LIMIT POINTS OF (SIGNLESS) LAPLACIAN SPECTRAL RADII OF LINEAR TREES

FRANCESCO BELARDO, ELISMAR R. OLIVEIRA, AND VILMAR TREVISAN

Abstract. We study limit points of the spectral radii of Laplacian matrices of graphs. We adapted the method used by J. B. Shearer in 1989, devised to prove the density of adjacency limit points of caterpillars, to Laplacian limit points. We show that this fails, in the sense that there is an interval for which the method produces no limit points. Then we generalize the method to Laplacian limit points of linear trees and prove that it generates a larger set of limit points. The results of this manuscript may provide important tools for proving the density of Laplacian limit points in $[4.38+, \infty)$.

1. Introduction

The 1972 seminal paper of A. J. Hoffman [4] introduced the concept of limit points of eigenvalues of graphs. Let $\mathcal{A}$ be the set of all symmetric matrices of all orders, every entry of which is a non-negative integer and $R = \{\rho : \rho = \rho(A) \text{ for some } A \in \mathcal{A}\}$ where $\rho(A)$ is the largest eigenvalue of A. He asked which real number can be in R and showed that it is sufficient to consider matrices of $\mathcal{A}$ having only entries in $\{0, 1\}$ and 0 diagonal, e.g. adjacency matrices of graphs. Additionally, he determined all limit points of $R \leq \sqrt{2+\sqrt{5}}$. More precisely, let $\tau = \frac{1+\sqrt{5}}{2}$ (the golden mean). For $n = 1, 2, \ldots$, let $\overline{\beta}_n$ be the positive root of

$$Q_n(x) = x^{n+1} - \left(1 + x + x^2 + \cdots + x^{n-1}\right).$$

Let $\bar{\alpha}_n = \bar{\beta}_n^{1/2} + \bar{\beta}_n^{-1/2}$. Then

$$2 = \bar{\alpha}_1 < \bar{\alpha}_2 < \cdots$$

are all the limit points of R smaller than $\lim_{n\to\infty} \bar{\alpha}_n = \tau^{1/2} + \tau^{-1/2} = \sqrt{2+\sqrt{5}}(= 2.05+)$.

In 1989, a remarkable result due to J. B. Shearer [13] extended the work of Hoffman. He showed that every real number larger than $\sqrt{2+\sqrt{5}}$ is a limit point of R.

In his original paper, Hoffman asks about limit points of other eigenvalues of graphs, and this originated several interesting results on this topic culminating with the fact that every real number is a limit point of some eigenvalue of a graph (see Zhang & Chen [16]). We remark that Estes [2] proved in 1992 that every totally real algebraic integer – which are roots of monic integral polynomials having only real roots – is an eigenvalue of a graph. Since eigenvalues are totally real algebraic integers (roots of the characteristic polynomial), it follows that this set is dense in the real line. What is perhaps surprising, is that only eigenvalues of trees are needed to be considered for these results. Salez [12], extending Estes' result, showed that any totally real algebraic integer is an eigenvalue of a tree, while both Hoffman and Shearer found sequences of trees whose spectral radii were limit points. It seems remarkable that the set composed only by the largest eigenvalue of the adjacency matrix of trees is dense in the interval $[\sqrt{2+\sqrt{5}}, \infty)$.

It is worth noticing that in 2003, Kirkland [8] showed that any nonnegative real number is a limit point for the algebraic connectivity (the second smallest Laplacian eigenvalue). The same author [9] considered limit points for the positive eigenvalues of the normalized Laplacian matrix of a graph. Specifically, he shows that the set of limit points for the j-th smallest such eigenvalues is equal to [0, 1], while the set of limit points for the j-th largest such eigenvalues is equal to [1, 2]. Wang et al. [14] proved that any nonnegative real number is a limit point for some eigenvalue of the signless Laplacian matrix of graphs.

In this paper, we are interested in the problem originated by Hoffman's question, which deals only with limit points of the spectral radius of graphs. More specifically, we want to study the Laplacian

2020 *Mathematics Subject Classification.* 05C50, 05C05, 15A18.
Key words and phrases. spectral radius; tree; limit points;

version of Hoffman and Shearer's results, that is, what real numbers are limit points of the spectral radii of Laplacian matrices of graphs. The converse of this problem may also be viewed as *which sequences of graphs have Laplacian spectral radius with limit points* (we refer to the next section for the precise definitions).

In order to explain our results, we recall the work of Guo [3]. Let

$$\omega = \frac{1}{3}(\sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}} + 1),$$

$\beta_0 = 1$ and $\beta_n,\ n \geqslant 1$ be the largest positive root of

$$P_n(x) = x^{n+1} - \left(1 + x + \cdots + x^{n-1}\right)(\sqrt{x}+1)^2.$$

Let $\alpha_n = 2 + \beta_n^{\frac{1}{2}} + \beta_n^{-\frac{1}{2}}$. Then

$$4 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots$$

are all the limit points of Laplacian spectral radii of graphs smaller than $\lim_{n\to\infty} \alpha_n = 2 + \omega + \omega^{-1}$ $(= 4.38+)$.

By analogy to the adjacency case, it is natural to ask whether any real number $\mu \geq 2 + w + w^{-1} = 4.38+$ is the limit point of the Laplacian spectral radii of graphs. We refer to Figure 1 for an illustration of the current state of knowledge.

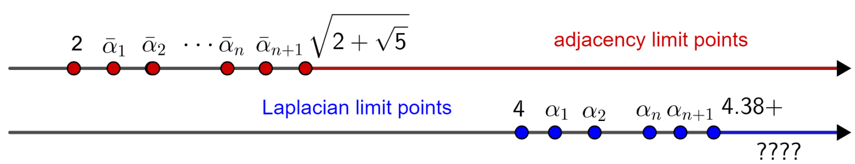

FIGURE 1. Current status of (Laplacian) limit points

Our contribution in this paper is the development of some analytical tools allowing one to study the density of Laplacian spectral radius in $[4.38+, \infty)$. We first adapt Shearer's method to the Laplacian case, verifying that it is not sufficient to prove density. In spite of the fact that the method produces sequence of caterpillars having a limit point, we show that this limit point is not always the *desired* number. As a consequence, we find a whole interval where the method produces no Laplacian limit points. We then extend Shearer's method to the class of linear trees, e.g. we define sequences of linear trees whose Laplacian spectral radius has limit points. This provides a generalization of Shearer's process, since caterpillars is a subclass of linear trees.

It is tempting to conjecture that any $\mu \geq 4.38+$ is a limit point of Laplacian spectral radii of graphs, however, it seems that Laplacian spectral radii are a bit slick! Our generalization improves Shearer's process in the sense that we are able to find a larger set of limit points. We do not know whether sequences of linear trees suffice to prove the density conjecture (if true).

The rest of the paper is organized as follows. In the next section, we present the necessary notation, definitions and the main tools used in the paper. In particular, we explain Shearer's procedure using our tools. In Section 3, we apply Shearer's approach to Laplacian case, verifying that it fails. In Section 4 we study the convergence of the Laplacian spectral radius of sequences of linear trees. In Section 5 we use these results to generalize Shearer's work: we obtain a method that produces a sequence of linear trees for a given number μ in a way that the Laplacian spectral radius has a limit point $\leq \mu$. We call this *generalized Shearer's sequence* and show that the process provide a larger set of limit points. In Section 6 we study the Laplacian limit points given by the method. We present some numerical evidence of the density of these limit points. We also prove a density result, showing that a truncated sequence of linear trees have algebraic limit points that is dense in a subset of the limit points. In Section 7, we apply some analytical tools to obtain approximations, providing quicker ways to obtain sequence of linear trees having a specified limit point. Finally, in Section 8, we suggest a few open problems and make final considerations.

