K-STABLE VALUATIONS AND CALABI-YAU METRICS ON AFFINE SPHERICAL VARIETIES

TRAN-TRUNG NGHIEM

Abstract. After providing an explicit K-stability condition for a $\mathbb{Q}$-Gorenstein log spherical cone, we prove the existence and uniqueness of an equivariant K-stable degeneration of the cone, and deduce uniqueness of the asymptotic cone of a given complete K-invariant Calabi-Yau metric in the trivial class of an affine G-spherical manifold, K being the maximal compact subgroup of G.

Next, we prove that the valuation induced by K-invariant Calabi-Yau metrics on affine G-spherical manifolds is in fact G-invariant. As an application, we point out an affine smoothing of a Calabi-Yau cone that does not admit any K-invariant Calabi-Yau metrics asymptotic to the cone. Another corollary is that on $\mathbb{C}^3$, there are no other complete Calabi-Yau metrics with maximal volume growth and spherical symmetry other than the standard flat metric and the Li-Conlon-Rochon-Székelyhidi metrics with horospherical asymptotic cone. This answers the question whether there is a nontrivial asymptotic cone with smooth cross section on $\mathbb{C}^3$ raised by Conlon-Rochon when the symmetry is spherical.

1. Introduction

1.1. **Background.** The Yau-Tian-Donaldson correspondence establishes an equivalence between the existence of canonical metrics and an algebro-geometric stability condition. Large progress has recently been made for Ricci-flat Kähler cone metrics (also called conical Calabi-Yau metrics) on a *Fano cone*, which is basically an affine cone with respect to a polarization over a log Fano base, hence comes with an effective complex torus action.

In base-independent terms, given a complex algebraic torus T, a Fano cone Y is a $\mathbb{Q}$-Gorenstein klt T-affine variety with an effective T-action and a unique fixed point under T [LWX21]. The *Reeb cone* of Y consists of elements ξ in the compact Lie algebra of T acting with positive weights on non-zero elements of $\mathbb{C}[Y]$.

A *conical Calabi-Yau metric* on (Y, J_Y) is a $\partial_{J_Y}\overline{\partial}_{J_Y}$-exact (weak) Ricci-flat metric ω with potential r^2, compatible with the weak complex structure J_Y, and homogeneous under the scaling vector field generated by r, i.e.

$$\mathcal{L}_{r\partial_r}\omega = 2\omega.$$

In particular, the $\xi = -J_Y(r\partial_r)$ is a Reeb vector generating a holomorphic isometric action of a compact torus $T_{\xi,c}$ on Y [DS17, Lemma 2.17].

Fano cones offer very rich geometry as they contain contact geometric structures, as well as underlying Fano orbifold structures. They serve as asymptotic models for Calabi-Yau metrics on affine manifolds in [Li19] [CR21] [Szé19] [BD19] [Ngh24], but also as local tangent cones to Kähler-Einstein metrics [HS17].

Through the pioneering works of [CS18], [CS19], [HL23], [Li21], it is now established that a Fano cone has a Ricci-flat Kähler cone metric if and only if it is K-stable. More precisely, when the cone has a unique singularity, K-stability of a polarized cone

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(Y, ξ) is shown in [CS19] to be equivalent to a K-stability condition that extends the Fano orbifold stability of Ross-Thomas [RT11].

The general $\mathbb{Q}$-Gorenstein case was solved by C. Li [Li21] by using the equivalence between (weak) Ricci-flat Kähler cone metrics on (Y, ξ) and certain g-solitons over quasi-regular Fano orbifold quotients of Y. The g-soliton equations have moreover the same form when passing from one Reeb vector to another while keeping the underlying CR structure. The K-stability of the Fano cone is then equivalent to weighted K-stability of all quasi-regular quotients of Y.

Varieties with low complexity have been known to provide concrete examples to test K-stability criteria [Del20a], [IS17]. The complexity of a variety with a regular action of a reductive group G is basically the codimension of a generic Borel orbit. Normal varieties with complexity zero are called *spherical varieties.* Equivalently, a G-variety is spherical if and only if it has a open dense orbit under the action of a Borel subgroup of G.

A simple G-spherical affine variety Y is said to be a *G-spherical cone* if its unique closed orbit is the fixed point of G. In fact, a $\mathbb{Q}$-Gorenstein G-spherical cone is always a Fano cone with respect to the action of a torus compatible with G [Ngh23].

1.2. **K-stable degeneration and K-stability of spherical cones.** Let Y be an n-dimensional $\mathbb{Q}$-Gorenstein conical embedding of a spherical space G/H with colored cone $(\mathcal{C}_Y, \mathcal{D}_Y)$ and set of G-invariant divisors $\mathcal{V}_Y$ (identified with their G-invariant valuations). Let $T_H = \mathrm{Aut}_G(Y)^0 \simeq (N_G(H)/H)^0$ be the connected component of the automorphism group of Y compatible with G.

Our first goal is to extend the main result on existence of log Calabi-Yau metrics on toric cones *with an isolated singularity* of de Borbon-Legendre [dBL22] to the spherical context with more general singularities.

Define

$$D := \sum_{\nu \in \mathcal{V}_Y} (1 - \gamma_\nu) D_\nu$$

to be a $G \times T_H$-invariant divisor (which has simple normal crossing support by construction) with $\gamma = (\gamma_\nu)_{\nu \in \mathcal{V}_Y}$ satisfying $0 < \gamma_\nu \leq 1$ such that the naturally $G \times T_H$-linearized divisor $-L := K_Y + D$ is $\mathbb{R}$-*Cartier.* The latter is equivalent to the existence of $\varpi_\gamma \in \mathrm{int}(\mathcal{C}_Y^\vee)$ such that

$$\langle \varpi_\gamma, \nu \rangle = \gamma_v, \ \forall D_\nu \in \mathcal{V}_Y, \quad \langle \varpi_\gamma, \rho(d)) \rangle = a_d, \forall d \in \mathcal{D}_Y.$$

The set of such elements ϖ_γ are called *angles.* The pair (Y, D) is said to be a *spherical log cone* and (Y, D, ξ) is a *polarized spherical log cone.* Moreover, D as a closed subvariety is also a G-spherical cone.

Given any Reeb vector ξ, one can build a (weak) cone metric $\omega_\xi = \sqrt{-1}\partial\overline{\partial} r_\xi^2$ following [HS16]. We say that a cone metric ω_ξ on Y is a *log Calabi-Yau metric with Reeb vector ξ* if

$$\mathrm{Ric}(\omega_\xi) = D,$$

which is equivalent to

$$(\sqrt{-1}\partial\overline{\partial} r_\xi^2) = \frac{dV_Y}{\prod |s_\nu|^{2(1-\gamma_\nu)}}, \tag{1}$$

where s_ν is the canonical G-equivariant section of D_ν. In particular, ω_ξ restricts to a bona fide (singular) Ricci-flat Kähler metrics on $Y \backslash \mathrm{Supp}(D)$. We also expect that ω_ξ has *conic singularities* of angles $2\pi\gamma_\nu$ along D_ν in the log smooth locus of Y (conditionally on an analogue of Guenancia-Paun's result [GP16]).

Theorem A (Prop. 3.2)**.** *Let (Y, D, ξ) be a polarized spherical log cone with angles $\gamma = (\gamma_\nu)_{\nu \in \mathcal{V}_Y}$, $0 < \gamma_\nu \leq 1$, such that (Y, D) has klt singularities. Then the following are equivalent*

- *Y has log Calabi-Yau metrics with Reeb vector ξ.*
- *(Y, D, ξ) is K-stable.*
- *$bar_{DH}(\Delta_\xi) - \varpi_\gamma \in (-\mathcal{V})^\vee$, $\quad \Delta_\xi := \{p \in \mathcal{C}_Y^\vee, \langle p, \xi \rangle = n\}$.*

The K-stability criterion generalizes largely our previous work on horospherical cones [Ngh23], which is based on solving an explicit real Monge-Ampère equation through variational approach. Here we explore the algebro-geometric method by constructing explicit G-equivariant test configurations of a polarized cone via description of G-equivariant degenerations in [BP87], [Del20a]. Any central fiber of such configuration admits a further equivariant degeneration to a horospherical central fiber [Pop86], and the Futaki invariant remains constant throughout (Lemma 2.25).

We then conclude based on an explicit computation of the Futaki invariant of a horospherical cone in Lemma 2.26, and the fact that G-equivariant K-stability over special test configurations is equivalent to K-stability, see Theorem 3.1.

Remark 1.1. *One can compute the generalized δ-invariant for spherical log cones, then use the valuative criterion for K-stability in Kai Huang's PhD thesis [Hua22]. Our approach is more geometric in nature and independent of the works in [LLW22] [Yin24].*

Remark 1.2. *Based on Pasquier's result on horospherical pairs [Pas16], we expect that any spherical log pair (Y, D) defined as above with $0 < \gamma_\nu < 1$ has automatically klt singularities.*

Our next main result is the following.

Theorem B (Prop. 3.3)**.** *Any K-semistable spherical log cone (W, D, ξ) admits an equivariant degeneration to a K-stable spherical log cone (C, D_0, ξ), unique up to equivariant isomorphisms.*

This result might be of independent interest in K-stability theory. In fact, the existence and uniqueness of the *G-equivariant* K-stable degeneration is known in the Fano case [Zhu21], but a proof for log Fano cones is still lacking, since the argument in [Zhu21] supposes the existence of a good moduli space for K-(semi)stable Fano varieties, which has not yet been shown to exist for Fano cones in all dimension.

By the time this article is being prepared, Xu-Zhuang has proved the boundedness property for K-semistable log Fano cones [XZ24], which is a crucial ingredient for the construction of the moduli spaces. However, our proof is rather direct and solely based on the combinatorial information in the spherical cone.

1.3. **K-semistable valuations and Calabi-Yau metrics.** Let (M, ω) be a n-dimensional complete Calabi-Yau manifold with *maximal volume growth*, i.e. for every ball $B_r(p)$ of radius $r > 0$ centered as p, there is $\kappa > 0$ satisfying

$$\mathrm{vol}(B_r(p)) \geq \kappa r^{2n}.$$

A *metric cone* $C := C(Z)$ over some compact metric space (Z, d_Z) is the metric completion of $]0, +\infty[\times Z$ with respect to the metric

$$d((r_1, z_1), (r_2, z_2)) = \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2 \cos \max\{d_Z(z_1, z_2), \pi\}}.$$

The seminal work of Cheeger-Colding [CC97] shows that, given a sequence $(M_i, \omega_i) = (M, \lambda_i \omega)$ with $\lambda_i \to 0$, after passing to a subsequence we obtain a metric cone C, called the *asymptotic cone* (or *tangent cone at infinity*) of M.

In the Kähler context, consider the set $\mathcal{K}(n, \kappa)$ of complete n-dimensional polarized Kähler manifolds (X, L, ω, p), where L is a Hermitian holomorphic line bundle over X with curvature $-i\omega$ and p a chosen base point, such that (X, ω) is Einstein with Euclidean volume growth as in [DS17].

As remarked by Székelyhidi [Szé20], if we suppose that $\omega = \sqrt{-1}\partial\overline{\partial}\varphi$ for some smooth psh function φ on M, then we can readapt the powerful theory of Donaldson-Sun in the noncompact setting by choosing (L_i, h_i) as trivial line bundles over M_i with Hermitian metric $h_i = e^{-\lambda_i \varphi}$ so that the sequence (M_i, L_i, ω_i) lies in the class $\mathcal{K}(n, \kappa)$. The same arguments in [DS17, Section 3.4] then show that the tangent cone at infinity is independent of the chosen subsequence, has a *complex normal affine cone* structure (C, J_0) with $\mathbb{Q}$-*Gorenstein klt singularities*, and the *metric singular set* of C (in the sense of Cheeger-Colding) in fact coincides with the algebraic singular set of C [DS17]. Moreover, (C, J_0) has a (weak) *conical Calabi-Yau structure* (so C is in particular K-stable).

It is generally very hard to classify all the tangent cone at infinity of a given Calabi-Yau affine manifold, even under the maximal volume growth condition. From Donaldson-Sun theory, at least we know that such cone can be obtained from (M, ω) via a *2-step degeneration* as follows.

First, a complete $\partial\overline{\partial}$-exact Calabi-Yau metric ω on M induces a negative valuation ν_ω on the ring $R(M)$ of holomorphic functions with polynomial growth of M (see Section 4 for the precise definition). This valuation moreover induces a filtration on the ring $R(M)$ and a degeneration of M to a K-semistable Fano cone (W, ξ) with the K-semistable Reeb valuation ν_ξ induced by ν_ω in a natural way. The K-semistable cone (W, ξ) then degenerates to the K-stable cone (C, ξ) via a further test configuration. It was recently shown by Sun-Zhang that when C has smooth link, then (M, ω) degenerates to (C, ξ) in a single step and is moreover *asymptotically conical* in the sense of Conlon-Hein [CH13] [SZ22].

Definition 1.3. *We say that the valuation ν_ω is* K-stable *(resp.* K-semistable*) if the graded ring of $R(M)$ by ν_ω is finitely generated and defines a* K-stable *(resp.* K-semistable*) Fano cone with the K-stable (resp. K-semistable) Reeb valuation induced by ν_ω.*

In [SZ22], the authors propose a four-steps scheme to classify complete Calabi-Yau metrics with Euclidean growth in the trivial Kähler class on noncompact manifolds. The scheme consists of

(1) Given an affine manifold M, classifying all K-(semi)stable valuations on M. More precisely, given a complete $\partial\overline{\partial}$-exact Calabi-Yau metric with maximal volume growth ω on M, determine the space of all possible K-(semi)stable valuations on M.
(2) Given a K-stable valuation ν on M, determining the space $\mathcal{M}_\nu$ of all compatible Calabi-Yau metric ω on M such that $\nu_\omega = \nu$.
(3) For any $\omega_1, \omega_2 \in \mathcal{M}_\nu$, finding a constant $c > 0$ such that $c^{-1}\omega_2 \leq \omega_1 \leq c\omega_2$.
(4) Let $\mathcal{N}_\nu$ be the space of conical Calabi-Yau metrics on the asymptotic cone C_ν. The natural map $\mathcal{M}_\nu \to \mathcal{N}_\nu$, defined by taking the rescaled limit of the Kähler form under the weighted cone construction, is bijective.

Our philosophy, which is rather natural, is that if we impose a large symmetry on the metric, the scheme should be considerably simplified. We thereby achieve Step (1) for semistable valuations of a Calabi-Yau *spherical manifold*.

Theorem C (Prop. 5.9 and 5.10). *If M is a G-spherical affine manifold and ω is K-invariant complete Calabi-Yau metric with maximal volume growth in the trivial Kähler class of M, then*

- *the asymptotic cone (C, ξ) of M is a G-spherical cone and unique up to an isomorphism preserving the K-stable Reeb vector ξ.*
- *the negative valuation ν_ω is G-invariant and restricts to the K-stable valuation $-\nu_\xi$ in the Cartan algebra of M and C. In particular, there can be only finitely many G-invariant K-stable valuations on a G-spherical affine manifold.*

Here are some remarks on this theorem.

