -i_{\frac{k-2}{2}}, ..., -i_1, 0, i_1, ..., i_{\frac{k-2}{2}}\}$

By Lemma 3 the following equality takes place

$$PfF_I = \sum_{i \in I \setminus \{-n\}} \sum_{I' \sqcup I'' = I \setminus \{i, -n\}} \frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} (-1)^{(I' - n i I'')} PfF_{I'} F_{ni} PfF_{I''}.$$

To find the image in the Mickelson-Zhelobenko algebra of the sum $\sum_{I' \sqcup I'' = I \setminus \{i, -n\}}$ divide the summands into three groups: 1) those for which $i = 0$; 2) those for which $i < 0$; 3) those for which $i > 0$.

Let us find the image of summands for which $i = 0$. In this case $PfF_{I'}$ and $PfF_{I'}$ commute with F_{n0}. Note that $(-1)^{(I' - n0I'')} = (-1)^{(I'I'')}(-1)^{\frac{k}{2}-1}$ (to prove this firstly move $-n, 0$ to two first places and then move 0 to the right place, then the signs $(-1)^{|I'|}$, $(-1)^{|I'|-1}$, $(-1)^{\frac{k}{2}}$ appear).

Using the corollary 3 one gets that the image sum these summands equals

$$(-1)^{\frac{k}{2}-1} \sum_{I' \sqcup I'' = I \setminus \{-n, 0\}} \frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} (-1)^{(I'I'')} PfF_{I'} PfF_{I''} F_{n0} =$$
$$\frac{1}{\frac{k}{2}} (-1)^{\frac{k}{2}-1} PfF_{I \setminus \{-n0\}} F_{n0} = (-1)^{\frac{k}{2}-1} PfF_{I \setminus \{-n,0\}} F_{n0}.$$

Since the sets of indices $\pm(I \setminus \{-n0\})$ and $\{-n, 0\}$ do not intersect, one can apply the projector p and equivalence $mod J'$ to each multiple.

Thus the image of these summands is

$$\check{z}_{n0}(pPf_{I \setminus \{-n0\}} mod J').$$

Find the image of summands

$$\frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} (-1)^{(I' - n i I'')} PfF_{I'} F_{ni} PfF_{I''},$$

for which $i \neq 0$. Let $i > 0$. Then change F_{ni} and $PfF_{I''}$. One obtains an expression

$$\frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!}(-1)^{(I'-niI'')}(PfF_{I'}PfF_{I''}F_{ni} - PfF_{I'}[PfF_{I''}, F_{ni}]).$$

Now let $i < 0$. Change F_{ni} and $PfF_{I'}$, one gets

$$\frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!}(-1)^{(I'-niI'')}(F_{ni}PfF_{I'}PfF_{I''} - [PfF_{I'}, F_{ni}]PfF_{I''}).$$

Consider the case $i > 0$. In the last expression the first summand has a zero image in the Mickelsson-Zhelobenko algebra by the following reason. The sum of roots corresponding to the elements F_{ij} that participate in the expression for $PfF_{I'}F_{ni}PfF_{I''}$ equals to e_n. The element F_{ni} corresponds to the root $e_n - e_i$. Thus the sum of roots corresponding to the elements $PfF_{I'}PfF_{I''}$ equals $-e_i$. Express $PfF_{I'}PfF_{I''}$ as a sum of normally ordered products. Since $i > 0$ than in every obtained normally product there is a negative root element of the algebra $\mathfrak{o}_{2n-1}$. Thus after applying the extremal projector p the expression $PfF_{I'}PfF_{I''}$ vanishes.

In the case $i < 0$ it is similarly proved that the first summand has a zero image in the Mickelsson-Zhelobenko algebra.

Now consider the second summand

$$-PfF_{I'}[PfF_{I''}, F_{ni}]$$

in the case $i > 0$ or

$$-[PfF_{I'}, F_{ni}]PfF_{I''}$$

in the case $i < 0$.