Remark 1.1. *The results of this paper are stated for the Laplacian matrix, however, as all the graphs involved are trees, they naturally extend to signless Laplacian matrix, since the spectrum of these two matrices are equal for bipartite graphs.*

2. Notation and preliminaries

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. If $|V| = n$, then its adjacency matrix $A(G) = [a_{ij}]$ is the $n \times n$ matrix of zeros and ones such that $a_{ij} = 1$ if and only if v_i is adjacent to v_j (that is, there is an edge between v_i and v_j). A value λ is an eigenvalue if $\det(\lambda I_n - A) = 0$, and since A is real symmetric its eigenvalues are real. In this setting, we denote by $\rho_A(G)$ the largest eigenvalue of $A(G)$, which is called the spectral radius of G. One can also consider the Laplacian matrix $L(G)$ of a graph, which is given by $L(G) := D(G) - A(G)$, where $D(G) = [d_{ij}]$ is the $n \times n$ diagonal matrix with $d_{ii} = \text{degree}(v_i)$. A value μ is an eigenvalue if $\det(\mu I_n - L) = 0$, and since L is positive semi-definite, its eigenvalues are non-negative. We denote by $\rho_L(G)$ the largest eigenvalue of $L(G)$ which is called the Laplacian spectral radius of G. One can also consider the signless Laplacian associated to a graph, which is denoted $Q(G) := D(G) + A(G)$.

Generalizing the concept of limit point of graphs, we say that a real number γ is an M-limit point of the M-spectral radius of graphs if there exists a sequence of graphs $\{G_k \mid k \in \mathbb{N}\}$ such that

$$\lim_{k\to\infty} \rho_M(G_k) = \gamma,$$

where $\rho_M(G_i) \neq \rho_M(G_j),\ i \neq j$ and M is a class of matrices associated with a graph such as adjacency, Laplacian, signless Laplacian, etc. (See Wang & Brunetti [15]).

In this note we will extend Shearer's ideas for the adjacency matrix to the Laplacian version. The background of this process can be recovered by the use of modern methods such as the Jacobs-Trevisan diagonalization algorithm [5], and properties of some recurrence equations from [10]. Those are our main tools and we briefly explain here, so that the note is self contained.

Given a symmetric matrix A whose underlying graph is a tree, the following algorithm of Figure 2 outputs a diagonal matrix that is congruent to $A + xI$ and hence, the sign of its diagonal values determine the number of eigenvalues that greater than/equal to/larger than x. More precisely, we state the following result for future reference.

Lemma 2.1. *([5]) Let A be a symmetric matrix whose underlying graph is a tree and let D be the matrix output by* Diagonalize(A, −x). *The number of eigenvalues of A smaller than/equal to/larger than x is, respectively, the number of negative/zero/positive diagonal values of D*

Input: matrix $A = (a_{ij})$ and underlying tree T whose vertices $v_1, \ldots, v_n$
are ordered bottom-up and real x
Output: diagonal matrix $D = \texttt{diag}(d_1, \ldots, d_n)$ congruent to $A + xI$

Algorithm Diagonalize(A, x)
  initialize $d_i \longleftarrow a_{ii} + x$, for all i
  for $k = 1$ to n
    if v_k is a leaf **then** continue
    else if $d_c \neq 0$ for all children c of v_k **then**
      $d_k \longleftarrow d_k - \sum \frac{(a_{ck})^2}{d_c}$, summing over all children of v_k
    else
      select one child v_j of v_k for which $d_j = 0$
      $d_k \longleftarrow -\frac{(a_{jk})^2}{2}$
      $d_j \longleftarrow 2$
      if v_k has a parent v_ℓ, delete the edge $\{v_k, v_\ell\}$.
  end loop

Figure 2. Diagonalizing A for a symmetric matrix A with an underlying tree.

Applying the tree algorithm on a path of a tree, produces certain numerical rational sequences. They all may be seen in a unified elementary form given by

$$x_{j+1} = \varphi(x_j), j \geq 1 \tag{1}$$

where $\varphi(t) = \alpha + \frac{\gamma}{t}$, for $t \neq 0$, $\alpha, \gamma \in \mathbb{R}$ are fixed numbers ($\gamma \neq 0$) and x_1 is a given initial condition. The explicit solution is a function $j \to f(j)$ such $x_j = f(j)$ for $j \geq 1$ and has been studied in [10].

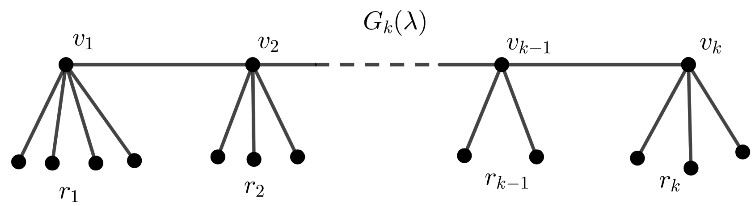

FIGURE 3. Representation of G_k

2.1. **Revisiting Shearer's work for adjacency limit points.** In [13] Shearer has proved the density of the limit points for adjacency matrices in the interval $[\sqrt{2+\sqrt{5}},\infty)$. The proof is based on an explicit construction of a sequence of graphs. For any given $\lambda \geq \sqrt{2+\sqrt{5}}$, Shearer constructs a sequence of caterpillars whose spectral radii converge to λ. We now explain, in our framework, why this method works.

Recall that a *caterpillar* is a tree in which the removal of all its leaves transforms it into a path. An arbitrary caterpillar with k back nodes $v_1,\ldots,v_k$, where v_i has r_i leaves may be represented by

$$G_k := [r_1, r_2, ..., r_k],\ k \geq 2.$$

We refer to Figure 3 for an illustration.

We notice that, if we replace each r_i by a star tree with r_i rays, we have an alternative representation for G. For instance

$$[3,1,0,2,2] = [[1,1,1],[1],[0],[1,1],[1,1]]$$

are different representations of the same tree.

For each $\lambda \in [\sqrt{2+\sqrt{5}},\infty)$ Shearer finds a caterpillar $G_k(\lambda) = [r_1, r_2, ..., r_k],\ k \geq 2$, in such a way that the number of leaves in each vertex satisfies a recurrence related to the number $\frac{-\lambda\pm\sqrt{\lambda^2-4}}{2}$.

Consider a given number $\lambda \in [\sqrt{2+\sqrt{5}},\infty)$ and an arbitrary caterpillar $G_k(\lambda) := [r_1, r_2, ..., r_k],\ k \geq 2$. We apply the algorithm Diagonalize$(A(G_k),-\lambda)$, the adjacency matrix of G_k. Each leaf is initialized with $-\lambda < 0$. In the vertex v_1 we obtain the value

$$R_1 := -\lambda - \frac{r_1}{-\lambda} = -\lambda + \frac{r_1}{\lambda},$$

and in the next vertex v_2 we obtain the value

$$R_2 := -\lambda - \frac{r_2}{-\lambda} - \frac{1}{R_1} = -\lambda - \frac{1}{R_1} + \frac{r_2}{\lambda},$$

and so on.

We observe that the sequence of equations may be seen as the recurrence relation

$$\begin{cases} x_{j+1} = \varphi(x_j) \\ x_1 = -\lambda, \end{cases} \tag{2}$$

where $\varphi(t) = -\lambda - \frac{1}{t},\ t \neq 0$, disturbed by a drift factor $\delta_j := \frac{r_j}{\lambda} \geq 0$.

From [10] we know that the iterates x_j obeying Equation (2) are asymptotic to $\theta = \frac{-\lambda-\sqrt{\lambda^2-4}}{2}$, which is a fixed point $\varphi(t) = t$, satisfies $t^2 + \lambda t + 1 = 0$. It is shown that $\theta := \frac{-\lambda-\sqrt{\lambda^2-4}}{2}$ is an attracting point and $\theta' := \frac{-\lambda+\sqrt{\lambda^2-4}}{2} = \theta^{-1} > \theta$ is a repelling point, with respect to the iteration process (2).