- An immediate corollary is that the only Calabi-Yau metrics with maximal volume growth and horospherical symmetry are the conical Calabi-Yau metrics on horospherical cones.
- The valuation doesn't uniquely determine the Calabi-Yau metric, but only up to a family. An explicit example of a 2-parameters family of Calabi-Yau metrics on $\mathbb{C}^3$ with asymptotic cone $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$ was constructed by Chiu [Chi22]. The fourth step in Sun-Zhang classification scheme predicts a family of Calabi-Yau metrics depending on as many parameters as the automorphisms group of the asymptotic cone.
- As for uniqueness of the asymptotic cone, an approach independent of K-stability theory is to use the equivariant Hilbert scheme constructed by Alexeev-Brion in [AB04], generalizing the Haiman-Sturmfels' Hilbert scheme for diagonalizable group [HS04], used in [DS17, Section 3.3].
- The G-invariance of ν_ω and uniqueness of C as an G-affine cone hold for any K-invariant Calabi-Yau metric in our context (cf. Remark 5.12), but it is not clear how to compare ν_ω and ν_ξ as in the spherical case.

Since every K-invariant Calabi-Yau metric on non Hermitian symmetric spaces is necessarily $\partial\overline{\partial}$-exact of maximal volume growth, and that the K-stable valuation induced by such metric lies outside the Weyl chamber in the G_2 case by explicit computations in [Ngh24], we obtain directly the following non-existence result as announced therein.

Corollary D. *There is no complete K-invariant Calabi-Yau metric with horospherical asymptotic cone on the symmetric spaces of type G_2.*

In particular, it follows from [BD19] that there can only be a unique possible G-spherical asymptotic cone and there exists complete K-invariant Calabi-Yau metric on G_2 with this asymptotic cone.

- This provides the first example of a non-rigid singular Calabi-Yau cone that cannot be realized as the tangent cone at infinity of a given *equivariant affine smoothing*. On the other hand, the existence of a AC Calabi-Yau metric on an affine smoothing of a *smooth Calabi-Yau cone* is *always guaranteed* because any affine manifold is Kähler [CH22, Theorem A].
- Since the G_2-cones in the case of multiplicity 2 have canonical singularities, the non-existence result also suggests that a general existence theorem à la Conlon-Hein [CH22, Theorem A,B] should involve finer properties of the cone's singularities.
- Given a Calabi-Yau cone C, it is expected that there are only two ways to obtain Calabi-Yau manifolds: either by smoothing C or crepantly resolving

C. This turns out to be the case for smooth Calabi-Yau cones [CH22]. If both ways work, one can shrink the exceptional divisor on the Calabi-Yau crepant resolution $\check{X}$ to C, then smoothly deforming C to a Calabi-Yau manifold $\widehat{X}$ with a different complex structure. This phenomenon is called *geometric transition* which is of interest to physicists [CdlO90] [Ros06]. In our context, no *equivariant geometric transition* phenomenon can occur through this cone, since there is no equivariant crepant resolution of the G_2-asymptotic cone in the first place (cf. Lemma 6.4).
- Ronan Conlon pointed out to me that there is not yet any counterexample when the asymptotic cone has smooth link. It would be interesting to ask whether there exists at all any equivariant Calabi-Yau smoothing of the G_2-horospherical asymptotic cones.

Finally, another motivation of our work comes from the author's remark that many known examples of Calabi-Yau manifolds of maximal volume growth with singular tangent cones so far are in fact *affine spherical manifolds* with respect to the complexified action of the given isometry on the metric. This includes the Li-Conlon-Rochon-Székelyhidi (LCRS) metrics on $\mathbb{C}^{n+1}, n \geq 2$ with asymptotic cone $\mathbb{C} \times A_1$ [Li19] [CR21] [Szé19], Biquard-Delcroix-Gauduchon's metrics with horosymmetric tangent cones [BD19], [BG97], and the metrics with horospherical tangent cones constructed by the author in [Ngh24]. Note however that on $\mathbb{C}^{n+1}$, there exist also metrics with non-spherical symmetry [Szé19] [CR21].

Every G-affine spherical manifold M is G-isomorphic to $G \times_H V$, where H is a reductive connected spherical subgroup of G (in particular G/H is an affine spherical space), and V is a spherical H-module [KVS06, Corollary 2.2].

Example 1.4. *The complex symmetric spaces G/H are all affine spherical manifolds. On the other hand, $\mathbb{C}^{n+1}$ is a rank two $\mathrm{SO}_{n+1} \times \mathbb{C}^*$-nonsemisimple symmetric cone with open orbit $\mathrm{SO}_{n+1} / \mathrm{SO}_n \times \mathbb{C}^*$.*

The LCRS metrics with spherical symmetry on $\mathbb{C}^3$ are invariant by the maximal compact subgroup $K = \mathrm{SO}_{n+1}(\mathbb{R}) \times \mathbb{S}^1$ and of horospherical tangent cones at infinity $A_1 \times \mathbb{C}$. By Székelyhidi's uniqueness theorem [Szé20], any complete Calabi-Yau metric on $\mathbb{C}^3$ asymptotic to the cone is unique up to scalings and biholomorphisms. Note that any K-invariant metric on the symmetric cone $\mathbb{C}^3$ is $\partial\overline{\partial}$-exact and has maximal volume growth (cf. [Del20b]). This fact combined with Székelyhidi's uniqueness and Theorem C implies the following.

Corollary E (Prop. 6.7)**.** *The only possible asymptotic cones of complete Calabi-Yau metrics with spherical symmetry on $\mathbb{C}^3$ are*
- *the horospherical asymptotic cone $A_1 \times \mathbb{C}$ of the LCRS metrics,*
- *and the asymptotic cone $\mathbb{C}^3$ itself of the standard flat metric.*

In particular, there are only two distinct families of complete Calabi-Yau metrics with spherical symmetry of $\mathbb{C}^3$.

1.4. **Organization.** The paper is organized as follows. In Section 2, we describe the test configurations and compute the Futaki invariant of spherical cones. Main Theorems A and B are proved in Section 3.

Section 4 contains a summary of Donaldson-Sun theory. The proof of Theorem C is given in Section 5. Examples of explicit K-stable valuations on spherical Calabi-Yau manifolds and proof of Corollary E are given in Section 6.

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2. Test configurations and Futaki invariant of spherical cones

2.1. Generalities on spherical cones.

Main references for this section are [Bri97b], [Kno91]. A spherical space is a homogeneous space G/H containing a Zariski-open orbit under the action of a Borel subgroup $B \subset G$. A G-spherical variety X is a G-equivariant embedding of a spherical space. A spherical variety is said to be *simple* if it contains a unique closed G-orbit. Each simple spherical variety contains an open B-stable affine subset X_B that intersects the closed orbit along an open B-stable orbit. Every spherical variety can be covered by simple spherical varieties.

Let $\mathcal{M}(G/H)$ be the lattice of characters of $\mathbb{C}(G/H)$ as a B-representation, and $\mathcal{N}(G/H)$ be its dual lattice. Denote by $\mathcal{V}(G/H)$ the set of G-invariant valuations on $\mathbb{C}(G/H)^*$. When the spherical space is clear from the context, we will just denote them by $\mathcal{M}, \mathcal{N}, \mathcal{V}$.

Theorem 2.1. *[BLV86] Let Q be the parabolic subgroup of G that stabilizes the open B-orbit BH (or equivalently, stabilizes all the colors in $\mathcal{D}$).*

There is a choice of a Lévi subgroup $L \subset Q$ and of a maximal torus $T \subset L$ (this is also the maximal torus of G) such that one can identify $\mathcal{M}$ and $\mathcal{N}$ with the character lattice of the adapted torus *$T/T \cap H$ and its dual. The dimension of this torus is the* rank *of G/H.*

For every valuation $\nu \in \mathcal{V}$, there exists an injective natural map $\rho : \mathcal{V} \to \mathcal{N}_{\mathbb{R}}$, such that $\rho(\nu)(f_\chi) = \langle \chi, \nu \rangle$ where $f_\chi \in \mathbb{C}(G/H)$ is an eigenvector of B with character χ.

Definition 2.2. *The set of reduced and irreducible B-stable divisors in G/H is called the* colors *of G/H, denoted by $\mathcal{D}$.*

A color *of a G/H-spherical embedding X is an element of $\mathcal{D}$ whose closure in X contains a closed orbit. The set of colors of a spherical embedding X is denoted by $\mathcal{D}_X$. The natural map ρ sends $\mathcal{D}$ to a subset of $\mathcal{N}$, but ρ is not injective on $\mathcal{D}$ in general.*

Let $\mathcal{V}_X$ be the set of G-invariant divisors of X. The injective map $\rho : \mathcal{V}_X \to \mathcal{N}_{\mathbb{R}}$ that sends a divisor to its valuation identifies $\mathcal{V}_X$ with a finite subset in $\mathcal{V}$.

To each simple embedding X, we can associate a pair $(\mathcal{C}_X, \mathcal{D}_X)$, where $\mathcal{C}_X$ is the strictly convex cone generated by $\mathcal{V}_X \cup \rho(\mathcal{D}_X)$, called a *colored cone* in the following sense.

Definition 2.3. *A* colored cone *$(\mathcal{C}, \mathcal{F})$ is the data of $\mathcal{C} \subset \mathcal{N}_{\mathbb{R}}$ and $\mathcal{F} \subset \mathcal{D}$, where $0 \notin \rho(\mathcal{F})$, $\mathcal{C}$ is a strictly convex cone generated by $\rho(\mathcal{F})$ and a finite number of elements of $\mathcal{V}$, and $\mathcal{F}$ is called the set of* colors *of $(\mathcal{C}, \mathcal{F})$.*

Theorem 2.4 (Luna-Vust)**.** *The map $X \to (\mathcal{C}_X, \mathcal{D}_X)$ is a bijection between the set of isomorphism classes of simple G/H-embeddings and the set of colored cones.*

Theorem 2.5. *[Ngh23] Let G/H be a spherical space. Let Y be a simple G-equivariant embedding of G/H with colored cone $(\mathcal{C}_Y, \mathcal{D}_Y)$. Then Y is a spherical cone if and only if $\mathcal{V}(G/H)$ has a linear part, $\mathcal{C}_Y$ is of maximal dimension, and $\mathcal{D} = \mathcal{D}_Y$ (i.e. all the colors of G/H contains the unique closed orbit of Y).*

The $\mathbb{Q}$-Gorenstein assumption on a spherical cone implies that is has at worst klt singularities. We refer the reader to [Pas17] for a survey on singularities of spherical varieties. Any $\mathbb{Q}$-Gorenstein spherical cone is in particular a Fano cone.

Theorem 2.6. *[Los09] Let $\Gamma_Y := \mathcal{C}_Y^\vee \cap \mathcal{M}$ be the weight monoid of the spherical cone Y. Then Y is uniquely determined up to G-isomorphisms by (Γ_Y, Σ_Y).*

Let $T_H := \mathrm{Aut}_G(Y)^0 \simeq (N_G(H)/H)^0$ be the neutral component of the automorphisms of Y that commutes with G. Since every $\sigma \in \mathrm{Aut}_G(G/H)^0$ can be extended to a G-equivariant isomorphism of (Y, y) to $(Y, \sigma(y))$, we have $T_H \simeq \mathrm{Aut}_G(G/H)^0$. Moreover, $\dim(T_H) = \dim \mathrm{lin}\mathcal{V} \geq 1$, and the noncompact Lie algebra of $\mathfrak{t}_H$ can be identified with $\mathrm{lin}\mathcal{V}$, hence $\mathcal{N}(T_H) = \mathrm{lin}\mathcal{V} \cap \mathcal{N}$.

Example 2.7. *Every toric space (i.e. $G = T$ and $H = \{1\}$) admits conical embeddings, while this is not the case for every spherical space. Indeed the symmetric space SL_2/T does not embed into any symmetric cone, since $N_{SL_2}(T)/T \simeq \mathbb{Z}_2$. However, the space $SL_2/T \times \mathbb{C}^*$ has a conical embedding.*

Under the T_H-action, the coordinate ring of Y decomposes as

$$R := \mathbb{C}[Y] = \bigoplus_{\alpha \in \Gamma} R_\alpha,$$

where $\Gamma := \{\alpha \in \mathcal{M}(T_H), R_\alpha \neq 0\}$ is a finitely generated monoid in $\mathcal{M}(T_H)$. The cone $\sigma^\vee$ generated by Γ is strictly convex and of maximal dimension in $\mathcal{M}(T_H)_\mathbb{R}$. By duality, the dual cone $\sigma = (\sigma^\vee)^\vee$ is also a strictly convex cone of maximal dimension.

Remark 2.8. *Since the right action of T_H commutes with G, every B-module $R^{(\alpha)}$ can be identified* as a one-dimensional $\mathbb{C}$-vector space *with a T_H-module R_{α_H} such that $\alpha|_{\mathcal{N}(T_H)} = \alpha_H$.*

Definition 2.9. *The interior of σ is called the* (algebraic) Reeb cone *of Y, denoted by $\mathcal{C}_R$. A couple (Y, ξ) with $\xi \in \mathcal{C}_R$ is said to be a* polarized cone.

An element $\xi \in \mathcal{N}(T_H)_\mathbb{Q}$ is said to be quasi-regular, *and* irregular *otherwise.*

Every Reeb vector induces a monomial valuation ν_ξ on $\mathbb{C}[Y]$, centered on the unique fixed point of Y, such that

$$\nu_\xi(f) = \min_{\alpha \in \Gamma} \{\langle \alpha, \xi \rangle, R_\alpha \neq 0\}.$$

Note that when the cone Y has smooth link, then the algebraic Reeb cone can be identified with the symplectic Reeb cone as follows. Let J be a complex structure on $Y^* := Y \setminus \{0\}$. A Kähler metric ω on (Y, J) is compatible with a Reeb element $\xi \in \mathcal{C}_R$ if there exists a ξ-invariant smooth psh function $r : Y^* \to \mathbb{R}_{>0}$ such that $\omega = \frac{1}{2} i\partial\overline{\partial} r^2$ and $\xi = J(r\partial r)$.

Given a quasi-regular Reeb vector field $\xi_0 \in \mathcal{N}(T_H)_\mathbb{Q}$, it can be shown that Y always admits a ξ_0-compatible metric and a dual 1-form η_0 on Y^* such that $\eta_0(\xi_0) = 1$. In this case, the *symplectic Reeb cone*

$$\mathcal{C}'_R := \{\xi \in \mathfrak{t}_H, \eta_0(\xi) > 0 \text{ on } Y^*\}$$

turns out to be exactly the algebraic Reeb cone $\mathcal{C}_R$ (cf. [CS18, Prop. 2.7]). In particular, it is independent of the choice of ξ_0 and η_0.

Definition 2.10. *A spherical space G/H is called* horospherical *if H contains a maximal unipotent subgroup of G.*

Let S the set of simple roots of G with respect to a Borel subgroup B and W the Weyl group of G. Recall that there is a bijection between the subsets of S and (the conjugacy classes of) parabolic subgroups of G as follows. For every $I \subset S$, let W_I the subgroup of W generated by the reflection $s_\alpha, \alpha \in I$. The parabolic subgroup $P_I \subset G$ is defined as the group generated by B and W_I.