In the case $i > 0$ if $-i \notin I''$ the expression is zero and otherwise it equals to

$$-PfF_{I'}PfF_{I''} \mid_{-i \mapsto -n} .$$

In the case $i < 0$ if $-i \notin I'$ the expression is zero and otherwise it equals to

$$-PfF_{I'} \mid_{-i \mapsto -n} PfF_{I''}.$$

Thus the image of summands for which $i \neq 0$ equals to the image of the expression

$- \sum_{i\in I\setminus\{-n\}, i>0} \sum_{I'\sqcup I''=I\setminus\{-n,i\}, -i\in I''} \frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!}(-1)^{(I'-niI'')} PfF_{I'} PfF_{I''} |_{-i\mapsto -n}$
$- \sum_{i\in I\setminus\{-n\}, i<0} \sum_{I'\sqcup I''=I\setminus\{-n,i\}, -i\in I'} \frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!}(-1)^{(I'-niI'')} PfF_{I'} |_{-i\mapsto -n}$
$PfF_{I''}$

Let us prove a proposition.

Proposion 4. *The expression above equals*

$$-2 \sum_{t=-\frac{k}{2}, \neq 0}^{\frac{k}{2}} (-1)^{\frac{k}{2}-t-1} \sum_{J'\sqcup J''=I\setminus\{\pm i\}} \frac{(\frac{|J'|}{2})!(\frac{|J''|}{2})!}{(\frac{k}{2})!}(-1)^{(J'J'')} PfF_{J'} PfF_{J''}. \quad (3)$$

Proof. To prove this let us firstly calculate the sign $(-1)^{(I'-niI'')}$. The sign $(-1)^{(I'-niI'')}$ differs from the sign $(-1)^{(I'I'')}$ by the sign of the permutation which moves $-n, i$ to their right places. This permutation can be done as follows: first of all move $-n, i$ to two last places, then move i to it's right place. If $i = i_t$, then

$$(-1)^{(I'-niI'')} = (-1)^{(I'I'')}(-1)^{(|I'|+|I'|+\frac{k-2}{2}-t)} = (-1)^{\frac{k}{2}-t-1}(-1)^{(I'I'')}.$$

Secondly compare $PfF_{I'} |_{-i\mapsto -n}$, $PfF_{I''} |_{-i\mapsto -n}$ and $PfF_{(I'\setminus\{-i\})\cup\{-n\}}$, $PfF_{(I''\setminus\{-i\})\cup\{-n\}}$ respectively. Here it is assumed that $i \in I'$ and $i \in I''$. In all these expressions at first, the index $-i$ is changed to n, but then in the last two expressions the new set of indices is naturally ordered. Thus $PfF_{I'} |_{-i\mapsto -n}$ and $PfF_{(I'\setminus\{-i\})\cup\{-n\}}$, $PfF_{I''} |_{-i\mapsto -n}$ and $PfF_{(I''\setminus\{-i\})\cup\{-n\}}$, differ by the sign of this ordering.

For summands in the sum $\sum_{i\in I\setminus\{-n\}, i<0} \sum_{I'\sqcup I''=I\setminus\{-ni\}, -i\in I''}$ denote

$$J' := (I' \setminus \{-i\}) \cup \{-n\}, \;\; J'' := I''.$$

One obtains

$$(-1)^{(I'I'')} PfF_{I'} |_{-i\mapsto -n} PfF_{I''} = (-1)^{(J'J'')} PfF_{J'} PfF_{J''}.$$

The sign that appears after the ordering is contained in $(-1)^{(J'J'')}$.

Analogously for the summands in the sum $\sum_{i\in I\setminus\{-n\}, i<0} \sum_{I'\sqcup I''=I\setminus\{-ni\}, -i\in I''}$, denote

$$J' := I', \;\; J'' := (I'' \setminus \{-i\}) \cup \{-n\}.$$

One obtains that

$$(-1)^{(I'I'')}PfF_{I'}\mid_{-i\mapsto -n} PfF_{I''} = (-1)^{(J'J'')}PfF_{J'}PfF_{J''}.$$

In both cases one has $J' \sqcup J'' = I \setminus \{\pm i\}$. Also $|J'| = |I'|$, $|I''| = |J''|$.

Note that a pair of sets J', J'' occurs twice. First as $(I' \setminus \{-i\}) \cup \{-n\}$, I'', second as I', $(I'' \setminus \{-i\}) \cup \{-n\}$.