From Lemma 2.1, it follows that $\rho_A(G_k) < \lambda$ if and only if $R_j < 0$ for all j. As the attracting interval for θ is $(-\infty,\theta']$ we need to require that $R_j \leq \theta'$ otherwise $R_{j'} > 0$ for some $j' > j$ contradicting $\rho_A(G_k) < \lambda$.

This is the basis for the Shearer's construction (see [13] equations (1)-(4)), that is, the sequential choice of $r_1, r_2, ..., r_k$:
If $-\lambda + \frac{r_1}{\lambda} \leq \theta'$ then $R_1 < 0$, thus we choose

$$r_1 := \max_{r\geq 0}\{-\lambda + \frac{r}{\lambda} \leq \theta'\} \Leftrightarrow r_1 := \lfloor \lambda(\theta' - (-\lambda)) \rfloor.$$

If $\varphi(R_1) + \frac{r_2}{\lambda} \leq \theta'$, then $R_2 < 0$, thus we choose

$$r_2 := \max_{r \geq 0}\{\varphi(R_1) + \frac{r_2}{\lambda} \leq \theta'\} \Leftrightarrow r_2 := \lfloor \lambda(\theta' - \varphi(R_1)) \rfloor,$$

and so on.

Definition 2.2. *Given* $\lambda \geq \sqrt{2+\sqrt{5}}$ *we define the adjacency Sharer's sequence as the sequence of caterpillars* $G_k(\lambda) := [r_1, r_2, ..., r_k],\ k \geq 2$ *where*

$$\begin{cases} r_1 := \lfloor \lambda(\theta' - (-\lambda)) \rfloor; \\ R_1 = -\lambda + \frac{r_1}{\lambda}; \\ r_j := \lfloor \lambda(\theta' - \varphi(R_{j-1})) \rfloor, 2 \leq j \leq k; \\ R_j = \varphi(R_{j-1}) + \frac{r_j}{\lambda}, 2 \leq j \leq k. \end{cases}$$

The adjacency Sharer's sequence $G_k,\ k \geq 2$ is such that $\rho_A(G_k) < \rho_A(G_{k+1}) < \lambda$. Indeed, the part $\rho_A(G_k) < \lambda$ is a trivial consequence of the construction because $R_1, ..., R_k < \theta' < 0$ then the spectral radius is necessarily smaller than λ (see Theorem 3, from [5]). The part $\rho_A(G_k) < \rho_A(G_{k+1})$ is a consequence of the interlacing property because G_k is always a subgraph of G_{k+1} since the choice of $r_k := \lfloor \lambda(\theta' - \varphi(R_{k-1})) \rfloor$ is the same for both graphs.

In particular there exists

$$\lim_{k\to\infty} \rho_A(G_k) = \gamma(\lambda) \leq \lambda.$$

Shearer [13] showed that for every real number $\lambda \geq \sqrt{2+\sqrt{5}}$, the limit point $\gamma(\lambda) = \lambda$, that is $\lim\limits_{k\to\infty} \rho_A(G_k) = \lambda$.

In this note we will adapt Shearer's construction to the Laplacian matrix. For this, we set next some notation that is going to be used throughout the paper.

2.2. **Notation for Laplacian Matrices.** From Guo's result [3], the problem for limit points of spectral radii of Laplacian matrices that remains to study is the interval $\mu \geq 4.38+$. More precisely, the question is whether any real number $\mu \in [4.38+, \infty)$ is a limit point of Laplacian spectral radius of graphs.

In the next section, we extend Shearer's method of caterpillar construction for Laplacian matrices. We will see the necessity to consider other trees. Here, we set the notation for this larger set of trees.

We recall that a *linear tree* is a tree obtained by attaching a starlike tree (that has at most one vertex of degree larger than 2) at each vertex of a path. We denote a linear tree by $G = [T_1, \ldots, T_k]$, for $k \geq 1$, where all high degree vertices (HDVs) are located in one single path $P = \{v_1 \to v_2 \to \cdots \to v_k\}$, called the main path. At each vertex v_j (we call then back nodes), we attach a number of paths $P_{q_1^j}, \ldots, P_{q_{r_j}^j}$ forming a starlike tree, which is denoted by $T_j = [q_1^j, ..., q_{r_j}^j], q_i^j \leq q_{i+1}^j$, for $1 \leq j \leq k$ (see Figure 4 for an illustration).

The class of linear trees was introduced by Johnson, Li & Walker in [6] and further studied by Johnson & Saiago (see [7] Chapter 10). Some special cases of linear trees are caterpillars, where all the HDVs have a number of pendant leafs P_1, and the open quipus, where all the HDVs have degree at most three.

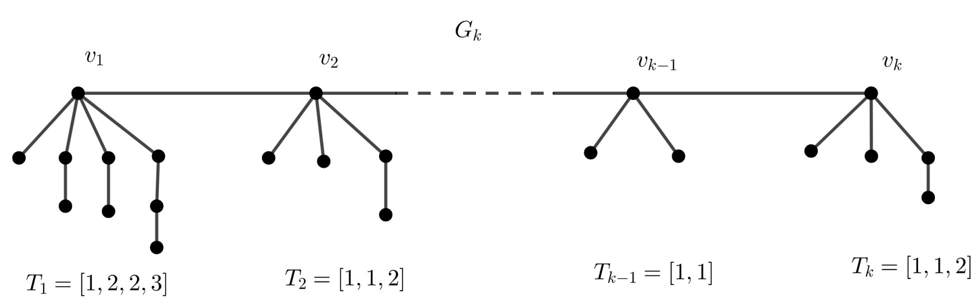

FIGURE 4. Linear tree $G_k = [[1, 2, 2, 3], [1, 1, 2], \ldots, [1, 1], [1, 1, 2]]$

We notice that the representation $T_j = [0]$ means that the graph T_j is formed just by the vertex v_j, or that T_j is empty.

We denote by $\ell(G_k) = k$, the *length* of $G = [T_1, \ldots, T_k]$, as the number of back nodes in the main path, by $\omega(T_j) = r_j$, the *width* of $T_j = [q_1^j, \ldots, q_{r_j}^j]$, as the number of paths it is composed, and by $h(T_j) = q_{r_j}^j$, the height of $T_j = [q_1^j, ..., q_{r_j}^j]$, as the maximum length of paths in it.

The representation $G_k = [T_1, ..., T_k]$ is almost never unique unless $T_1 = T_k = [0]$ and k is fixed, because we may choose another main path by picking paths from T_1 and/or T_k. For instance $[[2,2,2],[1,2],[1,1]]$ and $[[0],[0],[2,2],[1,2],[1,1]]$ are representations of the same linear tree. Although, in any new representation the length could increase at most by $h(T_1) + h(T_k)$.

We consider applying Diagonalize$(L, -\mu)$, for $\mu \geq 4.38+$ and L the Laplacian matrix of a linear tree with k back nodes $G_k = [T_1, T_2, \ldots, T_k]$.

When executing Diagonalize$(L, -\mu)$, we first apply in T_i, $i = 1, \ldots, k$, towards the vertex v_k, which is the root. We remark that T_i may be empty and in this case there is nothing to be done. If T_i is a non-empty starlike, for each path of T_i, we have that leaves receive the value $b_1 = 1 - \mu$, while the remaining values satisfy $b_j = 2 - \mu - \frac{1}{b_{j-1}}$, for $j > 1$. We refer to Figure 5 for an illustration.

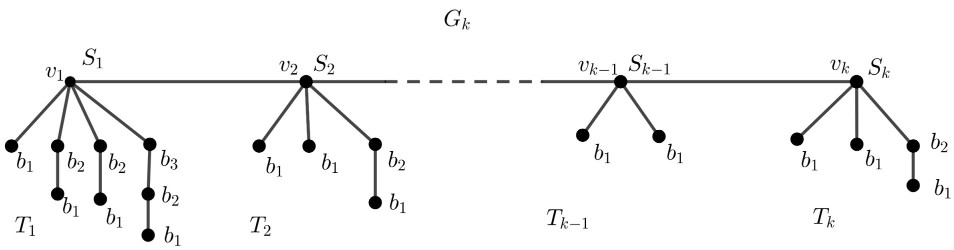

FIGURE 5. Linear tree with the values obtained by Diagonalize$(L, -\mu)$ depicted in each vertex.