Given a dominant weight λ, we have $\lambda = \sum_{\alpha \in S} x_\alpha \varpi_\alpha, x_\alpha \geq 0$. Then define the parabolic subgroup $P(\lambda)$ as P_I, where $I = \{\alpha \in S, x_\alpha = 0\}$. In particular, $P(\varpi_\alpha) = P_{S \backslash \{\alpha\}}$, and

$$\cap_{\alpha \in S \backslash I} P(\varpi_\alpha) = P_I.$$

Proposition 2.11. *[Pas06] A G horospherical space is uniquely determined by a couple $(\mathcal{M}, I)$ where $I \subset S$, and $\mathcal{M}$ is a sublattice of $\mathcal{M}(T)$ such that for all $\chi \in \mathcal{M}$ and $\alpha \in I$, $\langle \chi, \alpha^\vee \rangle = 0$. The isotropy subgroup is then*

$$H = \cap_{\chi \in \mathcal{M}(P_I)} ker(\chi).$$

Furthermore, P_I is the right-stabilizer of the open Borel orbit, and coincides with $N_G(H)$, and G/H is an equivariant torus bundle over G/P_I with fiber the torus P_I/H. The colors $\mathcal{D}$ of G/H are in bijection with the roots in $S \backslash I$ and

$$\rho(\mathcal{D}) = \left\{ \alpha^\vee|_{\mathcal{M}_I}, \alpha \in S \backslash I \right\}.$$

Note that P_I is the opposite parabolic subgroup of the (left-)stabilizer Q. When Y is a conical embedding of a horospherical space G/H with colored cone $(\mathcal{C}_Y, \mathcal{D}_Y)$, the group T_H coincides with P_I/H, but since the action is reverse, the Reeb cone is exactly

$$\mathcal{C}_R = -\mathrm{int}(\mathcal{C}_Y).$$

Horospherical cones can be obtained systematically as follows.

Proposition 2.12. *[VP72, Theorem 1] Let $V(\lambda)$ be a simple G-module of highest weight λ and eigenvector v_λ. The variety*

$$X(\lambda) := \overline{Gv_\lambda} \subset V(\lambda)$$

is then a rank one horospherical cone over the corresponding Grassmannian $G/P(\lambda)$ in $\mathbb{P}(V(\lambda))$ where $I = \{\alpha \in S, \langle \lambda, \alpha^\vee \rangle = 0\}$ and $P_I = P(\lambda)$ is the stabilizer of $[v_\lambda] \in \mathbb{P}(V(\lambda))$. Moreover, $\mathbb{C}[X] \simeq V(\lambda)^*$.

Example 2.13. *As an application, one can take $G = \mathrm{SO}_3$ with the unique fundamental weight $\lambda = 2\omega$, where ω is the fundamental weight of SL_2. Then $X(2\omega)$ is isomorphic to the ordinary double point, which is the Stenzel asymptotic cone of the rank one symmetric space $\mathrm{SO}_3 / \mathrm{SO}_2$. Indeed, $V(2\omega)^* \simeq S^2V^* \simeq \mathbb{C}[x^2, xy, y^2]$, which is the coordinate ring of the ordinary double point. On the other hand, $X(\omega)$ is simply $\mathbb{C}^2$.*

2.2. **Test configurations of spherical cones.** Recall that by a result of Knop [Kno91], if the vertex of a Fano cone Y is fixed by a reductive group G acting effectively on Y, then there is a $\mathbb{C}^*$-action on Y commuting with G.

Definition 2.14. *Let (Y, D, ξ) be any polarized log Fano cone, endowed with an effective action of a reductive group G that fixes the vertex of Y, and a compatible action of a complex torus T containing T_ξ, preserving D.*

A $G \times T$-equivariant test configuration of (Y, D, ξ) consists of

- *a $G \times T$-equivariant flat affine family $\pi : (\mathcal{Y}, \mathcal{D}) \to \mathbb{C}$, where $\mathcal{D}$ is an effective divisor not containing any component of $Y_0 = \pi^{-1}(0)$ such that each fiber away from 0 is isomorphic to (Y, D).*
- *a $\mathbb{C}^*$-holomorphic action on $(\mathcal{Y}, \mathcal{D})$ generated by $\zeta \in \mathfrak{t}$ and commuting with the $G \times T$-action such that π is $\mathbb{C}^*$-equivariant for this action, and that there is a $G \times \mathbb{C}^*$-equivariant isomorphism $(\mathcal{Y}, \mathcal{D}) \backslash (Y_0, D_0) \simeq (Y, D) \times \mathbb{C}^*$.*

The test configuration is said to be special *if $K_{\mathcal{Y}}+\mathcal{D}$ is $\mathbb{R}$-Cartier and that the central fiber (Y_0, D_0) is a klt pair.*

Finally, the test configuration is said to be trivial *if there is a T-equivariant isomorphism $(\mathcal{Y}, \mathcal{D}) \simeq (Y, D) \times \mathbb{C}$ and $\zeta = \zeta_0 + t\partial_t$ where ζ_0 generates a $\mathbb{C}^*$-holomorphic vector field that commutes with the action of ξ, and t is an element of the compact Lie algebra of $\mathbb{C}$.*

Remark 2.15. *It is well-known by Hironaka's lemma [Har77, II.9.12] that the configuration $(\mathcal{Y}, \mathcal{D})$ is itself a klt pair if the central fiber (Y_0, D_0) is.*

In the spherical context, if the test configuration is special, then the central fiber (Y_0, D_0, ξ) is also a polarized G-spherical log cone that inherits an action of T_H and a new action of $\mathbb{C}^*$ that commutes with $G \times T_H$. The action of the automorphism group $\mathrm{Aut}_{G\times\mathbb{C}^*}(G/H \times \mathbb{C}^*)^0 \supset T_H \times \{1\}$ extends automatically on $(\mathcal{Y}, \mathcal{D})$, hence a G-equivariant test configuration of (Y, D, ξ) is also a $G \times T_H$-test configuration for (Y, D, ξ). Moreover, since it suffices to check K-stability over special test configurations (cf. Theorem 3.1), we will mainly focus on *special* G-equivariant configurations.

Definition 2.16. *An* elementary embedding *is a G-equivariant embedding of G/H with a unique closed orbit of codimension 1. A* $\mathbb{C}^*$-equivariant degeneration *of G/H is a $G \times \mathbb{C}^*$-equivariant elementary embedding E of $G/H \times \mathbb{C}^*$ together with a $\mathbb{C}^*$-equivariant morphism $E \to \mathbb{C}$.*

Every couple $(\lambda, m) \in \mathcal{V} \oplus \mathbb{Q}^*$ determines an equivariant degeneration, and vice versa: a primitive generator of the colored cone of E is of the form $(\lambda, m) \in \mathcal{V} \oplus \mathbb{Q}^*$. The closed orbit of E can be identified with G/H_0, where H_0 is a spherical subgroup of G. If $\lambda \in \mathrm{int}(\mathcal{V})$, then G/H_0 is horospherical. Moreover, G/H_0 has the same left-stabilizer of the open Borel-orbit as well as the same adapted Levi subgroup as G/H.

For simplicity, we only describe here the test configuration of a polarized spherical cone, as the description for a log pair follows almost word-by-word. We first need the following result on spherical morphisms.

Proposition 2.17. *[Kno91] There exists a morphism between two G/H-embeddings X and X' if and only if for every colored cone $(\mathcal{C}, \mathcal{F})$ of X, there is a colored cone $(\mathcal{C}', \mathcal{F}')$ of X' such that $\mathcal{C} \subset \mathcal{C}'$ and $\mathcal{F} \subset \mathcal{F}'$.*

Theorem 2.18. *Let (Y, ξ) be a polarized $\mathbb{Q}$-Gorenstein spherical cone.*

(1) *To each G-equivariant special test configuration of (Y, ξ) with G-spherical central fiber Y_0, there exists $(\nu, m) \in \mathcal{V} \oplus \mathbb{N}^*$ and a spherical subgroup $H_0 \subset G$ such that Y_0 is a G/H_0-spherical embedding, and that the action of $\mathbb{C}^*$ on G/H_0 is*
$$e^{\tau}.gH_0 = g\nu(e^{-\tau/m})H_0.$$
(2) *Conversely, let $\nu \in \mathcal{V}$ and $m \in \mathbb{N}^*$. Let G/H_0 be the central fiber of the equivariant degeneration induced by (ν, m). Then there exists a G-equivariant test configuration (and a special one after a suitable base change) such that the central fiber Y_0 is a conical embedding of G/H_0, and that the $\mathbb{C}^*$-action can be described as above.*

    In particular, every polarized G-spherical cone admits a test configuration with G-horospherical central fiber.
(3) *Up to lattice isomorphisms, the lattices and weight monoids of Y and Y_0 are the same.*
(4) *$(\mathcal{Y}, \xi; \nu)$ is trivial if and only if ν belongs to the linear part of $\mathcal{V}$.*

Proof. Before going through the proof, remark that the spherical space $G/H \times \mathbb{C}^*$ has character lattices $\mathcal{M}(G/H \times \mathbb{C}^*) = \mathcal{M} \oplus \mathbb{Z}$ and valuation cone $\mathcal{V}(G/H \times \mathbb{C}^*) = \mathcal{V} \oplus \mathbb{R}$, which clearly has non-trivial linear part. The colors of $G/H \times \mathbb{C}^*$ are exactly $\{d \times \mathbb{C}^*, d \in \mathcal{D}\}$.

Every test configuration induces a $\mathbb{C}^*$-equivariant degeneration of G/H, hence there exists $\lambda \in \mathcal{V}$ and $m \in \mathbb{N}^*$ such that the ray generated by $(\lambda, m) \in \mathcal{V} \oplus \mathbb{N}^*$ is the colored cone of the equivariant degeneration. The $\mathbb{C}^*$-action on G/H_0 is described in [BP87], [Del20a]. Moreover this action commutes with ξ, so this action must lie in $\mathrm{lin}\mathcal{V} \cap \mathcal{C}_R$.

Conversely, let $(\nu, m) \in \mathcal{V} \oplus \mathbb{N}^*$ and consider the conical embedding defined by the generators of $\mathcal{C}_Y$ and (ν, m), with all the colors of $G/H \times \mathbb{C}^*$. This defines clearly a $G \times \mathbb{C}^*$-spherical cone $\mathcal{Y}$, and the projection to $\mathbb{R}_{\geq 0}(0,1)$ with $0 \in \mathcal{N}$ in the Euclidean spaces gives an affine $G \times \mathbb{C}^*$-equivariant morphism $\pi : \mathcal{Y} \to \mathbb{C}$ by classification of spherical morphisms recalled in Prop. 2.17.

The central fiber Y_0 corresponds then to the divisor of $\mathcal{Y}$ determined by the ray (ν, m). The latter can also be seen as an elementary embedding of $G/H \times \mathbb{C}^*$, hence an equivariant degeneration of G/H to G/H_0. The special test configuration is obtained after changing the lattice $\mathcal{N} \oplus \mathbb{Z}$ to $\mathcal{N} \oplus \frac{1}{k}\mathbb{Z}$ for a suitable k, while keeping the colored cone of $\mathcal{Y}$.

Remark that the coordinate rings R, R_0 of Y, Y_0 are isomorphic as G-modules, hence $\mathcal{M} \simeq \mathcal{M}_0$. Furthermore, $R^{(B)}$ is B-isomorphic to $R_0^{(B)}$ [Pop86, Proposition 4], hence $\Gamma_Y \simeq \Gamma_{Y_0}$.

Finally, taking ν that projects to the interior of $\mathcal{V}$ then yields a test configuration with horospherical central fiber. The last statement results from [Del20a]. □

We will denote from now on $(\mathcal{Y}, \xi; \nu)$ the G-equivariant test configuration of (Y, ξ) with respect to $\nu \in \mathcal{V}$.

Remark 2.19. *The embedding data of the central fiber Y_0 of $(\mathcal{Y}, \xi; \nu)$ can be obtained as follows. The weight lattice $\mathcal{M}_0$ of Y_0 can be identified with*

$$\mathcal{M}_0 := (\nu^\perp \cap \mathcal{M}) \oplus \mathbb{Z}\chi \simeq \mathcal{M},$$

where $\chi \in \mathcal{M}$ is such that $\langle \chi, \nu \rangle = 1$. In particular, if we let $\pi : \mathcal{N} \to \mathcal{N}_0$ be the dual map of the isomorphism $\mathcal{M}_0 \simeq \mathcal{M}$, then

$$\mathcal{V}_0 = \mathbb{R}\nu \oplus \pi(\mathcal{V}).$$

Since the weight monoids of Y and Y_0 are the same, their colored cones have the same support, and the colors of Y_0 can be determined using [GH15b].

2.3. **Futaki invariant.** Let us recall briefly the construction of Futaki invariant by Collins-Székelyhidi [CS18] through index character and the equivalent characterization of Li-Wang-Xu in terms of normalized volume and log discrepancy [LWX21]. Let (Y, ξ) be a n-dimensional polarized spherical cone and

$$\mathbb{C}[Y] = \oplus_{\alpha \in \Gamma} R_\alpha$$

be the decomposition of $\mathbb{C}[Y]$ as a T_H-representation. For any $t \in \mathbb{C}$ and $\xi \in \mathcal{N}_\mathbb{R}$, the index character is defined as

$$F(t, \xi) := \sum_{\alpha \in \Gamma} e^{-t\langle \alpha, \xi \rangle} \dim R_\alpha.$$

This is a meromorphic function on $\mathbb{C}$ with poles along imaginary axis, and decomposes near $t = 0$ as

$$F(t, \xi) = \frac{a_0(\xi) n!}{t^{n+1}} + \frac{a_1(\xi)(n-1)!}{t^n} + O(t^{1-n}).$$

where $a_0, a_1 : \mathcal{C}_R \to \mathbb{R}$ are smooth functions.

Let $d_\xi f(\nu)$ be the directional derivative of a function at a point ξ along the vector ν. The Futaki invariant of the test configuration $(\mathcal{Y}, \xi; \nu)$ is defined by

$$\mathrm{Fut}_\xi(\mathcal{Y}, \nu) = \frac{a_0(\xi)}{n} d_\xi \left(\frac{a_1}{a_0}\right)(\nu) + \frac{a_1(\xi) d_\xi a_0(\nu)}{n(n+1)a_0(\xi)}.$$

In particular, the Futaki invariant of a test configuration depends only on the coordinate ring of the central fiber as a representation of T_H. For computational reason, we shall use the definition of the Futaki invariant by Li-Wang-Xu in terms of normalized volume and log discrepancy, but note that this is the Futaki invariant of [CS18] up to a positive constant (see for example [LLX20] for details).

Let Y be a klt normal variety. The *log discrepancy function* of Y is a positive function A_Y over the set of valuations that admit a center on Y. For practical reason, we only give the definition of the log discrepancy for a divisorial valuation and refer the reader to [LLX20, Theorem 2.2, Theorem 3.5] for the general definition for a pair (Y, D). Let E be the exceptional divisor over a proper birational model $\mu : Y' \to Y$, and w_E the associated valuation over $\mathbb{C}(Y') = \mathbb{C}(Y)$, the log discrepancy is then

$$A_Y(w_E) := 1 + w_E(K_{Y'} - \mu^* K_Y).$$

The general discrepancy for a quasimonomial valuation is then defined in an obvious way, and for a general valuation centered on Y by using the retraction map from Val_Y to the set of quasimonomial valuations over any log smooth model of Y.