Thus one obtains that the considered sum of images of summands for which $i \neq 0$ is given by the expression 3

$$2\sum_{t=-\frac{k}{2},\neq 0}^{\frac{k}{2}} (-1)^{\frac{k}{2}-t-1} \sum_{J'\sqcup J''=I\setminus\{\pm i_t\}} \frac{(\frac{|J'|}{2})!(\frac{|J''|}{2})!}{(\frac{k}{2})!}(-1)^{(J'J'')}PfF_{J'}PfF_{J''}$$

The proposition is proved. $\square$

By Corollary 3 this expression 3 equals

$$2\sum_{t=-\frac{k}{2},\neq 0}^{\frac{k}{2}} (-1)^{\frac{k}{2}-t-1}PfF_{I\setminus\{\pm i_t\}} = 4\sum_{t=1}^{\frac{k}{2}}(-1)^{\frac{k}{2}-t-1}PfF_{I\setminus\{\pm i_t\}}.$$

Finally in $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ one has

$$PfF_I = (-1)^{\frac{k}{2}-1}\check{z}_{n0}(pPfF_{I\setminus\{-n,0\}}modJ') - 4\sum_{t=1}^{\frac{k}{2}}(-1)^{\frac{k}{2}-t-1}PfF_{I\setminus\{\pm i_t\}} \quad (4)$$

Note that $PfF_{I\setminus\{-n,\pm i_t\}}$ is a pfaffian PfF_{I^t} for a new indexing set $I^t = I \setminus \{\pm i_t\}$.

This set is of the same type as I. Apply to each pfaffian PfF_{I^t} the equality (4). For each t there appears a summand

$$(-1)^{\frac{k-2}{2}-1}\check{z}_{n0}pPfF_{I^t\setminus\{-n0\}} = (-1)^{\frac{k}{2}-2}\check{z}_{n0}pPfF_{I\setminus\pm i,0,-n}.$$

Also there appear summands

$$PfF_{I^t\setminus\{\pm i_s\}} = \pm PfF_{I\setminus\{\pm i_t,\pm i_s\}}.$$

But the sum of these summands over t and s is zero. Let $0 < t < s$. If this summand comes from the summand $PfF_{I\setminus\{\pm i_s\}}$ in (4), then it appears with the sign $(-1)^{\frac{k}{2}-s-1}(-1)^{(\frac{k}{2}-1)-t-1}$. If it comes from the summand $PfF_{I\setminus\{\pm i_t\}}$ in (4) then it has the sign $(-1)^{\frac{k}{2}-t-1}(-1)^{(\frac{k}{2}-1)-(s-1)-1}$. The sum of these signs is zero.

Hence in $Z(\mathfrak{o}_{2n+1},\mathfrak{o}_{2n-1})$ one has

$$PfF_I=(-1)^{\frac{k}{2}-1}\check{z}_{n0}pPfF_{I\setminus\{-n,0\}}-4\sum_{t=1}^{\frac{k}{2}}(-1)^{\frac{k}{2}-t-1}(-1)^{\frac{k}{2}-2}\check{z}_{n0}pPfF_{I\setminus\pm i,0,-n}.$$

Apply the obtained formulae to $I=\hat{n}$. Recall that according to Proposition 3 one has $pPfF_{\widehat{0,\pm n}}=D_{n-1}(F_{11},...,F_{(n-1)(n-1)})$, and $pPfF_{\widehat{\pm i,0,\pm n}}=D_{n-2}(F_{11},...,\widehat{F_{ii}},...,F_{(n-1)(n-1)})$. Thus one proves Theorem. $\square$

5.5 Action of pfaffians in the Gelfand-Tsetlin-Molev base.

The Gelfand-Tsetlin-Molev base of a $\mathfrak{o}_{2n+1}$-representation is a base of a Gelfand-Tsetlin type, whose construction is based on restrictions $\mathfrak{o}_{2n+1}\downarrow\mathfrak{o}_{2n-1}$, in contrast to the classical Gelfand-Tsetlin base, whose construction is based on restrictions $\mathfrak{o}_N\downarrow\mathfrak{o}_{N-1}$.

One can give the following formal definition. The Gelfand-Tsetlin-Molev base is a base of a $\mathfrak{o}_{2n+1}$-representation, for different n the procedures of constructions of bases must be coherent in the following sense. A base of a $\mathfrak{o}_{2n+1}$-representation is a union a bases in $\mathfrak{o}_{2n-1}$-representation, into which a $\mathfrak{o}_{2n+1}$-representation splits when one restricts $\mathfrak{o}_{2n+1}\downarrow\mathfrak{o}_{2n-1}$.

All notations below are taken from [18].