It will be convenient to denote, for future reference, the function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(t) := 2 - \mu - \frac{1}{t}. \tag{3}$$

With this notation, the values in each path of T_i satisfy

$$\begin{cases} b_1 = 1 - \mu \\ b_{j+1} = \psi(b_j), \quad j > 1. \end{cases} \tag{4}$$

From [10] we know that the iterates b_j obeying Equation (4) have the following properties. Let θ and θ' be fixed points of $\psi(t) = t$. They are given by

$$\theta := \frac{-(\mu-2) - \sqrt{(\mu-2)^2 - 4}}{2} \text{ and } \theta' := \frac{-(\mu-2) + \sqrt{(\mu-2)^2 - 4}}{2} = \theta^{-1} > \theta. \tag{5}$$

They are attracting and repelling points, respectively, meaning that for $b_1 \leq \theta'$, we have b_j converging to θ. Since $\mu > 4$, we have that $b_1 < \theta' < 0$ and, in particular, that all $b_j < 0$. We process each path of T_i until the vertices adjacent to v_i. We notice that to process the vertex v_i we need the value resultant from processing T_i. We denote by $\omega(T_i) = r_i$ the number of paths of T_i and, for $m = 1, \ldots, r_i$, by $b_{i,m}$ the last value output by the algorithm at the m-th path of T_i. The net result of the starlike T_i is given by

$$\delta(T_i, \mu) := \begin{cases} \sum_{j=1}^{\ell_i} \left(1 - \frac{1}{b_{i,j}}\right), \text{ for non-empty } T_i \\ 0 \text{ if } T_i \text{ is empty.} \end{cases} \tag{6}$$

Continuing applying the algorithm, we go from v_1 towards v_k, the root of G_k. We denote S_i, for $i \in \{1, \ldots, k\}$, the value output by Diagonalize$(L, -\mu)$. We have $S_1 = 1 - \mu + \delta(T_1, \mu)$, $S_2 = 2 - \mu + \delta(T_2, \mu) - \frac{1}{S_1}$, $S_3 = 2 - \mu + \delta(T_2, \mu) - \frac{1}{S_2}$ and so on. For S_k the expression is $S_k := 1 - \mu - \frac{1}{S_{k-1}} + \delta(T_k, \mu)$ (see Figure 5 for an illustration). Using the function ψ, we notice that S_i obeys the following recurrence.

$$\begin{cases} S_1 = 1 - \mu + \delta(T_1, \mu), \\ S_i = \psi(S_{i-1}) + \delta(T_2, \mu), \ 2 \leq i \leq k-1, \\ S_k = -1 + \psi(S_{k-1}) + \delta(T_k, \mu). \end{cases} \tag{7}$$

We are interested in determining whether μ is smaller than, equal to, or larger than $\rho_L(G_k)$, the Laplacian spectral radius of G_k. Since the values of all b_j's are negative, by Lemma 2.1, only the signs of the values S_j suffice to determine this. More precisely, consider the function

$$\Pi := (G_k, \mu) \to (S_1, S_2, \dots, S_k)\,,$$

and let $\mathrm{sign}(\Pi)(G_k,\mu) = (\mathrm{sign}(S_1), \mathrm{sign}(S_2), \dots, \mathrm{sign}(S_k))$. From Lemma 2.1 we can state the following result.

Lemma 2.3. *With the established notation, we have*

(a) $\rho_L(G_k) < \mu \Leftrightarrow$ *sign*$(\Pi)(G_k,\mu) = (-,-,\dots,-,-)$*;*
(b) $\rho_L(G_k) = \mu \Leftrightarrow$ *sign*$(\Pi)(G_k,\mu) = (-,-,\dots,-,0)$*;*
(c) $\rho_L(G_k) > \mu \Leftrightarrow$ *sign*$(\Pi)(G_k,\mu)$ *has a + entry.*

3. Applying Shearer's approach for Laplacian limit points

In the previous section, we set a framework to apply Diagonalize$(L(G_k), -\mu)$, where G_k is a linear tree and μ is any real number larger than 4. We are going to use the notation established there, but we observe a subtle difference. Here, from a given number $\mu \geq 4.38+$, we construct a prescribed caterpillar $G_k(\mu) := [r_1, r_2, ..., r_k],\ k \geq 2$ (see Figure 3) by adapting Shearer's method to Laplacian matrices. Additionally, we will study the convergence of $\rho_L(G_k(\mu))$.

Let us then fix a real number $\mu \in [4.28+, \infty)$. Shearer's approach needs to determine numbers $r_1, r_2, ..., r_k$, such that $G_k(\mu) = [r_1, r_2, ..., r_k]$ has $\rho_L(G_k) < \mu$. In the language of Section 2.2, the linear trees T_i are simply stars, whose paths have length one. As $T_i = [1, 1, \dots, 1]$, we have $\delta(T_i, \mu) := \sum_{j=1}^{r_i}\left(1 - \frac{1}{b_1}\right) = \sum_{j=1}^{r_i}\left(1 - \frac{1}{1-\mu}\right) = r_i\frac{\mu}{\mu-1} = \delta_i$. We recall that, from Lemma 2.3, $\rho_L(G_k) < \mu$ if and only if $S_j < 0$ for all $j < k$. As the attracting interval for θ, given by Equation (5), is $(-\infty, \theta']$ we need to require that $S_j \leq \theta'$ otherwise $S_{j'} > 0$ for some $j' > j$ contradicting $\rho_L(G_k) < \mu$.

Following the reasoning of Shearer's construction, we determine the sequential choice of $r_1, r_2, \dots, r_k$ as follows.
Since $S_1 < 0$ if $1 - \mu + r_1\frac{\mu}{\mu-1} \leq \theta'$, we choose

$$r_1 := \max_{r\geq 0}\left\{1 - \mu + r\frac{\mu}{\mu-1} \leq \theta'\right\} \Leftrightarrow r_1 := \left\lfloor \frac{\mu-1}{\mu}(\theta' - (1-\mu))\right\rfloor.$$

We point out that here we deal with real inequalities, such as $S_1 < 0$, or equivalently $1-\mu+r_1\frac{\mu}{\mu-1} \leq \theta'$. Of course, this inequality is true for $r_1 = 0$. Moreover, if we increase r by one, then the left hand side of the inequality increases by $\frac{\mu}{\mu-1} > 0$. Thus, there will be a maximum number r satisfying the inequality. If we assume that r could be a real number, then the equality happens for the real number $\tilde{r} := \frac{\mu-1}{\mu}(\theta' - (1-\mu))$. Clearly, the largest positive integer satisfying this property is $r_1 := \lfloor \tilde{r} \rfloor$.
And this is the choice for r_1. As $S_2 < 0$ if $\psi(S_1) + r_2\frac{\mu}{\mu-1} \leq \theta'$ we choose

$$r_2 := \max_{r\geq 0}\left\{\psi(S_1) + r\frac{\mu}{\mu-1} \leq \theta'\right\} \Leftrightarrow r_2 := \left\lfloor \frac{\mu-1}{\mu}(\theta' - \psi(S_1))\right\rfloor,$$

and so on. We notice that r_k has a distinct expression. Since $S_k < 0$ if $-1 + \psi(S_{k-1}) + r_k\frac{\mu}{\mu-1} \leq \theta'$, we choose

$$r_k := \max_{r\geq 0}\left\{-1 + \psi(S_{k-1}) + r\frac{\mu}{\mu-1} \leq \theta'\right\} \Leftrightarrow r_k := \left\lfloor \frac{\mu-1}{\mu}(\theta' + 1 - \psi(S_{k-1}))\right\rfloor.$$