Proposition 2.20. *[CS19], [Li18] Let (Y, D) be a spherical log cone with angles γ and m be an integer such that $m(K_Y + D)$ is Cartier. Let s be a $G \times T_H$-equivariant nowhere-vanishing holomorphic section of $-m(K_Y + D)$. Then there exists a linear function $\varpi_\gamma : \mathcal{C}_R \to \mathbb{R}$ such that*

$$\mathcal{L}_\xi s = m \langle \varpi_\gamma, \xi \rangle s.$$

Moreover, the log discrepancy of w_ξ is exactly

$$A_{(Y,D)}(w_\xi) = \langle \varpi_\gamma, \xi \rangle.$$

If (Y, D, ξ) has log Calabi-Yau cone metrics, then $A_{(Y,D)}(w_\xi) = n$.

Definition 2.21. *Let (Y, D) be a log spherical cone. Let ξ be an element in the Reeb cone $\mathcal{C}_R$ and $\varpi_\gamma : \mathcal{C}_R \to \mathbb{R}$ the linear function as above. The* (algebraic) volume *of (Y, ξ) is defined as*

$$\mathrm{vol}_Y(\xi) = \lim_{k \to \infty} \frac{\dim \left(\bigoplus_{\langle \alpha, \xi \rangle < k} R_\alpha\right)}{k^n / n!}.$$

The normalized volume of a spherical log cone (Y, D) is a function that takes $\xi \in \mathcal{C}_R$ to

$$\widehat{\mathrm{vol}}_{(Y,D)}(\xi) := A_{(Y,D)}(w_\xi)^n \,\mathrm{vol}_Y(\xi) = \langle \varpi_\gamma, \xi \rangle^n \,\mathrm{vol}_Y(\xi).$$

Remark 2.22. *It has been established that* vol *is a continuous function, see e.g. [CS18, Theorem 4.10] for the case where Y is smooth, and [Li21, Theorem 2.8] for the general case. From the differentio-geometric point of view,* vol *is the g-weighted volume of the (log) Fano base. More precisely, when ξ is quasi-regular, $\mathrm{vol}_Y(\xi)$ is exactly the volume of the quasi-regular quotient with respect to the transverse Kähler form for ξ.*

Definition 2.23. *Let $(\mathcal{Y},\mathcal{D},\xi;\nu)$ be any special test configuration of the polarized spherical log cone (Y,D,ξ) with angles γ and central fiber (Y_0,D_0,ξ). Let $A := A_{(Y_0,D_0)}$ be the log discrepancy of the central fiber. The Futaki invariant of $(\mathcal{Y},\mathcal{D},\xi;\nu)$ is defined as*

$$\mathrm{Fut}_\xi(\mathcal{Y},\mathcal{D},\nu) := \frac{d_\xi \widehat{\mathrm{vol}}_{(Y_0,D_0)}(\nu)}{nA_{(Y,D)}(\xi)^{n-1}\,\mathrm{vol}_{Y_0}(\xi)} = \langle \varpi_\gamma,\nu\rangle + \frac{\langle \varpi_\gamma,\xi\rangle}{n}\frac{d_\xi\,\mathrm{vol}_Y(\nu)}{\mathrm{vol}_Y(\xi)}.$$

Definition 2.24. *We say that a polarized spherical log cone (Y,D,ξ) is G-equivariantly K-semistable if for every special G-equivariant test configuration defined by $\nu\in\mathcal{V}$, $\mathrm{Fut}_\xi(\mathcal{Y},\mathcal{D},\nu)\geq 0$.*

Moreover, (Y,D,ξ) is G-equivariantly K-stable (or K-polystable in [LWX21]) if it is K-semistable and that $\mathrm{Fut}_\xi(\mathcal{Y},\mathcal{D},\nu)=0$ only if $(\mathcal{Y},\mathcal{D},\xi;\nu)$ is a trivial test configuration.

The following lemma allows to prove the main theorem by reducing to the computation of the Futaki invariant of a horospherical cone.

Lemma 2.25. *Let $(\mathcal{Y},\mathcal{D},\xi;\nu)$ be a degeneration with horospherical central fiber (Y_0,D_0). The Futaki invariant of (Y,D,ξ) is the same as the Futaki invariant of (Y_0,D_0,ξ).*

Proof. Since the Futaki invariant as defined by Collins-Székelyhidi only depends on the moment cone of Y (that is the convex cone generated by the weights of T_H), and that the central fiber Y_0 has the same moment cone as Y by a theorem of Knop [Kno90, Satz 5.4], the result then follows. □

Let us now compute the Futaki invariant of a pair associated to a horospherical conical embedding $G/H\subset Y$. Recall that G/H is a equivariant torus bundle over G/P, where $P := N_G(H)$ is the right-stabilizer of the open Borel orbit. Denote by Φ_{P^u} the root system of the reductive part P^u. By Brion's description of the canonical divisor, K_Y can be represented by

$$-K_Y = \sum_{\nu\in\mathcal{V}_Y} D_\nu + \sum_{d\in\mathcal{D}_Y} a_d d.$$

where $\mathcal{V}_Y$ is the set of G-stable divisors of Y and $\mathcal{D}_Y$ the set of colors of Y, and a_d are coefficients that depend only on G/H.

Lemma 2.26. *Let (Y,D,ξ) be a polarized horospherical log cone with angles γ, colored cone $\mathcal{C}_Y$ and Reeb cone $\mathcal{C}_R := -int(\mathcal{C}_Y)$. Let $\Delta_\xi = \{\langle .,\xi\rangle = n\}\cap\mathcal{C}_Y^\vee$ and $bar_{DH}(\Delta_\xi)$ be the barycenter of Δ_ξ with respect to the Duistermaat-Heckman measure*

$$P(p)d\lambda(p) := \prod_{\alpha\in\Phi_{P^u}} \langle\alpha,p\rangle\, d\lambda(p)$$

For every $\xi\in\mathcal{C}_R$ and $\nu\in\mathcal{V}$, the Futaki invariant of (Y,D,ξ) can be written as

$$Fut_\xi(Y,D,\nu) = \left\langle -\frac{\langle\varpi_\gamma,\xi\rangle}{n} bar_{DH}(\Delta_\xi) + \varpi_\gamma,\nu\right\rangle,$$

where ϖ_γ can be interpreted as the B-weight of the canonical section of the Cartier divisor $-m(K_Y+D)$.

Proof. Let us first work with an usual cone Y. A horospherical cone is $\mathbb{Q}$-Gorenstein if and only if there exists a linear function $l\in\mathcal{M}_\mathbb{Q}$ on $\mathcal{C}_Y$ such that

$$\langle l,\nu\rangle = 1,\ \langle l,\rho(d)\rangle = a_d.$$

This linear function is exactly the B-weight $-\varpi$ of the canonical section of K_Y

$$-\varpi = \sum_{\alpha \in \Phi_{Pu}} \alpha.$$

Moreover, one can show as in [Ngh23] that the unique T_H-equivariant holomorphic section s of the Cartier divisor $-mK_Y$ satisfies

$$\mathcal{L}_\xi s = -m \langle l, \xi \rangle s.$$

It follows from the description of the log discrepancy in terms of s that $A_Y(w_\xi) = -\langle l, \xi \rangle = \langle \varpi, \xi \rangle$ for every $\xi \in \mathcal{C}_R$. The case of (Y, D) follows by replacing $(-K_Y, \varpi)$ with $(-(K_Y + D), \varpi_\gamma)$.

We now compute the volume of (Y, ξ). By continuity of the volume, it suffices to compute $\mathrm{vol}_Y(\xi)$ for a quasiregular Reeb vector $\xi \in (\mathcal{C}_R)_\mathbb{Q}$. Let $X := Y // \langle \xi \rangle$ be the GIT orbifold quotient of Y. It is naturally a log Fano spherical variety endowed with a Hamiltonian action of the torus $T_H / \langle \xi \rangle$, and the moment polytope for this action after normalizing is exactly Δ_ξ. The Duistermaat-Heckman measure on this polytope coincides with $P d\lambda$. This measure is moreover independent of the choice of ξ, cf. [Li21]).

In particular, for a horospherical cone Y polarized by a quasi-regular Reeb element ξ,

$$\mathrm{vol}_Y(\xi) = n! \int_{\Delta_\xi} P(p) d\lambda(p).$$

Using the definition of the Gamma function

$$\Gamma(n+1) = n! = \int_{s>0} s^n e^{-s} ds,$$

and a Fubini argument, we obtain

$$\mathrm{vol}_Y(\xi) = \int_{s>0} \int_{\langle ., \xi \rangle = s} e^{-\langle p, \xi \rangle} \langle p, \xi \rangle^n P(p) d\lambda(p) ds = \int_{\mathcal{C}_R^\vee} e^{-\langle p, \xi \rangle} P(p) d\lambda(p).$$

Finally, a direct computation yields

$$d_\xi (\log \mathrm{vol}_Y)(\nu) = - \langle \mathrm{bar}_{DH}(\Delta_\xi), \nu \rangle.$$

The lemma then follows from the definition of the Futaki invariant in terms of normalized volume. □

3. Proof of Theorem A and B

3.1. Proof of Theorem A.

Theorem 3.1. *The following conditions are equivalent.*

- *A polarized log Fano cone (Y, D, ξ) admits log Calabi-Yau cone metric with Reeb vector ξ.*
- *(Y, D, ξ) is K-stable.*

Moreover, it suffices to test these stability conditions over G-equivariant special test configurations, where G is a reductive group acting effectively and holomorphically on (Y, D, ξ). In particular, a G-spherical cone (Y, D, ξ) admits K-invariant log Calabi-Yau cone metrics iff (Y, D, ξ) is G-equivariantly K-stable.

Proof. This was essentially proved in [Li21], see [Li21, Theorem 2.9], also [HL23, Theorem 1.7]. For the reader's convenience, we provide a sketch of proof.

Let η be the (weak) contact form associated to ξ. The log Calabi-Yau cone equation on (Y, D, ξ) can be shown (cf. Equation (78) [Li21]) to be equivalent to an equation of the form

$$g(\eta)(d\eta)^{n-1} \wedge \eta = dV_Y^{\xi},$$

where g is a positive smooth function on the link $\left\{r_\xi^2 = 1\right\}$. Now let $\xi_0 := \xi - \xi'$ be any other quasi-regular Reeb vector field, and $\eta_0 = \eta/\eta(\xi_0)$ be the contact form with respect to ξ_0. The Reeb vector ξ_0 generates a $\mathbb{C}^*$-action and we identify the Fano orbifold quotient $(Y, D)// \langle v_{\xi_0} \rangle$ with a log Fano variety (X, D_X), where D_X takes into account the ramified divisor. If (Y, D) is $G \times \mathbb{C}^*$-equivariant, then (X, D_X) is G-equivariant. Translating the above equation in terms of η_0, ξ_0, we obtain

$$g(\eta_0)(d\eta_0)^{n-1} \wedge \eta_0 = dV_Y,$$

which is a g-soliton equation on the quotient $(Y, D)// \langle v_{\xi_0} \rangle = (X, D_X)$ (cf. Equation (104) [Li21]). In particular, (Y, D, ξ) admits a weak log Calabi-Yau cone metric if and only if any quasi-regular quotient admits a g-soliton.

Let $\zeta := \xi_0 + t\partial_t$, where $t\partial_t$ is the holomorphic vector field generating the $\mathbb{C}^*$-action. The quotient $(\mathcal{Y}, \mathcal{D})/ \langle v_\zeta \rangle = (\mathcal{X}, \mathcal{D}_{\mathcal{X}}, -(K_{\mathcal{X}} + \mathcal{D}_{\mathcal{X}}))$ is a test configuration of $(X, D_X, -(K_X + D_X))$. Here, the Cartier divisor $-(K_{\mathcal{X}} + \mathcal{D}_{\mathcal{X}})$ is the multiple of the polarizing orbifold line bundle $\mathcal{L}$ (viewed as a $\mathbb{Q}$-Cartier divisor) such that $\mathcal{L}^* \backslash \mathcal{X} \simeq \mathcal{Y} \backslash \{0\}$.

Conversely, any test configuration of (X, D_X) induces a test configuration of (Y, D) (by taking the fiberwise cones over X with respect to the polarization $-(K_X + D_X)$). Moreover, the correspondence sends special test configurations to special test configurations, and G-equivariant test configurations of Y to G-equivariant test configurations of X (if the action of ξ_0 is compatible with G).

Next, can show that the Ding invariant of $\mathcal{Y}$ is exactly the weighted Ding invariant of any quotient test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$. The work of Han-Li [HL23] establishes that (X, D_X) admits a g-soliton if and only if it is g-weighted Ding stable. It follows that (Y, D, ξ) is Ding-stable iff (Y, D, ξ) admits a weak log Calabi-Yau cone metric, iff any quasi-regular quotient is g-weighted Ding-stable.

Finally, since it is enough to check g-weighted Ding stability of a quasi-regular quotient over G-equivariant special test configurations [HL23, Theorem 7.3], [Li21, Theorem 1.15], the polarized cone (Y, D, ξ) is Ding-stable iff it is Ding-stable over all G-equivariant special test configurations for a given G. Finally for a special test configuration, the Ding invariant of the polarized cone (Y, D, ξ) coincides with the Futaki invariant, and the theorem follows. $\square$

Theorem 3.2. *Recall that* $\Sigma := (-\mathcal{V})^\vee$. *A polarized spherical log cone* (Y, D, ξ) *with angles* γ *is* K*-stable if and only if*

$$bar_{DH}(\Delta_\xi) - \varpi_\gamma \in \mathrm{RelInt}(\Sigma).$$

Proof. This follows from Theorems 2.18, 3.1, and our computation of a horospherical cone's Futaki invariant. For simplicity, we work with a $\mathbb{Q}$-Gorenstein G-spherical cone Y. Given any G-equivariant special test configuration $(\mathcal{Y}, \xi; \nu)$, $\nu \in (-\mathcal{V})$ of (Y, ξ) with central fiber Y_0, we can construct another test configuration of Y_0 with horospherical central fiber Y_0'. The Futaki invariant of (Y_0', ξ) is the same as (Y_0, ξ) by Lemma 2.25, hence the K-semistability condition is equivalent to

$$\frac{\langle \varpi, \xi \rangle}{n} \langle \mathrm{bar}_{DH}(\Delta_\xi), \nu \rangle \geq \langle \varpi, \nu \rangle, \ \forall \nu \in (-\mathcal{V}).$$

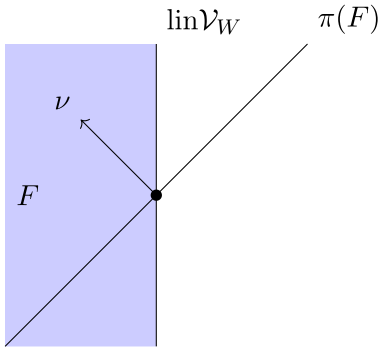


FIGURE 1. Degeneration of (W,ξ) along a valuation ν in the Futaki vanishing locus F on $\mathcal{V}_W$.