Let V be given a $\mathfrak{o}_{2n+1}$-representation with the highest weight $(\lambda_1,...,\lambda_n)$, where

$$0\geq\lambda_1\geq...\geq\lambda_n.$$

Base vectors are indexed by tableaus Λ of type

$\lambda_{n,1},\lambda_{n,2},...,\lambda_{n,n}$
$\sigma_n,\lambda'_{n,1},\lambda'_{n,2},...,\lambda'_{n,n}$
$\sigma_{n-1},\lambda_{n-1,1},\lambda_{n-1,2},...,\lambda_{n-1,n-1}$
...
λ_{11}
σ_1,λ'_{11}

The restrictions on these numbers are the following:

1. $\lambda_{ni} = \lambda_i$
2. $\sigma_k = 0, 1$
3. The inequalities hold:
   $\lambda'_{k1} \geq \lambda_{k1} \geq \lambda'_{k2} \geq ... \geq \lambda'_{k,k-1} \geq \lambda'_{kk} \geq \lambda_{kk}$ for $k = 1, ..., n$, and
   $\lambda'_{k1} \geq \lambda_{k-1,1} \geq \lambda'_{k2} \geq ... \geq \lambda'_{k,k-1} \geq \lambda_{k-1,k-1} \geq \lambda'_{kk}$ for $k = 2, ..., n$.
4. If $\lambda'_{k1} = 0$, then $\sigma_k = 0$.

The first row is the highest weight of a $\mathfrak{o}_{2n+1}$-representation V.

The third row is a weight of a $\mathfrak{o}_{2n-1}$-representation that appears if one restricts $\mathfrak{o}_{2n+1} \downarrow \mathfrak{o}_{2n-1}$ and that contains the base vector.

The second row is set of indices of base vectors in V^+_μ. And so on.

Now we obtain formulae of the action of $PfF_{\widehat{n}}$ on a base in V^+_μ and then in the Gelfand-Tsetlin-Molev base in V.

Write the equality $V = \sum_\mu V^+_\mu \otimes V(\mu)$. From one other hand, the pfaffian commutes with $\mathfrak{o}_{2n-1}$. Thus the action on $V = \sum_\mu V^+_\mu \otimes V(\mu)$ is written as $\sum_\mu (PfF_{\widehat{n}}\,|_{V^+_\mu}) \otimes id$. From the other using the theorem we relate the action of $PfF_{\widehat{n}}$ on V^+_μ with the action of z_{0n}. The action of z_{0n} on V^+_μ is described in [18].

Using the description we obtain the following theorem.

Let $\rho_i = i - \frac{1}{2}$ for $i > 0$, and $\rho_{-i} = -\rho_i$,

$$\gamma_i = \lambda'_{2i} + \rho_i + \frac{1}{2},$$

$$\overline{\sigma} = \sigma + 1\, mod\, 2.$$

Theorem 4. *On the vector ξ_Λ the pfaffian $PfF_{\widehat{2}}$ acts as follows.*

Let $\Lambda = (\lambda, \sigma, \lambda', \Lambda')$, where λ is the first row of Λ, σ, λ' is the second row of Λ and Λ' is the remaining part of Λ.

If $\sigma = 0$, then

$$PfF_{\widehat{2}}\xi_{\sigma,\lambda',\Lambda'} = C\xi_{\bar{\sigma},\lambda,\Lambda'},$$

where C is some constant.

If $\sigma = 1$, then

$$PfF_{\widehat{2}}\xi_{\sigma,\lambda,\Lambda'} = (-1)^n \sum_{j=1}^{n} \Pi^n_{t=1,t\neq j} \frac{-\gamma_t^2}{\gamma_j^2 - \gamma_t^2} \xi_{\bar{\sigma},\lambda'+\delta_j,\Lambda'},$$

Where $\lambda' + \delta_j$ is the row λ' with 1 added to the j-th component.
Here $C = \frac{C_n(\lambda_{n-1,1},\ldots,\lambda_{n-1,n-1})}{\prod_{i=1}^{n-1}(\lambda_{n-1,i}+i-1)}$ (see the definition 5).

Note that in the case $n = 2$, which corresponds to $\mathfrak{o}_5$ one has $C = \frac{C_2(\lambda_{1,1})}{(\lambda_{1,1})} = 1$ and we obtain the theorem 1.

6 Conclusion

We have constructed a quantum number k. Using it together with the quantum numbers N (the number of particles), T, τ_0 (the isospin and its projection) we can classify the states of the five-dimensional quasi-spin. The quantum number k is defined through its creation operator. As the creation operator we use the noncommutative pfaffian associated with the algebra $\mathfrak{o}_5$, which is isomorphic to the quasi-spin algebra. We describe the action of the pfaffian on the other quantum numbers. In particular the pfaffian increases by one the number of particles and conserves the isospin and its projection.

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