Definition 3.1. *Given $\mu \geq 4.38+$, let $\theta' := \frac{-(\mu-2)+\sqrt{(\mu-2)^2-4}}{2}$. For $\psi(t) = 2 - \mu - \frac{1}{t}$, we define the classic Laplacian Sharer's sequence as the sequence of caterpillars $G_k(\mu) := [r_1, r_2, ..., r_k],\ k \geq 2$ where*

$$\begin{cases} r_1 := \left\lfloor \frac{\mu-1}{\mu}(\theta' - (1-\mu))\right\rfloor. \\ S_1 = 1 - \mu + r_1\frac{\mu}{\mu-1} \\ r_j := \left\lfloor \frac{\mu-1}{\mu}(\theta' - \psi(S_{j-1}))\right\rfloor \\ S_j = \psi(S_{j-1}) + r_j\frac{\mu}{\mu-1} \\ r_k := \left\lfloor \frac{\mu-1}{\mu}(\theta' + 1 - \psi(S_{k-1}))\right\rfloor \\ S_k = -1 + \psi(S_{k-1}) + r_k\frac{\mu}{\mu-1}. \end{cases} \tag{8}$$

Example 3.2. *Consider applying Definition 3.1 for $\mu = 5.4$. We have $\theta := \frac{-(\mu-2)-\sqrt{(\mu-2)^2-4}}{2} \approx -3.074772708$ and $\theta' := \frac{-(\mu-2)+\sqrt{(\mu-2)^2-4}}{2} = \theta^{-1} \approx -0.32522729 > \theta$. We obtain by computation the first 11 terms, as follows.*
$r_1 := \left\lfloor \frac{\mu-1}{\mu}(\theta' - (1-\mu)) \right\rfloor$, *gives* $r_1 = 3$
$S_1 = 1 - \mu + r_1 \frac{\mu}{\mu-1}$, *gives* $S_1 \approx -0.718181818$
$r_2 := \left\lfloor \frac{\mu-1}{\mu}(\theta' - \psi(S_1)) \right\rfloor$, *gives* $r_2 = 1$
$r_j := \left\lfloor \frac{\mu-1}{\mu}(\theta' - \psi(S_{j-1})) \right\rfloor$, *gives* $r_j = 1$ *for* $j = 2, \ldots, 10$, *while*
$r_{11} := \left\lfloor \frac{\mu-1}{\mu}(\theta' + 1 - \psi(S_{10})) \right\rfloor$, *where* $S_{10} \approx -1.503801894$, *giving* $r_{11} = 2$.

Hence for $k = 11$ our $G_k(5.4)$ is given by

$$G_k := [[1,1,1],[1],[1],[1],[1],[1],[1],[1],[1],[1],[1,1]].$$

Before we proceed with the analysis of this process, it is of great importance to understand how it evolves, in particular we want to obtain a relation between the sequence S_j for $G_\mu(\mu) := [r_1, r_2, ..., r_{k-1}, r_k]$ comparatively with the one for $G_{k+1}(\mu)$. Let us denote it by $\tilde{S}_j$ and $G_{k+1}(\mu) := [\tilde{r}_1, \tilde{r}_2, ..., \tilde{r}_{k-1}, \tilde{r}_k, \tilde{r}_{k+1}]$.

We claim that, if $r_k \geq 1$ then

$$\tilde{r}_k = r_k - 1.$$

To prove this, we first notice that, by construction $r_j = \tilde{r}_j$ and $S_j = \tilde{S}_j$ for $1 \leq j \leq k-1$. We also recall that

$$r_k := \max_{r \geq 0} \left\{ -1 + \psi(S_{k-1}) + r\frac{\mu}{\mu-1} \leq \theta' \right\}. \tag{9}$$

We notice that this is equivalent to

$$\theta' - \frac{\mu}{\mu-1} < -1 + \psi(S_{k-1}) + r_k\frac{\mu}{\mu-1} \leq \theta'. \tag{10}$$

Equation (10) is equivalent to

$$-1 + \psi(S_{k-1}) + r_k\frac{\mu}{\mu-1} \leq \theta' \text{ and } -1 + \psi(S_{k-1}) + (r_k+1)\frac{\mu}{\mu-1} > \theta',$$

which are equivalent to

$$\psi(S_{k-1}) + (r_k-1)\frac{\mu}{\mu-1} + \frac{1}{\mu-1} \leq \theta' \text{ and } \psi(S_{k-1}) + r_k\frac{\mu}{\mu-1} + \frac{1}{\mu-1} > \theta'.$$

Rewriting these, we have

$$\psi(S_{k-1}) + (r_k-1)\frac{\mu}{\mu-1} \leq \theta' - \frac{1}{\mu-1}, \text{ and } \psi(S_{k-1}) + r_k\frac{\mu}{\mu-1} > \theta' - \frac{1}{\mu-1}.$$

We claim that these inequalities imply

$$\psi(S_{k-1}) + (r_k-1)\frac{\mu}{\mu-1} \leq \theta', \text{ and } \psi(S_{k-1}) + r_k\frac{\mu}{\mu-1} > \theta'. \tag{11}$$

Indeed, as $\theta - \frac{1}{\mu-1} \leq \theta'$, the first inequality follows. Now if $\psi(S_{k-1}) + r_k\frac{\mu}{\mu-1} \leq \theta'$, the first inequality implies that $\frac{\mu}{\mu-1} \leq 0$, which is a contradiction. It follows that, by comparing Equation (11) with Equation (10) we can write

$$r_k - 1 = \max_{r \geq 0} \left\{ \psi(S_{k-1}) + r\frac{\mu}{\mu-1} \leq \theta' \right\}. \tag{12}$$

Now we notice that, because $\psi(\tilde{S}_{k-1}) = \psi(S_{k-1})$, the expression for $\tilde{r}_k$ is

$$\tilde{r}_k := \max_{r \geq 0} \left\{ \psi(S_{k-1}) + r\frac{\mu}{\mu-1} \leq \theta' \right\} = r_k - 1, \tag{13}$$

in view of Equation (12).

Analogously to the adjacency case, the procedure given in Definition 3.1, always works. In the sense that it produces a sequence of caterpillars

$$G_k(\mu) := [r_1, r_2, ..., r_k], \; k \geq 2$$

such that the spectral radius $\rho_L(G_k(\mu)) := \rho_L(G_k)$ has a limit point smaller than μ. More precisely, the sequence $\rho_L(G_k)$ is increasing and bounded by μ. Indeed, the part $\rho_L(G_k) < \mu$ is a trivial consequence of the construction because $S_1, \ldots, S_k < \theta' < 0$ then the spectral radius for the Laplacian matrix of G_k is necessarily smaller than μ by Lemma (2.3). The part $\rho_L(G_k) < \rho_L(G_{k+1})$ is not quiet obvious because the choice of r_k for G_k is possibly different from the one for G_{k+1}, unless $r_k = 0$. In that case G_k is necessarily a subgraph of G_{k+1} and $\rho_L(G_k) < \rho_L(G_{k+1})$. Thus, we assume that $r_k \geq 1$. In order to verify the inequality $\rho_L(G_k) < \rho_L(G_{k+1})$, we recall that, from Equation (13), we have $\tilde{r}_k = r_k - 1$ meaning that G_k is necessarily a subgraph of G_{k+1} and then $\rho_L(G_k) < \rho_L(G_{k+1})$ again. In particular there exists $\lim\limits_{k\to\infty} \rho_L(G_k) = \gamma(\mu) \leq \mu$. We summarize the findings of this section, stating the following result.

Theorem 3.3. *Let $\mu \in [4.38+, \infty)$. The sequence $G_k(\mu)$ of caterpillars given by Equation (8) produces a limit point of L-spectral radii: there exists*

$$\lim_{k\to\infty} \rho_L(G_k(\mu)) = \gamma(\mu) \leq \mu.$$

The question is: what are the limit points produced by this process?