The fact that a Ricci-flat Kähler cone (Y,ξ) satisfies $\langle\varpi,\xi\rangle = n$ (cf. Prop. 2.20) simplifies further this condition to

$$\mathrm{Fut}_\xi(Y_0',\nu) = \langle \mathrm{bar}_{DH}(\Delta_\xi) - \varpi, \nu\rangle \geq 0,\ \forall \nu \in (-\mathcal{V}).$$

Recall following fact:

$$\mathrm{RelInt}(\Sigma) = \{\sigma, \langle\sigma,\nu\rangle > 0, \forall\nu\in(-\mathcal{V})\backslash \mathrm{lin}(-\mathcal{V})\}\,.$$

The combinatorial condition in the statement

$$\mathrm{bar}_{DH}(\Delta_\xi) - \varpi \in \mathrm{RelInt}(\Sigma)$$

holds if and only if $\mathrm{Fut}_\xi(Y,\nu) > 0, \forall\nu\in(-\mathcal{V})\backslash\mathrm{lin}(-\mathcal{V})$. Under this condition, (Y,ξ) is clearly K-semistable, and the vanishing of $\mathrm{Fut}_\xi(Y,\nu)$ implies that $\nu\in\mathrm{lin}(\mathcal{V})$, hence the test configuration defined by ν is a trivial test configuration by Theorem 2.18. Conversely, suppose that (Y,ξ) is K-stable and $\mathrm{bar}_{DH}(\Delta_\xi) - \varpi \notin \mathrm{RelInt}(\Sigma)$. Then there is $\nu\notin\mathrm{lin}\mathcal{V}$ such that $\mathrm{Fut}_\xi(Y,\nu)=0$, i.e. there is a non-trivial test configuration with vanishing Futaki invariant, a contradiction. The theorem is then proved. By replacing ϖ with ϖ_γ, one obtains directly the K-stability criterion for a log pair. $\square$

3.2. **Proof of Theorem B.**

Proposition 3.3. *Let (W,ξ) be any strictly K-semistable G-spherical cone. Then there is a G-equivariant special degeneration of (W,ξ) with K-stable central fiber. Any other such degeneration has G-isomorphic central fiber. The analogue holds for a strictly K-semistable G-spherical log pair (W,D,ξ).*

Proof. Let F be the vanishing locus of Fut_ξ on $\mathcal{V}_W$, which is a face of $\mathcal{V}_W$ containing the linear part $\mathrm{lin}\mathcal{V}_W$. We degenerate (W,ξ) along a valuation $\nu\in\mathrm{RelInt}(F)$ (cf. Figure 1). The resulting central fiber (W',ξ) then remains K-semistable (cf. Lemma 2.25) with vanishing locus of Fut_ξ contained in $\mathrm{lin}\mathcal{V}_{W'}$, hence K-stable.

Indeed, $\mathcal{V}_{W'}$ can be identified with

$$\mathcal{V}_{W'} := \mathbb{R}\nu \oplus \pi(\mathcal{V}_W),$$

where π is the quotient map $\mathcal{N}_{W,\mathbb{R}} = \mathcal{N}_{W',\mathbb{R}} \to (\mathcal{N}_W/\mathbb{Z}\nu)_\mathbb{R}$ (cf. Remark 2.19). Since $\nu\in\mathrm{RelInt}(F)$, $\pi(F)$ is a vector space in $(\mathcal{N}_W/\mathbb{Z}\nu)_\mathbb{R}$, and the new Futaki vanishing locus $\mathbb{R}\nu\oplus\pi(F)$ is contained in $\mathrm{lin}\mathcal{V}_{W'}$.

Uniqueness of the K-stable degeneration follows from [LWX21]: two K-stable central fibers are isomorphic as affine varieties, hence if any one of them is G-invariant, the other can be endowed with the G-action through the isomorphism. $\square$

3.3. **Examples.**

3.3.1. *Horosymmetric cones of rank one.* Let G/H be a semisimple horosymmetric space, i.e. an equivariant fibration $G/H \to G/P$ over a flag manifold with semisimple symmetric fiber $L/L \cap H$ (cf. [Del20b] for more details). Supopse that G/H admits a $\mathbb{Q}$-Fano embedding. For simplicity, we suppose that $\mathrm{rank}(G/H) = 1$, but our arguments extend easily to any rank.

Let Φ be the root system of G and Φ_L be the root system of L with involution σ. Let $\alpha, 2\alpha$ be the simple restricted roots with multiplicites n_1, n_2 induced by (Φ_L, σ) (where $n_2 = 0$ if 2α is not a restricted root). Denote by $\Phi_s^+ := \Phi_L^+ \backslash \Phi^\sigma$ and $\Phi_{Q^u} := \Phi^+ \backslash \Phi_L$.

Choosing the horosymmetric subgroup H such that $L \cap H = N_G(H)$, we have

$$\mathcal{M}(G/H) = \mathbb{Z}\alpha, \quad \mathcal{N}(G/H) = \mathbb{Z}(\alpha^\vee/2).$$

Let κ be the Killing form such that $\langle \alpha, \beta^\vee \rangle = 2\frac{\kappa(\alpha,\beta)}{\kappa(\beta,\beta)}$.

Let X be the $\mathbb{Q}$-Fano compactification of G/H (with all the colors) associated to the $\mathbb{Q}$-reflexive polytope Q_X [GH15a]. Let $m > 0$ be the minimal integer such that mK_X is Cartier. Take Y as the Fano cone over X, obtained by contracting the canonical line bundle mK_X along X.

By construction, Y is a $G \times \mathbb{C}^*$-spherical cone with open orbit isomorphic to $G/H \times \mathbb{C}^*$. Here the $\mathbb{C}^*$-action on Y comes from the natural $\mathbb{C}^*$-action on mK_X. For simplicity, we can suppose that $m = 1$ (so that K_X is Cartier).

We endow $\mathcal{M}(G/H \times \mathbb{C}^*)$ with the basis (α, η), where η is the weight of the $\mathbb{C}^*$-action on K_X. Let $\mathcal{N}(G/H \times \mathbb{C}^*)$ be the dual lattice. The valuation cone of $G/H \times \mathbb{C}^*$ can be identified with the half-space

$$\mathcal{V} := \{(x, y) \in \mathcal{N}(G/H \times \mathbb{C}^*)_\mathbb{R}, \quad x \leq 0\}.$$

and the cone of spherical roots with

$$\Sigma = (-\mathcal{V})^\vee = \mathbb{R}_{\geq 0}(\alpha^\vee/2).$$

Let ϖ be the weight of the canonical section of $-K_X$, which writes

$$\varpi := \sum_{\widehat{\alpha} \in \Phi_s^+ \cup \Phi_{Q^u}} \widehat{\alpha}.$$

The divisor $-K_X = \sum_{\nu \in \mathcal{V}_X} D_\nu + \sum_{d \in \mathcal{D}_X} a_d d$ defines a polytope in $\mathcal{M}(G/H)_\mathbb{R}$

$$Q_X^* := \{\chi \in \mathcal{M}(G/H)_\mathbb{R}, \langle \chi, \nu \rangle + 1 \geq 0, \ \langle \chi, \rho(d) \rangle \geq -a_d\},$$

which is the dual polytope of Q_X [GH15a]. The moment polytope Δ_X of $-K_X$ is then $\Delta_X = Q_X^* + \varpi$ [GH15a] and we can identify the colored cone of Y with

$$\mathcal{C}_Y = \mathrm{Cone}(Q_X \times \{1\}) = \mathrm{Cone}(Q_X^* \times \{1\})^\vee, \quad \mathcal{D}_Y = \left\{\overline{d \times \mathbb{C}^*}, d \in \mathcal{D}_X\right\}.$$

(cf. Figure 2 for an example). Note that $\rho(d \times \mathbb{C}^*) = (\rho(d), a_d)$, where a_d is the coefficient of d in $-K_X$. The linear function $(0, 1)$ then defines a linear function on $\mathcal{C}_Y$ making K_Y a Gorenstein divisor.

Since the equivariant automorphism group of G/H is discrete, as $\mathcal{V}(G/H)$ is only a half-line and $\dim \mathrm{Aut}_G(G/H) = \dim \mathrm{lin}\mathcal{V}$, the Reeb cone $\mathcal{C}_R$ of Y is one-dimensional and can be identified with the positive half-line $\mathbb{R}_{\geq 0}\eta$. Thus the K-stable Reeb vector of Y, if exists, is unique, so the unique polarization of Y is given by the polytope Q_X.

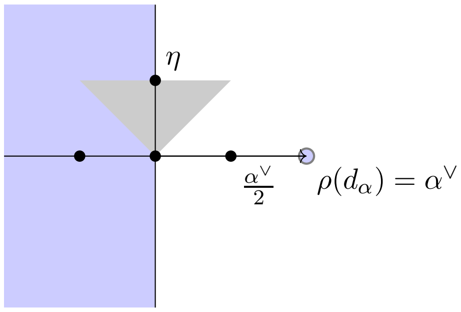


FIGURE 2. Colored cone $(\mathcal{C}_Y, \mathcal{D}_Y)$ of a symmetric rank one conical embedding of $G/H \times \mathbb{C}^*$ where $G = \mathrm{SL}_2$, $H = N_{SL_2}(T)$. Here the lattice of G/H is generated by the unique restricted root α $(= 2\widehat{\alpha}$, $\widehat{\alpha}$ being the unique root of SL_2). There is a unique color d_α of G/H with $\rho(d_\alpha) = \alpha^\vee$. The polytope Q_X is then $\{t\alpha^\vee 2, |t| \leq 1\}$, and $\mathcal{C}_Y^\vee$ is the cone over Q_X in $\mathbb{Z}(\alpha, \eta)_\mathbb{R}$. Note also that $\varpi = \alpha$ and $\Delta_X = \{t\alpha, 0 \leq t \leq 2\}$.

Setting $2\chi := \sum_{\beta \in \Phi_{Q^u}} (\beta + \sigma(\beta))$, the Duistermaat-Heckmann polynomial of Y is defined by

$$P_{DH}(p) := p^{n_1+n_2} \prod_{\beta \in \Phi_{Q^u}} \kappa(\beta, 2\chi - p\alpha).$$

Proposition 3.4. *The cone Y is K-stable if and only if*

$$\left\langle bar_{DH}(Q_X^*), \alpha^\vee/2 \right\rangle = \left\langle bar_{DH}(\Delta_X) - \varpi, \alpha^\vee/2 \right\rangle > 0,$$

i.e. iff X is K-stable as a $\mathbb{Q}$-Fano variety.

Note that if Y is K-stable then any Fano cone over X obtained by taking a root (or power) of mK_X and contracting along X is also K-stable. Repeating the arguments for any rank, we recover in particular the K-stability criterion for $\mathbb{Q}$-Fano semisimple horosymmetric varieties.

Example 3.5. *Consider the rank one symmetric space $G = \mathrm{SL}_2$, $H = N_{\mathrm{SL}_2}(T)$ and the Fano embedding X with $Q_X = \{t\alpha^\vee/2, |t| \leq 1\}$. Then $P_{DH}(p) = p$ and*

$$\left\langle bar_{DH}(Q_X^*), \alpha^\vee/2 \right\rangle = \frac{\int_{-1}^1 p^2 dp}{\int_{-1}^1 p dp} = \frac{1}{3} > 0.$$

3.3.2. *Horosymmetric cones over boundary divisors of canonical compactifications.* Let us recover state the K-stability result in [BD19] in terms of cone. Consider a rank two semisimple symmetric space O of rank two, with restricted root system R generated by long and short simple roots α_1, α_2 of multiplicities m_1, m_2, m_3 with m_3 being the multiplicity of $2\alpha_2$ which is 0 if $2\alpha_2 \notin R^+$. Let $P(p) := \prod_{\alpha \in R^+} \kappa(\alpha, p)$.

Let D be a reduced prime divisor in the boundary of the canonical compactification of a rank two semisimple symmetric space O. The divisor D is in fact always a rank one horosymmetric variety (but not Fano) [Del20b]. Consider the Fano blowdown $D^\vee$ of D along its unique closed orbit with moment polytope Δ, and take $\alpha, 2\alpha$ be the unique restricted positive roots with multiplicities n_1, n_2.

Proposition 3.6. *Let $C(D^\vee)$ be a Fano cone over $D^\vee$. Then $C(D^\vee)$ has a conical Calabi-Yau metric iff $\kappa(bar_P(\Delta) - \varpi, \alpha) > 0$ iff $D^\vee$ is K-stable.*

Proof. The blowdown $D \to D^\vee$ can be seen as the decoloration map, and the colored cone of $D^\vee$ is obtained by adding to the colored cone of D all the remaining colors.

From description of the data of D and $\mathbb{Q}$-Fano spherical variety [GH15a] [GH15b], the blowdown $D^\vee$ is then a Fano horosymmetric variety of rank one.

The combinatorial data of $D^\vee$ can then be deduced from the combinatorial data of the rank two symmetric space O following [BD19, Section 3.2]. With the same notation as above, we can take α to be the restricted root, say, α_1, and the weight lattice of $G/H \times \mathbb{C}^*$ can be identified with $\mathbb{Z}(\alpha_1, \eta = \omega + \lambda_1\alpha_1)$ (cf. (2)), while the valuation cone of G/H is a half-line, so the Reeb cone of $C(D^\vee)$ is one-dimensional, hence K-stability of $D^\vee$ is equivalent to that of $C(D^\vee)$.

The multiplicities of $\alpha, 2\alpha$ in G/H corresponds to their multiplicities as restricted roots in O, namely $n_1 = m_1, n_2 = 0$ ($n_1 = m_2, n_2 = m_3$ if taking $\alpha = \alpha_2$).

The anticanonical weight ϖ of $D^\vee$ then restricts to $\mathfrak{a}$ as

$$2\varpi = \sum_{\alpha \in R^+} m_\alpha \alpha.$$

Moreover, $2\varpi = (n_1 + 2n_2)\alpha_1 + 2\chi$ and

$$P_{DH}(p) = P(2\varpi - (n_1 + 2n_2 + p)\alpha_1).$$

The polytope Δ is the segment $\chi + [0, \lambda]\alpha_1$ where $\lambda := \lambda_2 - \lambda_1$ and $\lambda_{1,2}$ are the intersections of the line $\varpi + t\alpha_1$ with the walls of the Weyl chamber

$$\lambda_1 := -\frac{\kappa(\varpi, \alpha_2)}{\kappa(\alpha_1, \alpha_2)}, \quad \lambda_2 := -\frac{\kappa(\varpi, \alpha_1)}{\kappa(\alpha_1, \alpha_1)}. \tag{2}$$

Remark that $\kappa(\alpha_1, \chi) = 0$, hence χ is a multiple of the generator of the Weyl chamber. The K-stability criterion of $C(D^\vee)$ can finally be translated in terms of combinatorial data of O as

$$\begin{aligned}\kappa(\mathrm{bar}_P(\Delta) - \varpi, \alpha_1) &= \kappa(\mathrm{bar}_{DH}([0, \lambda]) - (n_1/2 + n_2)\alpha_1, \alpha_1) \\ &= \frac{\int_0^\lambda pP_{DH}(p)dp}{\int_0^\lambda P_{DH}(p)dp} - (n_1/2 + n_2) > 0.\end{aligned}$$

□

As a corollary, we have

Proposition 3.7. *[BD19, Section 3.3.3] Let α_1, α_2 be the long and short root of a rank two symmetric space with restricted root system G_2 and D_1, D_2 the divisors in the canonical compactification with restricted root system generated by α_1, α_2 respectively. The Fano cones $C(D_1^\vee)$, $C(D_2^\vee)$ are respectively K-unstable and K-stable.*

In fact the choices in Section 3.3 of [BD19] should read "$\alpha_2 = \alpha, \alpha_2 = \beta$" with α, β being their long and short roots.