Definition 3.4. *We denote by $\Omega_1 \subseteq [4.38+, \infty)$ the set of all Laplacian limit points produced by Shearer's method*

$$\Omega_1 := \{\gamma \mid \lim_{k\to\infty} \rho_L(G_k(\mu)) = \gamma\}.$$

Similar to the adjacency case studied by Shearer [13], where he shows that this method produces an A-limit point for any real number in $[\sqrt{2+\sqrt{5}}, \infty)$, it is natural to conjecture that $\Omega_1 = [4.38+, \infty)$.

Unlike the adjacency case, however, we show in the sequel that there are points where $\gamma(\mu) < \mu$. In fact, we will provide a whole interval I, for which the sequence of caterpillars $G_k(\mu)$, given by Definition 3.1 is the same for every $\mu \in I$. This means $G_k(\mu)$ has Laplacian spectral radius converging to the same value $\gamma < \mu$.

This only shows that the classic Shearer's caterpillar construction for the Laplacian version is not powerful enough to prove the density of Laplacian spectral radii in $[4.38+, \infty)$. In subsequent sections of this note, we extend this method and show how this extension obtains more accumulation points.

3.1. **A nasty interval.** In this section we study the spectral radius of the caterpillar G_k, $k \geq 3$ given by

$$G_k := [[1,1,1],[1],[1],\ldots,[1],[1,1]].$$

We notice this was obtained in Example 5.3 as the classic Laplacian Shearer's construction with $\mu = 5.4$ and $k = 11$. We first consider applying Diagonalize$(G_k, -\mu)$, for $\mu > 4.38+$, using the notation established in Section 2.2, disregarding the fact that this sequence is generated by some actual μ.

Since the starlike trees T_i are actually stars whose number of rays are $r_1 = 3, r_j = 1,\ 2 \leq j \leq k-1$, and $r_k = 2$, the application of Equation (7) gives $S_1 = 1 - \mu + 3\frac{\mu}{\mu-1}$, $\ S_j = 2 - \mu - \frac{1}{S_{j-1}} + \frac{\mu}{\mu-1}$, $2 \leq j \leq k-1$ and $S_k = 1 - \mu - \frac{1}{S_{k-1}} + 2\frac{\mu}{\mu-1}$. For convenience we let $\tilde{\psi}(t) = (2 - \mu + \frac{\mu}{\mu-1}) - \frac{1}{t}$, and rewrite as the following recurrence.

$$\begin{cases} S_1 = 1 - \mu + 3\frac{\mu}{\mu-1}, \\ S_j = \tilde{\psi}(S_{j-1}),\ 2 \leq j \leq k-1, \\ S_k = \tilde{\psi}(S_{k-1}) + \frac{1}{\mu-1} \end{cases} \tag{14}$$

The fixed points $\psi(t) = t$ are θ and θ' given by Equation (5), while the fixed points of $\tilde{\psi}(t) = t$ are given by

$$\begin{cases} \sigma = \frac{(2-\mu+\frac{\mu}{\mu-1}) - \sqrt{(2-\mu+\frac{\mu}{\mu-1})^2 - 4}}{2} \\ \sigma' = \frac{(2-\mu+\frac{\mu}{\mu-1}) + \sqrt{(2-\mu+\frac{\mu}{\mu-1})^2 - 4}}{2} = \sigma^{-1}. \end{cases} \tag{15}$$

Lemma 3.5. *Let $4 < \mu < 6$. Let μ_* and μ^* be, respectively, the solutions of $S_1 = \sigma'$ and $\theta^{-1} - \frac{\mu}{\mu-1} = \sigma$.*

(a) $\mu_* = \frac{5+\sqrt{33}}{2} = 5.3722+$ *is the largest root of the polynomial $x^2 - 5x - 2$;*
(b) $\mu^* = 5.4207+$ *is the largest root of $2x^4 - 14x^3 + 19x^2 - 10x - 1$.*

Proof. (*a*) $S_1 = \sigma'$ is equivalent to

$$1-\mu+3\frac{\mu}{\mu-1} = \frac{(2-\mu+\frac{\mu}{\mu-1}) + \sqrt{(2-\mu+\frac{\mu}{\mu-1})^2-4}}{2},$$

which can be transformed into the polynomial $\mu^2-5\mu-2$. In the range $4<\mu<6$, μ_* is the only root of the polynomial.
(*b*) Now, for $\theta^{-1}-\frac{\mu}{\mu-1}=\sigma$ we can perform the same computations transforming

$$\frac{(2-\mu)+\sqrt{(2-\mu)^2-4}}{2} - \frac{\mu}{\mu-1} = \frac{(2-\mu+\frac{\mu}{\mu-1}) - \sqrt{(2-\mu+\frac{\mu}{\mu-1})^2-4}}{2}$$

into a polynomial equation as prescribed. Performing that we obtain that μ^* is the largest root of $2\,\mu^4-14\,\mu^3+19\,\mu^2-10\,\mu-1$.

□

Lemma 3.6. *Let G_k be the caterpillar $G_k := [[1,1,1],[1],[1],\ldots,[1],[1,1]]$ and $\mu_k = \rho_L(G_k)$ be the spectral radius of G_k. Then*

$$\lim_{k\to\infty} \rho_L(G_k) = \mu_*.$$

Proof. We consider the auxiliary graphs $H_k(\mu) := [[1,1,1],[1],\ldots,[1],[1]]$, $k \geq 3$. By applying Diagonalize$(H_k,-\mu_k)$ with $\mu_k = \rho_L(G_k)$, we obtain the sequence of values $W_1,\ldots,W_{k-1},W_k$,

$$W_1 = 1-\mu_k+3\frac{\mu_k}{\mu_k-1} < 0$$

$$W_{j+1} := \tilde{\psi}(W_j),\ 1\leq j,$$

where $\tilde{\psi}(t) := (2-\mu_k+\frac{\mu_k}{\mu_k-1}) - \frac{1}{t},\ t\neq 0$. From [10] we know that a closed formula W_j is as follows.

$$W_j := \sigma_k + \frac{\sigma_k^{-1}-\sigma_k}{\beta_k\sigma_k^{2j-2}+1},\ 1\leq j,$$

where $\beta_k := \frac{\sigma_k^{-1}-\sigma_k}{W_1-\sigma_k} - 1$ and $\sigma_k = \frac{(2-\mu_k+\frac{\mu_k}{\mu_k-1}) - \sqrt{(2-\mu_k+\frac{\mu_k}{\mu_k-1})^2-4}}{2}$. Since H_k and G_k only differ in the last term, applying Diagonalize$(G_k,-\mu_k)$, we obtain a sequence of values $S_1 = W_1,\ldots,S_{k-1}=W_{k-1}$ and $S_k = W_k + \frac{1}{\mu_k-1} = 0$ because $\mu_k := \rho_L(G_k)$. An easy computation transforming this equation into a polynomial with respect to σ_k shows that taking limit when $k\to\infty$ in $W_k+\frac{1}{\mu_k-1}=0$, we obtain that the limit point $\mu := \lim_{k\to\infty}\rho_L(G_k)$ satisfies $(\sigma+\frac{1}{\mu-1})\cdot(S_1-\sigma^{-1}) = 0$.

Indeed,

$$S_k = W_k + \frac{1}{\mu_k-1} = 0 \Leftrightarrow W_k + \frac{1}{\mu_k-1} = 0$$

$$\sigma_k + \frac{\sigma_k^{-1}-\sigma_k}{\beta_k\sigma_k^{2j-2}+1} + \frac{1}{\mu_k-1} = 0$$

$$\sigma_k + \frac{\sigma_k^{-1}-\sigma_k}{\left(\frac{\sigma_k^{-1}-\sigma_k}{W_1-\sigma_k}-1\right)\sigma_k^{2j-2}+1} + \frac{1}{\mu_k-1} = 0.$$

Now we just transform this into a polynomial and use that $\sigma_k^{2j-2}\to\infty$. The remaining equation is $(\sigma_k+\frac{1}{\mu-1})\cdot(W_1-\sigma_k^{-1}) = 0$. As $W_1 = S_1$ and since $\sigma_k+\frac{1}{\mu-1}$ has no solution for $\mu>4$, we obtain the equation $S_1 = \sigma_k^{-1}$, which, by Lemma 3.5 is μ_*. □

Lemma 3.7. *Let $\mu_* = 5.3722+$ and $\mu^* = 5.4207+$ be as in Lemma 3.5 and let $I = [\mu_*,\mu^*]$. Then for all $\mu\in I$, the classic Laplacian Shearer's sequence given by Definition 3.1 is $G_k(\mu) := [[1,1,1],[1],[1],\ldots,[1],[1,1]]$.*

Proof. We consider two facts, the first one is

$$\theta^{-1} - \frac{\mu}{\mu-1} < 1-\mu+3\frac{\mu}{\mu-1} < \theta^{-1}$$

for $5.09807+ < \mu < 6.05505+$ (see Figure 6), meaning that in this interval every G_k begins with $[[1,1,1],?,?,\ldots]$ because $I\subset[5.09807+, 6.05505+]$. By applying Diagonalize$(G_k,-\mu)$ with $\mu\in I$, we obtain a sequence of values $S_1,...,S_{k-1},S_k$ where $S_1 = 1-\mu+3\frac{\mu}{\mu-1}\in(\theta^{-1}-\frac{\mu}{\mu-1},\theta^{-1}]$, that is $r_1 = 3$.

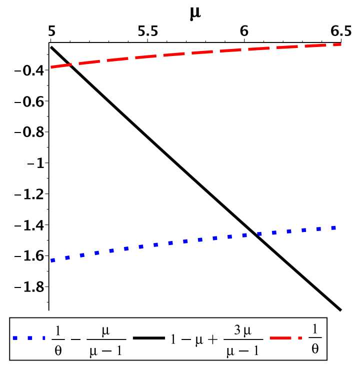

FIGURE 6. Inequality $\theta^{-1} - \frac{\mu}{\mu-1} < 1 - \mu + 3\frac{\mu}{\mu-1} < \theta^{-1}$.

The second fact is the inequality

$$\theta^{-1} - \frac{\mu}{\mu-1} < \sigma < 1 - \mu + 3\frac{\mu}{\mu-1} < \sigma^{-1} < \theta^{-1},$$

where σ and σ^{-1} are defined in Equation (15). This is easily verified from the graphics given in Figure 7, since those are functions of μ.

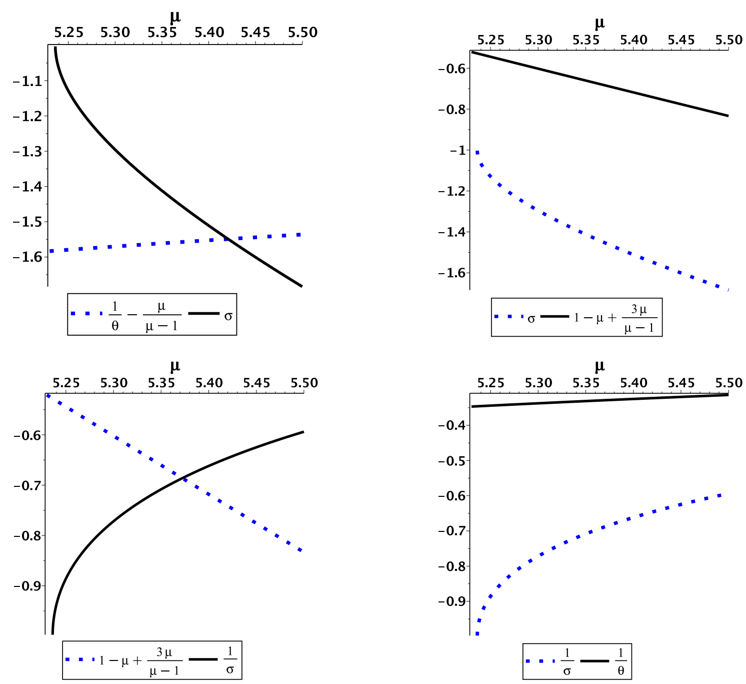

FIGURE 7. Inequality $\theta^{-1} - \frac{\mu}{\mu-1} < \sigma < 1 - \mu + 3\frac{\mu}{\mu-1} < \sigma^{-1} < \theta^{-1}$.

We know that $S_2 = 2-\mu-\frac{1}{S_1}+r_2\frac{\mu}{\mu-1}$ and we claim that $r_2 = 1$ because $2-\mu-\frac{1}{S_1}+\frac{\mu}{\mu-1} = \tilde{\psi}(S_1) \in (\sigma,\sigma^{-1}] \subset (\theta^{-1} - \frac{\mu}{\mu-1}, \theta^{-1}]$ meaning that $r_2 = 1$. In each step, $S_{j+1} = \tilde{\psi}(S_j)$ obeys the recurrence generated by $\tilde{\psi}(t) := (2-\mu+\frac{\mu}{\mu-1})-\frac{1}{t}$, $t \neq 0$. From [10] we know that, since $\Delta = (2-\mu+\frac{\mu}{\mu-1})^2-4 > 0$, we have

$$S_j := \sigma + \frac{\sigma^{-1} - \sigma}{\beta\sigma^{2j-2}+1},\ 1 \leq j \leq k-1,$$

where $\beta := \frac{\sigma^{-1}-\sigma}{S_1-\sigma} - 1$ and $\sigma = \frac{(2-\mu+\frac{\mu}{\mu-1})-\sqrt{(2-\mu+\frac{\mu}{\mu-1})^2-4}}{2}$. We notice that $\psi(S_j) \in (\sigma,\sigma^{-1}] \subset (\theta^{-1} - \frac{\mu}{\mu-1}, \theta^{-1}]$, meaning that $r_j = 1$ for $j \leq k-1$. We could extrapolate for $\psi(S_{k-1}) \in (\sigma,\sigma^{-1}] \subset$

$(\theta^{-1} - \frac{\mu}{\mu-1}, \theta^{-1}]$ obtaining $r_k = 1 + 1$.
Thus $G_k(\mu) := [[1,1,1],[1],...,[1],[1,1]],\ k \geq 3$, as claimed.

□

Remark 3.8. *We are not claiming that the points $\mu \in I$ are not Laplacian limit points, only that they can not be achieved by using caterpillar graphs according to the Shearer's process. This motivates us to consider a more general class of trees in the next section.*

4. Sequences of linear trees

Inspired by the Shearer's procedure from [13], we will consider increasing sequences of Laplacian spectral radii of graphs $\{G_k \mid k \in \mathbb{N}\}$, where G_k are linear trees, as introduced in Section 2.2.

We need some technical results. The first one is:

Lemma 4.1. [1, 17] *Let $G = (V, E)$ be a graph on n vertices and m edges. Let $V = \{v_1, v_2, \ldots, v_n\}$ and let $d(v_i)$ be the degree of the vertex v_i. Then*

$$\rho_L(G) \leq \max\{d(v_i) + d(v_j) \mid v_i, v_j \in E\}.$$

In particular, we can say that $\rho_L(G) \leq 2\Delta(G)$, where $\Delta(G)$ is the largest degree of G.

The second one is a bound on the spectral radii of $\{G_k \mid k \in \mathbb{N}\}$. We recall that $\omega(T_j)$ is the number of paths that compose the starlike T_j.

Lemma 4.2. *Consider a sequence of linear trees $\{G_k \mid k \in \mathbb{N}\}$. Then,*

$$\exists M > 0,\ \rho_L(G_k) \leq M, \forall k \in \mathbb{N} \Leftrightarrow \max_k \max_{1\leq j\leq k} \omega(T_j) < +\infty.$$

Proof. Assume that $\exists M > 0,\ \rho_L(G_k) \leq M, \forall k \in \mathbb{N}$, then necessarily $\max_k \max_{1\leq j\leq k} \omega(T_j) < +\infty$ otherwise if there exists a sequence of T_j with $\omega(T_j) \to \infty$ then, since $\rho_L(G_k) \geq 1 + \omega(T_j)$, would be a contradiction.