4. Valuations and asymptotic cones of Calabi-Yau manifolds

4.1. **Donaldson-Sun theory.** Let (M, ω) be a $\partial\overline{\partial}$-exact complete Calabi-Yau manifold of complex dimension n with maximal growth and asymptotic cone (C, ξ), with ξ being the K-stable Reeb vector. By [DS17, Appendix], we also have the Bando-Mabuchi-Matsushima theorem for cones.

Proposition 4.1. *[DS17, Propositions 4.8, 4.9] Let $G_\xi := \mathrm{Aut}_\xi(C)$ be the group of holomorphic transformations of C that preserves ξ. If there exists a Ricci-flat Kähler cone metric on C with Reeb vector ξ, then G_ξ is reductive, i.e. there is a maximal compact subgroup K_ξ such that*

$$G_\xi = K_\xi^{\mathbb{C}},$$

and the metric is unique up to the action of the identity component of G_ξ.

Following [DS17], the ring of holomorphic functions with polynomial growth $R(C)$ (with respect to ω_C) on C can be identified with its coordinate ring, and decomposes under the complexified T_c-action of the Reeb vector as

$$R(C) = \bigoplus_{\alpha \in \Gamma^*} R_\alpha,$$

where α are the $T := T_c^{\mathbb{C}}$-action weights. In order to embed C into $\mathbb{C}^N$ as an affine subvariety, one can use the local holomorphic embedding F_∞ at the unique fixed point O, and extend it globally to C using homogeneity under the T-action.

Proposition 4.2. *[DS17] If $x_1, \ldots, x_N$ are local holomorphic functions such that $F_\infty = (x_1, \ldots, x_N)$ is the local embedding near O, then the affine cone C agrees globally with the affine variety generated by $x_1, \ldots, x_N$, i.e. there is a finitely generated ideal I_C defined by algebraic relations between $x_1, \ldots, x_N$ such that $C = \mathbb{C}[x_1, \ldots, x_N]/I_C$. Under such embedding, the Reeb vector has an extension to $\mathbb{C}^N$ of the form $\xi = \Re(i \sum_{a=1}^N w_a z_a \partial_{z_a})$, where $w_a > 0$ for all a.*

For each $\alpha \in \Gamma^*$, the map sending α to $\langle \alpha, \xi \rangle$ is injective, so we can in fact redecompose $R(C)$ as

$$R(C) = \bigoplus_k R_{d_k},\ R_{d_k} := \left\{ f_{\alpha_k}, \langle \alpha_k, \xi \rangle = d_k \right\}.$$

Definition 4.3. *The set $\{0 = d_0 < d_1 < d_2 < \ldots\}$ is called the* holomorphic spectrum *of C, denoted by $\mathcal{S}$.*

Proposition 4.4. *[DS17, Theorem 3.3] The set $\mathcal{S} \subset \mathbb{R}_{\geq 0}$ consists of algebraic numbers and is independent of the converging subsequence of (M_i, ω_i). In particular, $\mathcal{S}$ is a finitely generated semigroup.*

Proof. The result in [DS17] is stated in the context of *local* tangent cone at a point, but the proof can be adapted almost verbatim for tangent cone at infinity. Fix $\lambda > 1$ and let (M_i, ω_i) be the rescaling of (M, ω) be a factor λ^{-i}. Denote by $\mathcal{C}_\infty$ the set of all sequential Gromov-Hausdorff limits of (M_i, ω_i) as $i \to +\infty$. The main ingredients of the proof are the following facts.

(1) $\mathcal{C}_\infty$ is compact connected, cf. [DS17, Lemma 3.2] for a proof which relies on the fact that $\mathcal{K}(n, \kappa)$ is compact Hausdorff (this is still true for $\partial\overline{\partial}$-exact Calabi-Yau metrics).
(2) From [DS17, Lemma 3.5], there is a dense subset $\mathcal{I}$ of $\mathbb{R}^+$ such that if $D \in \mathcal{I}$, then $N_D := \dim \oplus_{0<d<D} R_d$ is independent of $C \in \mathcal{C}_\infty$.
(3) For any $C \in \mathcal{C}_\infty$, we may arrange $\mathcal{S} \cap (0, D)$ with multiplicities in the increasing order as $d_1 \leq \cdots \leq d_{N_D}$, and the map

$$\iota_D : \mathcal{C}_\infty \to (\mathbb{R}^+)^{N_D}$$

sending C to the vector $(d_1, \ldots, d_{N_D})$ is in fact continuous.

Since $\mathcal{C}_\infty$ is connected, the image of ι_D must be a single point for all $D \in \mathcal{I}$, hence $\mathcal{S}$ is independent of $C \in \mathcal{C}_\infty$. □

Given a point $p \in M$, $\lambda > 0$, $B_i := B(p, \lambda^{2i})$, let f be a holomorphic function on M, and $\|f\|_i$ be the L^2-norm of $f|_{B_i}$ with respect to the normalized metric $\omega_i := \lambda^{-2i}\omega$ restricted to B_i. The *growth rate* of f on M with respect to ω is defined by

$$d_\omega(f) := \lim_{i \to +\infty} (\log \lambda)^{-1} \frac{\log \|f\|_{i+1}}{\log \|f\|_i}.$$

Proposition 4.5. *[DS17, Corollary 3.8] For every holomorphic function f on M, the rate $d_\omega(f)$ is either $+\infty$ or belongs to $\mathcal{S}$, and does not depend on the choice of p.*

This is stated in the context of local tangent cones, but can be specialized to the case of infinity tangent cones. We also have the following equivalent characterization:

$$d_\omega(f) = \lim_{r\to\infty} \frac{\sup_{B(p,r)} \log|f|}{\log r}.$$

Hence $d_\omega(f)$ can be seen as the vanishing order at infinity of f, measured with respect to the Calabi-Yau metric ω. Let $R(M)$ be the ring of holomorphic functions f with polynomial growth on M, i.e. $d_\omega(f) < +\infty$.

Proposition 4.6. *[DS17] [Liu21] The ring $R(M)$ is finitely generated, and*

$$\widetilde{M} := \mathit{Spec}(R(M))$$

has the structure of an affine variety with isolated singularities. Moreover, there is a map $\pi_M : M \to \widetilde{M}$ which is a crepant resolution of singularities.

One can easily check that

$$\nu_\omega := -d_\omega$$

extends to a nonpositive (hence never centered) valuation on the quotient field $\mathcal{K}(M)$ of $R(M)$, namely

- $\nu_\omega(\mathbb{C}^*) = 0,\ \nu_\omega(0) = +\infty$,
- $\nu_\omega(fg) = \nu_\omega(f) + \nu_\omega(g)$,
- $\nu_\omega(f+g) \geq \min\{\nu_\omega(f), \nu_\omega(g)\}$.

Proposition 4.7. *[DS17] The possible finite growth rates $0 = d_0 < d_1 < \dots$ on M coincide with $\mathcal{S}$ and ν_ω is a valuation on $\mathcal{K}(M)$ whose value group $\nu(\mathcal{K}(M)^*)$ is $\mathcal{S} \cup (-\mathcal{S}) \cup \{0\}$.*

The degree function d_ω induces a filtration

$$0 = I_0 \subset I_1 \subset \cdots \subset R(M)$$

on M, where $I_k = \{f \in R(M), d_\omega(f) \leq d_k\}$. Moreover, we have $\dim I_k = \dim \bigoplus_{j\leq k} R_{d_j}$.

Algebraically, C can be constructed by a *2-step degeneration* as follows. The graded ring

$$R(W) := \bigoplus I_{k+1}/I_k$$

is finitely generated, and can be seen as the central fiber of the filtration induced by the valuation ν_ω. The affine variety $W = \operatorname{Spec}(R(W))$ is the central fiber of a test configuration induced by ν_ω with generic fiber isomorphic to $\widetilde{M}$. The cone W is in fact a *weighted tangent cone* at infinity of $\widetilde{M}$.

Proposition 4.8. *[DS17] [SZ22] Let $B = B(O,1)$ the unit ball of C at the fixed point O, embedded in $\mathbb{C}^N$ using F_∞, and $B_i = B(p, 2^i) \subset (M, \omega)$ the unit ball on (M_i, ω_i). Let $\Lambda : \mathbb{C}^N \to \mathbb{C}^N$ be linear transformation on $\mathbb{C}^N$ defined by*

$$\Lambda(z_1, \dots, z_N) = ((1/\sqrt{2})^i z_1, \dots, (1/\sqrt{2})^i z_N),$$

which induces an action on F_i by

$$(\Lambda.F_i) = \Lambda(x_1^i, \dots, x_N^i).$$

Then there are holomorphic embeddings $F_i : \widetilde{M} \to \mathbb{C}^N$ and $G_i := \Lambda + \tau_i \in G_\xi$ for linear maps $\tau_i \to 0$, such that

- $F_{i+1} = G_i \circ F_i$

- *For any subsequence $i \to +\infty$, passing to a further subsequence we have $F_i(\pi_M(B_i)) \to h.F_\infty(B)$ in the Hausdorff sense in $\mathbb{C}^N$ for some $h \in K_\xi$.*

Moreover, if $M_i := F_i(\widetilde{M})$ and W_i is the weighted tangent cone at infinity of M_i, then $M_i \simeq M_j$ and $W_i \simeq W_j$ for all i, j in the sequence. The elements $(M_i)_{i\in\mathbb{N}}$ are generic fibers in the special test configuration with central fiber W.

We often identify $(\widetilde{M}, W)$ with $(F_1(\widetilde{M}), W_1)$. Geometrically, W can be realized by firstly embedding $\widetilde{M}$ as an affine variety into $\mathbb{C}^N$ using holomorphic functions $F_1 = (x_1, \dots, x_N)$, while diagonally linearizing the T_c-action on $\mathbb{C}^N$ with weight $w = (w_1, \dots, w_N) \in (\mathbb{R}_{>0})^N$. Define the weight of a monomial $x_1^{a_1} \dots x_N^{a_N}$ in $\mathbb{C}^N$ as $a_1w_1 + \dots a_Nw_N$. Let I be the polynomial ideal in $\mathbb{C}^N$ generating $\widetilde{M}$, which is of finite type. For each generator f of I (in the Gröbner basis of I with respect to the ordering induced by w for example), keep only the term f_w, which consists of monomials with highest weight. The ideal I_w generated by all the f_w then corresponds to W and $\mathcal{R} = \mathbb{C}[x_1, \dots, x_N]/I_w$. Then $\mathcal{R}$ admits a natural gradation by w as

$$\mathcal{R} = \bigoplus \mathcal{F}_{d_k}/\mathcal{F}_{d_{k+1}},$$

where $\mathcal{F}_{d_k} = \{f \in \mathcal{R}, w(f) \leq d_k\}$.

Proposition 4.9. *[DS17] The natural map $\mathcal{R} \to R(W)$ is an isomorphism and valuation-preserving, namely every element in $\mathcal{F}_{d_{k+1}}/\mathcal{F}_{d_k}$ is sent to an element in I_{k+1}/I_k.*

Remark 4.10. *We often identify the weighted valuation w on $\mathcal{R}$ with the valuation ν_ξ on $R(W)$.*

By [DS17, Prop 3.26], $R(W)$ has the same grading as $R(C)$, hence admits an action of T_c with the same Hilbert function as C.

Proposition 4.11. *[DS17] There is a special test configuration with generic fiber isomorphic to W and central fiber C. The varieties W_i are in fact generic fibers in the test configuration.*

Moreover, since (C, ξ) is K-stable, (W, ξ) is K-semistable by [LLX20, Prop. 5.5] and the K-semistable valuation ν_ξ coincides with the valuation induced by ν_ω on $R(W)$.

5. Proof of Theorem C

Before stating key propositions in this section, we make a brief digression to symplectic aspects of spherical varieties. Let (X, ω) be a Kähler manifold with K acting by holomorphic isometries. A vector field $\mathbf{X}$ on X is said to be *locally hamiltonian* if $\mathcal{L}_\mathbf{X}\omega = 0$. The set $\mathrm{Ham}_{\mathrm{loc}}(X)$ of locally hamiltonian vector fields on X is then naturally a Lie algebra. Every smooth function H on X defines a locally hamiltonian vector field $\mathbf{X}_H$ by $dH = i_{\mathbf{X_H}}\omega$, and there is also a Lie algebra structure on $C^\infty(X)$, called the Poisson structure. The morphism $\nu : C^\infty(X) \to \mathrm{Ham}_{\mathrm{loc}}(X), H \to \mathbf{X}_H$ is in fact a Lie algebra morphism.

The action of K is said to be *Poisson* if there is a Lie algebra morphism $\lambda : \mathfrak{k} \to C^\infty(X)$, called a *lifting*, such that the morphism $\nu \circ \lambda$ is exactly the natural Lie algebra morphism $\mathfrak{k} \to \mathrm{Ham}_{\mathrm{loc}}(X)$. Such a lifting exists iff K acts trivially on the Albanese variety of X [HW90, Proposition 1]. In particular, on a $G = K^{\mathbb{C}}$-projective manifold, $\mathrm{Alb}(X)$ is trivial (since $b_1(X) = 0$), hence the holomorphic-isometric action of K is always Poisson.

A compact connected Kähler manifold (X, ω) with a Poisson K-action is said to be a *spherical K-space* if the Lie subalgebra $C^\infty(X)^K$ is an abelian Lie algebra.

Theorem 5.1. *[HW90, Equivalence Theorem] A compact connected Kähler manifold (X,ω) with a Poisson K-action is a K-spherical space iff it is a projective $G = K^{\mathbb{C}}$-spherical manifold. The result is moreover independent of the Kähler structure.*

The following lemma will be useful to us.

Lemma 5.2. *[HW90, Restriction Lemma] Let X be a compact Kähler manifold with a Poisson action of a connected compact group K. If X is a spherical K-space, then every closed K-invariant subvariety of X is also a spherical K-space.*

Let us now make a brief recall of valuation theory. The reader may consult [ZS75] or the short notes of Stevensson [Ste17] for more information. Let $\mathcal{K}/\mathbb{C}$ be a finitely generated field extension (e.g. $\mathcal{K}$ is the function field of a complex variety). A complex variety X is said to be a *model* of $\mathcal{K}$ if $\mathbb{C}(X) = \mathcal{K}$.

Recall the following basic notions.

Definition 5.3. *Let ν be a valuation on $\mathcal{K}/\mathbb{C}$.*

(1) *The* valuation ring R_ν *of ν is defined as $R_\nu := \{f \in \mathcal{K}, \nu(f) \geq 0\}$. This is a local ring with maximal ideal $\mathfrak{m}_\nu = \{f \in \mathcal{K}, \nu(f) > 0\}$.*
(2) *The field $\kappa_\nu := R_\nu/\mathfrak{m}_\nu$ is said to be the* residue field *of ν.*
(3) *The abelian subgroup $\Gamma_\nu := \nu(\mathcal{K}^*) \subset \mathbb{R}$ is called the* value group *of ν.*
(4) *The* transcendence degree *of ν is $tr.deg(\nu) := tr.deg(\kappa_\nu/\mathbb{C})$.*
(5) *The* rational rank *of ν is $rt.rk(\nu) := \dim_{\mathbb{Q}}(\Gamma_\nu \otimes \mathbb{Q})$.*

Theorem 5.4 (Zariski-Abhyankar)**.** *If ν is a valuation on $\mathcal{K}/\mathbb{C}$, then*

$$tr.deg(\nu) + rt.rk(\nu) \leq tr.deg(\mathcal{K}/\mathbb{C}).$$

Definition 5.5. *A valuation ν on $\mathcal{K}/\mathbb{C}$ is said to be* Abhyankar *if $tr.deg(\nu) + rt.rk(\nu) = tr.deg(\mathcal{K}/\mathbb{C})$.*

Definition 5.6. *Let X be a model of $\mathcal{K}/\mathbb{C}$. If there is a (generally non-closed) point $x \in X$ and a local inclusion $\mathcal{O}_{X,x} \subset R_\nu$ of local rings, then the valuation ν is said to be* centered *on X, and x is called the* center *of ν on X, denoted by $c_X(\nu)$.*

By the valuative criterion for separatedness, if the center of ν on a model exists then it is unique, and the valuative criterion of properness guarantees the existence of a center on a proper model. We often identify the center $c_X(\nu)$ of a valuation with its closure $\overline{c_X(\nu)}$ inside of the model X on which the center exists.

Definition 5.7. *A valuation ν on $\mathcal{K}/\mathbb{C}$ is said to be* quasimonomial *if there exist*

(1) *a smooth model X of $\mathcal{K}/\mathbb{C}$,*
(2) *a (generally non-closed) point $x \in X$,*
(3) *a regular system of parameters $y = (y_1, \ldots, y_d)$ of the local ring $\mathcal{O}_{X,x}$ at x, such that $\nu_1, \ldots, \nu_d$ generate $\nu(\mathcal{K}^*) \cup \{0\} = \Gamma_\nu$ as an abelian group.*

One can in fact take x to be the *center* of the valuation ν on some proper model.

Theorem 5.8. *[ELS03, Proposition 2.8] The valuation ν is quasimonomial if and only if it is Abhyankar, i.e.*

$$tr.deg(\nu) + rt.rk(\nu) = tr.deg(\mathcal{K}/\mathbb{C}).$$

Proposition 5.9. *The valuation ν_ω induced by the a $\partial\overline{\partial}$-exact complete Calabi-Yau metric ω on a quasiprojective manifold M is quasimonomial.*

If M admits a G-spherical action, then ν_ω is moreover G-invariant and identifies with $-\nu_\xi$ in the Cartan algebra of M.

Proof. By assumption $\dim R(M) = \dim R(W)$, hence the quasimonomiality of ν_ω follows from a theorem due to Olivier Piltant (cf. [Tei03, Proposition 3.1] for an accessible reference).

Next, remark that ν_ω is K-invariant. Indeed, since the metric ω is K-invariant, every $k \in K$ defines an isometry between $B(p,r)$ and $B(kp,r)$ for any base point $p \in M$, hence for any meromorphic function f on M,

$$d_\omega(k.f) = \lim_{r\to+\infty} \frac{\sup_{B(p,r)} \log\left|f(k^{-1})\right|}{\log r} = \lim_{r\to+\infty} \frac{\sup_{B(kp,r)} \log|f|}{\log r},$$

which is exactly $d_\omega(f)$ as the growth rate at infinity does not depend on the given fixed point. It follows that ν_ω is a K-invariant valuation.

Let us now show that ν_ω is G-invariant. The arguments again use K-spherical space theory. Let Z be the center of ν_ω in a G-equivariant smooth projective compactification $\overline{M}$. In particular, $\overline{M}$ is a spherical K-space by Equivalence Theorem 5.1. Since ν_ω is K-invariant, Z is also a K-invariant closed subvariety of $\overline{M}$, hence a K-spherical space by Restriction Lemma 5.2, which is also G-spherical again by Equivalence Theorem.

Let ν' be any quasimonomial valuation with center Z. The latter means that there is a G-equivariant proper birational modification $Y \to \overline{M}$ with normal crossing divisors $E_1, \dots, E_m$ such that $\cap_{i=1}^{r\leq m} E_i$ contains the generic point o_Z of Z and ν' is a monomial valuation on Y with center Z.

Let $y_1, \dots, y_r \in \mathcal{O}_{Y,o_Z}$ be a system of local parameters such that $E_i = \{y_i = 0\}, 1 \leq i \leq r$ (by a well-known fact, such y_j can always be chosen since $E_1, \dots, E_m$ intersect transversally). By definition, there is a r-uple $(\alpha_1, \dots, \alpha_r) \in (\mathbb{R}^+_{\geq 0})^r$ satisfying $\nu' = \sum_{i=1}^r \alpha_i \mathrm{ord}_{E_i}$. Since E_i is G-invariant, ord_{E_i} is also G-invariant, hence ν' is G-invariant. Thus every quasimonomial valuation with center Z is G-invariant, hence ν_ω is G-invariant.

The fact that the valuation ν_ω corresponds to the valuation induced by the Reeb vector ξ of the K-stable cone (C,ξ) can be seen as follows. Since the K-semistable Reeb vector of W is the same as the K-stable Reeb vector of C, it is enough to show that $d_\omega = -\nu_\omega$ corresponds to the K-stable valuation ν_ξ of (W,ξ) induced by ξ.

Let $G/H \subset M$ and G/H_0 be the open G-orbits in M and W. Since $R(M)$ and $R(W)$ are isomorphic as G-modules by construction, their weight lattices are the same, i.e. $\mathcal{M}(G/H) = \mathcal{M}(G/H_0) =: \mathcal{M}$. Let $f_\infty \in I_{k+1}^{(\alpha)}/I_k^{(\alpha)} = R(W)^{(\alpha)}$ be any nonzero element and $f \in I_{k+1}^{(\alpha)}$ a lift. Since d_ω induces ν_ξ, we have

$$d_\omega(f) = \nu_\xi(f_\infty).$$

The equality is moreover independent of the choice of f. Finally, from Remark 2.8 it follows that

$$-\langle \alpha, \nu_\omega \rangle = d_\omega(f) = \nu_\xi(f_\infty) = \langle \alpha_H, \nu_\xi \rangle = \langle \alpha, \nu_\xi \rangle.$$

This terminates our proof. ☐

Proposition 5.10. *The semistable cone W in the two-steps degeneration is a G-spherical cone. In particular, the asymptotic cone of the K-invariant Calabi-Yau metric (M,ω) is a K-stable G-spherical affine cone (C,ξ), which is unique up to a G-equivariant isomorphism preserving ξ.*

Proof. Since M is a G-spherical manifold and that ν_ω is a G-invariant valuation, it is immediate that W is a G-spherical variety. Finally, by Prop. 3.3, there is a unique G-equivariant degeneration of (W,ξ) to (C,ξ), hence C is G-spherical. ☐

Remark 5.11. *It may be worth mentioning that to prove the uniqueness of the asymptotic cone, one can alternatively use the construction of the G-equivariant Hilbert scheme in [AB04] and then readopt the strategy of [DS17]. We explain briefly the main steps.*

(1) *First, since W_i and C have the same positive Hilbert function, the action of the torus T on W induces a T-action on C, and by [AB04] there is a projective $G \times T$-invariant Hilbert scheme $\mathbf{H}$ parametrizing polarized affine varieties in $\mathbb{C}^N$ such that for i large enough, W_i and C define points $[W_i]$ and $[C]$ in $\mathbf{H}$. After extracting a subsequence, one can show that $[W_i]$ converges to $[C]$ up to a K_ξ action.*
(2) *There is a small enough neighborhood $\mathcal{U}$ of C in $\mathcal{C}_\infty$ such that any $C' \in \mathcal{U}$ defines an element in $\mathbf{H}$. The argument uses compactness of $\mathbf{H}$.*
(3) *The stabilizer of $[C]$ in $\mathbf{H}$ is in fact $Aut(C)$, which is reductive by a Matsushima theorem for cones, i.e. there is a maximal compact subgroup such that $Aut(C) = K^{\mathbb{C}}$.*
(4) *We can apply the equivariant slice theorem for $([C], K^{\mathbb{C}})$, and show that $[C]$ and $[C']$ are in the same G_ξ orbit, hence isomorphic as Ricci-flat Kähler cones. We conclude by connectedness of $\mathcal{C}_\infty$.*

Remark 5.12. *A K-invariant good Calabi-Yau metric on any affine G-manifold induces in fact a G-invariant valuation. The arguments can run as follows. Let $G_\nu \subset G$ be the subgroup stabilizing the induced valuation ν. Then using the definition of ν, one can show that G_ν is in fact closed in G and contains K, hence coincides with G as a whole.*

Finally, using the Alexeev-Brion Hilbert scheme, one can build a G-equivariant degeneration of the K-semistable G-cone W to the K-stable G-cone C and show that it is unique.

6. Examples

6.1. **Smooth affine spherical varieties.** As mentionned in the introduction, any smooth affine G-spherical variety M is isomorphic to $G \times^H V$ where H is a reductive subgroup of G such that G/H is (affine) spherical and V is a H-module.

Our examples will deal with two extreme cases. The first is the case $V = 0$, i.e. M is homogeneous, the second is when $H = G$, or M is a spherical G-module. For simplicity, we only consider varieties of rank two. The description of K-stable valuations is as follows.

Proposition 6.1. *Let (M, ω) be a complete K-invariant Calabi-Yau smooth affine G-spherical manifold. Then the valuation ν_ω induced by ω corresponds to either*

- *the quasi-regular K-semistable Reeb vector of a non-horospherical asymptotic cones if $\nu_\omega \in \partial\mathcal{V}$;*
- *the K-stable Reeb vector of the unique horospherical asymptotic cone of M if $\nu_\omega \in int(\mathcal{V})$.*

Proof. By spherical theory and previous discussions, if $\nu_\omega \in \mathrm{int}(\mathcal{V})$, then there is a test configuration defined by ν_ω that degenerates M to a K-semistable horospherical cone (W, ν_ω), hence K-stable. By uniqueness of G-equivariant K-stable degeneration, W and C are G-equivariantly isomorphic.

If $\nu_\omega \in \partial\mathcal{V}$, then the cone (W, ν_ω) is K-semistable, and necessarily quasi-regular since its Reeb cone is a half-line. □

6.2. **K-stable valuations on indecomposable spherical spaces.** The following lemma allows us to simplify the problem of classifying K-stable valuations on affine homogeneous spaces by supposing that the open orbit is indecomposable.

Lemma 6.2. *Let (M, ω) be the affine spherical homogeneous space $G_1/H_1 \times \cdots \times G_k/H_k$, endowed with complete $K_1 \times \cdots \times K_k$-invariant $\partial\overline{\partial}$-exact Calabi-Yau metric ω, such that each factor G_i/H_i is affine indecomposable and admits a complete K_i-invariant $\partial\overline{\partial}$-exact Calabi-Yau metric ω_i.*

The K-stable valuation ν_ω induced by ω is then a product of K-stable valuations ν_{ω_i} on the factors. In particular, the asymptotic cone of (M, ω) is the product asymptotic cone.

Proof. Let Γ be the weight monoid of M and C the asymptotic cone. Since C is a G-equivariant degeneration of M, it has the same weight monoid as M, hence the Reeb cone of C is the interior of $(\mathbb{R}_{\geq}\Gamma)^\vee$. But Γ is the product of the $\Gamma_i s$, hence the Reeb cone of C is the product of the Reeb cones of all factors' asymptotic cones.

The Duistermaat-Heckman volume functional vol_{DH} then writes as the product of the volume functionals on each factor, and $-\nu_\omega$ can be identified with the unique minimizer, which is clearly the product of the $-\nu_{\omega_i}$. □

Proposition 6.3. *[BD19, Table 2] [Ngh24, Theorem 4.2] Let $\mathcal{W}$ be the restricted Weyl chamber of a rank two symmetric space, and $\widetilde{\alpha}_{1,2}$ the primitive generators.*

- *The unique K-stable valuation on decomposable symmetric spaces of rank two is the product of K-stable valuations on each rank one factor.*
- *On indecomposable symmetric spaces of rank two, there are 3 K-stable valuations on symmetric spaces of restricted root system $A_2, BC_2/B_2$ which correspond to some rational multiple of $\widetilde{\alpha}_{1,2}$ and the unique K-stable horospherical valuation.*

  The unique K-stable valuation on symmetric spaces of restricted root system G_2 is the valuation corresponding to a unique generator of the Weyl chamber.

Proof. The construction and K-stability of horosymmetric cones was already done in [BD19] (see also part 3.3.1 for translation in the cone language). For the reader's convenience, we recall here the construction of the horospherical G_2-asymptotic cones and the computation of the K-stable Reeb vector in [Ngh23] [Ngh24].

Construction of the asymptotic cone.

Let $\widehat{S}$ be the set of simple roots with respect to a choice of a Borel. The involution θ on the symmetric space induces an involution $\widehat{\theta}$ on $\widehat{S}$. Without loss of generality, we work on symmetric spaces G/G^θ, so that $\mathcal{M}$ is the lattices generated by the restricted fundamental weights.

Let α_1, α_2 be the short and long restricted roots and $\widehat{\alpha}_1, \widehat{\alpha}_2$ be the lifts on $\widehat{S}$ of α_1, α_2 in the same connected component of the Dynkin diagram.

Let $I := \widehat{S} \backslash \left\{ \widehat{\alpha}_1, \widehat{\theta}(\widehat{\alpha}_1), \widehat{\alpha}_2, \widehat{\theta}(\widehat{\alpha}_2) \right\}$. The open $(G_2 \times \mathbb{C}^*)$-orbit $(G_2/H_0) \times \mathbb{C}^*$ of the cone is uniquely determined by $\mathcal{M}_I = \mathcal{M}$ (=weight lattice of the symmetric space) and I (cf. Prop. 2.11 and Remark 2.19). Moreover, G/H_0 is a fibration over G/P_I where $P_I = P(\varpi_{\widehat{\alpha}_1}) \cap P(\varpi_{\widehat{\theta}(\widehat{\alpha}_1)}) \cap P(\varpi_{\widehat{\alpha}_2}) \cap P(\varpi_{\widehat{\theta}(\widehat{\alpha}_2)})$.

The colors $\mathcal{D}$ of $G_2/H_0 \times \mathbb{C}^*$ are in bijection with $\widehat{S} \backslash I$, and two colors of two roots in the same cycle of $\widehat{\theta}$ have the same image in $\mathcal{M}_I$. Let $\widehat{\alpha}_i^\vee, \alpha_i^\vee$ be the coroots and restricted coroots, $i = 1, 2$.

When $m = 1$ (e.g. G_2/SO_4), since there is no simple root of G_2 fixed by θ (i.e. all nodes in the Satake diagram are white), we have $\theta(\widehat{\alpha}) = -\widehat{\alpha}$, so $\widehat{\alpha}_i^\vee|_{\mathcal{M}} = 2\alpha_i^\vee$.

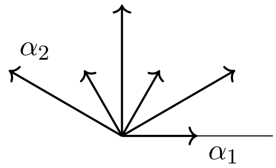


FIGURE 3. Restricted root system of G_2 symmetric spaces.

When $m = 2$ (for example $G_2 \times G_2/G_2$), $\theta(\widehat{\alpha}_i) = -\widehat{\theta}(\widehat{\alpha}_i)$, hence $\theta(\widehat{\alpha}_i)(\widehat{\alpha}_i) = 0$, so $\widehat{\alpha}_i^\vee|_{\mathcal{M}} = \alpha_i^\vee$.

It follows that

$$\rho(\mathcal{D}) = \left\{\widehat{\alpha}_1^\vee|_{\mathcal{M}}, \widehat{\alpha}_2^\vee|_{\mathcal{M}}\right\} = \begin{cases} \{2\alpha_1^\vee, 2\alpha_2^\vee\}, m = 1 \\ \{\alpha_1^\vee, \alpha_2^\vee\}, m = 2. \end{cases}$$

In both cases, the colored cone of C is $(\mathbb{R}_{\geq 0}\rho(\mathcal{D}), \mathcal{D})$.

Reeb vector computation.

Recall that $\kappa(\alpha_1, \alpha_1) = 1$, $\kappa(\alpha_2, \alpha_2) = 3$, and both roots have the same multiplicity $m \in \{1, 2\}$. The positive roots of G_2 are

$$\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_2 + 2\alpha_1, 2\alpha_2 + 3\alpha_1, \alpha_2 + 3\alpha_1$$

The half sum of the positive restricted roots (in the Cartan space) is just $\varpi = 10m\alpha_1 + 6m\alpha_2$. Recall the setup in [Ngh23] to compute the Reeb vector ξ. Set $\delta = \alpha_2 - t\alpha_1, t \in \mathbb{R}$ to be the vector orthogonal to ξ under κ. Identify the valuation cone $\mathcal{V}$ of the symmetric space with the negative restricted Weyl chamber and the Reeb cone with the positive restricted Weyl chamber $-\mathcal{V}$.

Let ν_ω be the valuation induced by the K-invariant Calabi-Yau metric on a G_2-symmetric space, then $\nu_\omega \in \mathcal{V}$. By our main Theorem C, this is only possible if $\xi \in -\mathcal{V}$, i.e. iff $t > 0$.

The moment polytope Δ_ξ can be identified with

$$\Delta_\xi := \{\varpi + p\delta, \lambda_- \leq p \leq \lambda_+\}, \quad \lambda_- = -\frac{2m}{t+2}, \quad \lambda_+ = \frac{2m}{2t+3}.$$

Moreover, the Duistermaat-Heckman polynomial restricted to the Cartan space can be written as

$$\begin{aligned} P(p) =& (2m - (2t+3)p)^m (6m + (3t+6)p)^m (8m + (t+3)p)^m \\ & (10m - tp)^m (12m - (3t+3)p)^m (18 + 3p)^m. \end{aligned}$$

Then the Reeb vector is a K-stable polarization iff t is a solution of

$$\int_{\lambda_-}^{\lambda_+} pP(p)dp = 0.$$

For $m = 1$ and $m = 2$, $\xi \in -\mathcal{V}$ iff t is the *positive* solution of the following respective polynomial equations

$$2376 + 9225t + 13407t^2 + 9357t^3 + 3179t^4 + 424t^5 = 0,$$

and

$$\begin{aligned} &20558772 + 134444448t + 374274594t^2 + 590688162t^3 + 587394519t^4 + \\ &383740299t^5 + 165293858t^6 + 45384306t^7 + 7221048t^8 + 507988t^9 = 0. \end{aligned}$$

Since all the coefficients are positive, there can be no positive solution. $\square$

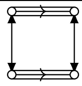
Type	Representative	R	Multiplicities	Satake diagram	Hermitian
G	G_2/SO_4	G_2	1		no
G_2	$G_2 \times G_2/G_2$	–	2		no

TABLE 1. Indecomposable symmetric spaces of restricted root system G_2. The involution $\widehat{\theta}$ relate two roots connected by the two-sided arrows in the Satake diagram.

As mentionned in the introduction, one can then wonder if there is a Calabi-Yau smoothing of the horospherical G_2-asymptotic cone, which would be obtained as the generic fiber of a G_2-equivariant deformation of the cone. If this is the case, one can further ask whether a geometric transition phenomenon can occur, that is to prove a crepant resolution of the cone is also Calabi-Yau. The metric would then form a mirror pair with the hypothetical Calabi-Yau smoothing of the cone. This happens for the conifold $\left\{(X, Y, Z, W) \in \mathbb{C}^4, XZ - YW = 0\right\}$ [Gro01] which is the unique Gorenstein toric cone of dimension 3 with an isolated *terminal* singularity.

In our case, even if we don't know whether a Calabi-Yau smoothing exists, we can at least affirmatively answer that there can be no G_2-equivariant geometric transition.

Lemma 6.4. *There is no equivariant crepant resolution of the horospherical asymptotic cone of G_2-symmetric spaces.*

Proof. We use the same notation as in Proposition 6.3. From [Bri97a] and [GH15b, Remark 4.3], the anticanonical line bundle of C can be represented as

$$-K_C = \sum_{\alpha \in \widehat{S} \setminus I} a_\alpha \overline{D}_\alpha, \quad a_\alpha = \left\langle \varpi, \alpha^\vee \right\rangle$$

Suppose that $\pi : X \to C$ is a crepant resolution, then there is a G_2-equivariant divisor $D \subset X$ (corresponding to the primitive vector d in $\mathcal{M}$) such that

$$-K_X = \sum_{\alpha \in \widehat{S} \setminus I} a_\alpha \overline{D}_\alpha + D = \pi^*(-K_{C_0}) = \sum_{\alpha \in \widehat{S} \setminus I} a_\alpha \overline{D}_\alpha + \frac{2\kappa(\varpi, d)}{\kappa(d, d)} D,$$

hence $2\kappa(\varpi, d) = \kappa(d, d)$. Let $d = x\alpha_1 + y\alpha_2$, with x, y being positive rationals. Then $2\kappa(\varpi, d) = \kappa(d, d)$ iff

$$2m(x - 3y) = x(x - 3y) + 3y^2 \iff x^2 - x(2m + 3y) + 6my + 3y^2 = 0.$$

It is easy to check by computing the discriminant that for every positive rational y, the equation in x does not have any solution. □

6.3. **K-stable valuations on spherical modules.** Let (ρ, V) be a regular representation of a connected linear reductive group G with the induced representation $(\widehat{\rho}, \mathbb{C}[V])$. Then (ρ, V) is said to be *multiplicity-free* if the decomposition of $\mathbb{C}[V]$ into simple G-modules contains at most one copy of each simple G-module. A representation (ρ, V) is multiplicity-free iff V is a (smooth affine) G-spherical variety.

The irreducible multiplicity-free representations were classified by Kac [Kac80] (see also [BR96, Theorem 1] [Lea98, Theorem 1.4]).

Theorem 6.5. *[Kac80] The list of multiplicity-free irreducible linear actions of connected reductive linear groups G is*

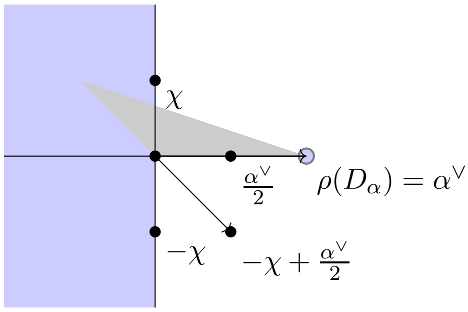


FIGURE 4. Colored cone $(\mathcal{C}, \mathcal{D})$ of the symmetric manifold $\mathbb{C}^3$ with open orbit $\mathrm{SO}_3 / \mathrm{SO}_2 \times \mathbb{C}^*$. The K-stable valuations correspond to the vectors of coordinates $(1,-1)$ and $(0,-1)$ in the lattice generated by $(\alpha^\vee/2, \chi)$.

1) SL_n, Sp_{2n}, $\Lambda^2 \mathrm{SL}_n$ *(n odd)*, $\mathrm{SL}_m \otimes \mathrm{SL}_n$ $(n \neq m \geq 2)$, $\mathrm{SL}_n \otimes \mathrm{Sp}_4$ $(n > 4)$, *Spin*$_{10}$ *when G is semisimple.*
2) $G \otimes \mathbb{C}^*$ *with G being*

$$\mathrm{SL}_n,\ \mathrm{Sp}_{2n}(n \geq 2),\ \mathrm{SO}_n(n \geq 3),\ \mathit{Spin}_7,\ \mathit{Spin}_9,\ \mathit{Spin}_{10},\ G_2,\ E_6,$$

and

$$S^2 \mathrm{SL}_n(n \geq 2),\ \Lambda^2 \mathrm{SL}_n(n \geq 4),\ \mathrm{SL}_m \otimes \mathrm{SL}_n(m, n \geq 2),$$
$$\mathrm{SL}_{2,3} \otimes \mathrm{Sp}_{2n}(n \geq 2),\ \mathrm{SL}_n \otimes \mathrm{Sp}_4(n \geq 4).$$

Here:

- *The index under each group is the dimension of the module.*
- *The representation of G corresponds to $V(\omega_1)$ where ω_1 is the first fundamental weight of G.*
- *$G \otimes G'$ (resp. S^2G, $\Lambda^2 G$) denote the action of $G \times G'$ on the tensor product $V(\omega_1) \otimes V(\omega_1')$ (resp. of G on $S^2V(\omega_1)$, $\Lambda^2 V(\omega_1)$).*

The result is extended to the reducible case independently by Benson-Ratcliff [BR96] and A. Leahy [Lea98]. This is done via classification of *indecomposable* spherical modules, namely G-representations (ρ, V) that are not equivalent to $(\rho_1, V_1) \oplus \rho_2, V_2)$, where (ρ_i, V_i) are multiplicity-free representations of G_i with $G = G_1 \times G_2$.

Lemma 6.6. *The only non-horospherical multiplicity-free G-action on a module V with underlying vector space $\mathbb{C}^3$ is given by $G = \mathrm{SO}_3 \otimes \mathbb{C}^*$, where SO_3 acts on $\mathbb{C}^3$ in the standard way.*

Proof. The classification in [BR96, Theorem 2], [Lea98, Theorem 2.5] shows that any indecomposable module must either have one factor (hence belongs to Kac's classification in Theorem 6.5), or two factors V_i each of dimension at least 2. It follows that any spherical module V with underlying vector space $\mathbb{C}^3$ is indecomposable with only one factor.

From the list in Theorem 6.5, the possible multiplicity-free representations (ρ, V) with underlying vector space $\mathbb{C}^3$ are

$$(\mathrm{SL}_3, V(\lambda)), (\mathrm{SL}_3 \otimes \mathbb{C}^*, V(\lambda)), (\mathrm{SO}_3 \otimes \mathbb{C}^*, V(2\omega)), (S^2 \mathrm{SL}_2 \otimes \mathbb{C}^*, S^2 V(\omega)),$$

where λ, ω are the fundamental weights of SL_3, SL_2. The first two are horospherical (cf. Prop. 2.12), while the last two are isomorphic via

$$(S^2 \mathrm{SL}_2, S^2 V(\omega)) \simeq (\mathrm{PSL}_2, V(2\omega) \simeq (\mathrm{SO}_3, V(2\omega)),$$

since $Z(\mathrm{SL}_2) = \{\pm 1\}$ fixes $S^2V(\omega) \simeq V(2\omega)$ and $\mathrm{PSL}_2 \simeq \mathrm{SO}_3$. □

Proposition 6.7. *The K-stable valuations of* $\mathrm{SO}_3(\mathbb{R}) \times \mathbb{S}^1$*-invariant Calabi-Yau metrics on* $\mathbb{C}^3$ *are*

- *the trivial valuation on the linear part of* $\mathcal{V}$*,*
- *the product of the K-stable valuations on the factors* $\mathrm{SO}_3 / \mathrm{SO}_2 \times \mathbb{C}^*$.

The former induces a trivial equivariant degeneration, while the latter lies in the interior of $\mathcal{V}$ *and induces a degeneration of* $\mathbb{C}^3$ *to the horospherical cone* $A_1 \times \mathbb{C}^*$ *where* A_1 *is the Stenzel asymptotic cone of* $\mathrm{SO}_3 / \mathrm{SO}_2$ *(cf. Example 2.13).*

Proof. Since asymptotic cones are central fibers of equivariant degenerations, one can identify the weight lattice of the cone with the open orbit $\mathrm{SO}_3 / \mathrm{SO}_3 \times \mathbb{C}^*$ of $\mathbb{C}^3$ (cf. Remark 2.19), which is generated by $\{\alpha^\vee/2, \chi\}$ where α is the positive (restricted) root of SO_3 and χ the weight of the $\mathbb{C}^*$-action on $\mathbb{C}^3$ (cf. Figure 4). The valuation cone is then

$$\mathcal{V} = \mathbb{R}_{\leq}\alpha^\vee \times \mathbb{R}\chi.$$

From Proposition 6.1, the K-stable valuations of $\mathbb{C}^3$ are either in the linear part (with trivial central fiber) or uniquely in $\mathrm{Int}\mathcal{V}$ (with horospherical cone as central fiber). Since the horospherical central fiber does not depend on the choice of $\nu \in \mathrm{Int}\mathcal{V}$, it must be $\mathrm{SO}_3 \times \mathbb{C}^*$-isomorphic to the cone $A_1 \times \mathbb{C}$. Indeed, an explicit equivariant test configuration can be given by

$$f = z_1^2 + z_2^2 + z_3^2 : \mathbb{C}^4_{z_0,z_1,z_2,z_3} \to \mathbb{C},$$

with central fiber $A_1 \times \mathbb{C} = f^{-1}(0)$. Here we view $\mathbb{C}^4$ as the spherical module $\mathbb{C}^3 \times \mathbb{C}$ with an action of $(\mathrm{SO}_3 \times \mathbb{C}^*) \times \mathbb{C}^*$, where SO_3 acts in the standard way.

Let ω be Li's metric on $\mathbb{C}^3$ with corresponding K-stable valuation ν_ω, asymptotic to $A_1 \times \mathbb{C}$ (endowed with the horospherical product conical Calabi-Yau metric). From explicit computation in [Ngh24], the metric on A_1 has Reeb vector $\xi = \alpha^\vee/2$.

The K-stable valuation of the metric on $A_1 \times \mathbb{C}$ is then $\nu_\xi = (\alpha^\vee/2, \chi)$, hence ν_ω corresponds to the vector $(-\alpha^\vee/2, -\chi)$ by Theorem C. □

If we consider any spherical module V with open orbit of the form $R_1 \times \mathbb{C}^*$ where $R_1 = G/H$ is any rank one symmetric space, then reasoning as above and using Székelyhidi's uniqueness theorem, one can show that the only Calabi-Yau metrics with the $G \times \mathbb{C}^*$-symmetry on V are the standard Calabi-Yau metric and the Li-Conlon-Rochon-Székelyhidi metrics.

In general, there may exist more of non-horospherical multiplicity-free symmetries of linear reductive groups on V, and one can get a complete list of such actions using [BR96] [Lea98]. However, to get a full classification of metrics with corresponding symmetry, the difficulty lies in proving a uniqueness theorem with asymptotic cones *not* of the type $C \times \mathbb{C}$ with C having an isolated singularity.

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Tran-Trung Nghiem, IMAG, Univ Montpellier, CNRS, Montpellier, France
Email address: tran-trung.nghiem@umontpellier.fr