Reciprocally, let $r_0 := \max_k \max_{1\leq j\leq k} \omega(T_j) < +\infty$ and $h_0 := \max_{1\leq j\leq k} h(T_j)$ then for any k, G_k is a subgraph of $F_k = [[h_0,\ldots,h_0],\ldots,[h_0,\ldots,h_0]]$, where $w([h_0,\ldots,h_0]) = r_0$ and $\ell(F_k) = k$. Thus, $\rho_L(G_k) \leq \rho_L(F_k)$, for any k.

A well known upper bound for the $\rho_L(F_k)$ is $\rho_L(F_k) \leq 2\Delta(F_k)$ (see Lemma 4.1), where $\Delta(F_k)$ is the largest degree of F_k. As the $\Delta(F_k) \leq r_0 + 2$, we have $\rho_L(G_k) \leq \rho_L(F_k) \leq 2(r_0 + 2)$ and the result follows.

□

Definition 4.3. *We denote by $\mathcal{L}_\infty$ the set of all sequences of starlike trees with bounded width, that is,*

$$\mathcal{L}_\infty := \left\{(T_1, T_2, ...) \mid T_j's \text{ are starlike trees and } \max_{k\geq 1} \max_{1\leq j\leq k} \omega(T_j) < +\infty\right\}.$$

Definition 4.4. *For each $\mathbb{T} = (T_1, T_2, ...) \in \mathcal{L}_\infty$ and $C = (C_2, C_3, ...) \in \mathcal{L}_\infty$ we can define a sequence of linear trees*

$$\begin{cases} G_1(\mathbb{T}) := [T_1] \\ G_k(\mathbb{T}) := [T_1, T_2, ..., T_{k-1}, C_k],\ k \geq 2, \end{cases}$$

called (Laplacian) generalized Shearer sequence *associated to $\mathbb{T}$ and C if $\rho_L(G_k(\mathbb{T})) < \rho_L(G_{k+1}(\mathbb{T}))$ for all $k \geq 1$.*

We actually should use the notation $G_j(\mathbb{T}, C)$, but we will omit the dependence in C when it is clear. A sufficient condition to have $\rho_L(G_k(\mathbb{T})) < \rho_L(G_{k+1}(\mathbb{T}))$ is $G_k(\mathbb{T})$ to be a subgraph of $G_{k+1}(\mathbb{T})$ for all $k \geq 1$, called subgraph condition.

Remark 4.5. *The canonical case we may deal is, $\mathbb{T} = (T_1, T_2, ...) \in \mathcal{L}_\infty$ and $C = (T_2, T_3, ...)$ a shift of $\mathbb{T}$ (thus $C \in \mathcal{L}_\infty$), then $G_1 = [T_1]$, $G_2 = [T_1, T_2]$ and $G_3 = [T_1, T_2, T_3]$, and so on. This sequence naturally satisfies the subgraph condition so it is a generalized Shearer sequence.*

Example 4.6. *Let us consider some cases for generalized Shearer sequences:*

a) *Consider* $\mathbb{T} = ([1,1,1],[1],[1],[1],[1],...) \in \mathcal{L}_\infty$ *and* $C = ([1,1],[1,1],[1,1],[1,1],...) \in \mathcal{L}_\infty$*, then* $G_1 = [[1,1,1]]$*,* $G_2 = [[1,1,1],[1,1]]$*,* $G_3 = [[1,1,1],[1],[1,1]]$*,* $G_4 = [[1,1,1],[1],[1],[1,1]]$*, and so on, is obviously a generalized Shearer sequence coincident with the one in Section 3.1.*

b) *Consider* $\mathbb{T} = ([1,1],[1],[0],[1],[0],...) \in \mathcal{L}_\infty$ *and* $C = ([0],[0],[0],[0],...) \in \mathcal{L}_\infty$ *then* $G_1 = [[1,1]]$*,* $G_2 = [[1,1],[0]]$*,* $G_3 = [[1,1],[1],[0]]$*,* $G_4 = [[1,1],[1],[0],[0]]$*,* $G_5 = [1,1],[1],[0],[1],[0]$*, and so on, is a generalized Shearer sequence.*

c) *Consider* $\mathbb{T} = ([1,1],[1,1],[1],[1],[1],...) \in \mathcal{L}_\infty$ *and* $C = ([1,1,1,1],[0],[0],[0],...) \in \mathcal{L}_\infty$ *then* $G_1 = [[1,1]]$*,* $G_2 = [[1,1],[1,1,1,1]]$ *and* $G_3 = [[1,1],[1,1],[0]]$*. Note that* G_2 *is not a subgraph of* G_3*, moreover,*

$$\rho_L(G_2) = 6.141336+ > 5.261802+ = \rho_L(G_3)$$

thus it is not a generalized Shearer sequence.

d) *Consider* $\mathbb{T} = ([1],[1,1],[1,1,1],[1,1,1,1],...)$ *and* $C = ([1],[1],[1],[1],...) \in \mathcal{L}_\infty$ *then* $G_1 = [[1]]$*,* $G_2 = [[1],[1]]$ *and* $G_3 = [[1],[1,1],[1]]$*,* $G_4 = [[1],[1,1],[1,1,1],[1]]$*, and so on. This sequence satisfies the subgraph condition however it is not a generalized Shearer sequence because* $\omega(T_j) = j-1 \to \infty$*, that is,* $C \in \mathcal{L}_\infty$ *but* $\mathbb{T} \notin \mathcal{L}_\infty$*.*

By construction, each generalized Shearer sequence satisfies the property that $\rho_L(G_k) \le \rho_L(G_{k+1})$ and by Lemma 4.2 we also know that $k \to \rho_L(G_k)$ is bounded. Thus, there exists $\lim\limits_{k\to\infty} \rho_L(G_k) := \gamma(\mathbb{T})$.

Definition 4.7. *We denote by* $\mathcal{S}$ *the set of all limit points of (Laplacian) generalized Shearer sequences, that is,*

$$\mathcal{S} := \{\gamma \mid \gamma = \gamma(\mathbb{T}) \text{ for some } \mathbb{T}, C \in \mathcal{L}_\infty\}.$$

5. Applications: Generalized random Shearer process

As the set $\Omega_1 \subseteq \mathcal{S} \subseteq [4.38+, \infty)$ (see Section 3), it is natural to believe that $\mathcal{S}$ is a larger set of Laplacian limit points. Indeed, we will show this. Neverthless, it will be useful to have a structured way to construct linear trees as generalized Shearer sequences, for a given parameter μ. The main goal of this section is provide one, which is an adaptation of the classical process used by Shearer to produce caterpillars. We denominate it as **Generalized random Shearer process**.

Fixed $\mu > 4.38+$, we will construct a generalized Shearer sequence in the following way. Let $\mathbb{T} = (T_1, T_2, \ldots)$ and $C = (C_2, C_3, \ldots)$ be arbitrary sequences of starlike trees. From that we define linear trees

$$G_k := [T_1, T_2, , T_{k-1}, C_k],\ k \ge 2,$$

with $G_1 := [T_1]$. We would like to have $\rho_L(G_k) < \mu$ and G_k a subgraph of G_{k+1}.

To achieve this property, we use Diagonalize$(G_k, -\mu)$ obtaining $\Pi(G_k, \mu) = (S_1, \ldots, S_k)$ (see Section 2.2). We recall that $\psi(t) = 2 - \mu - \frac{1}{t},\ t \ne 0,\ \theta := \frac{-(\mu-2)-\sqrt{(\mu-2)^2-4}}{2}$ is an attracting point and $\theta' := \frac{-(\mu-2)+\sqrt{(\mu-2)^2-4}}{2} = \theta^{-1} > \theta$ is a repelling one.

Thus, in order to obtain $S_j < 0$ for all j we successively choose the following starlike trees:

$$\begin{cases} T_1 \text{ such that } 1 - \mu + \delta(T_1, \mu) < \theta^{-1}; \\ T_j \text{ such that } \psi(S_{j-1}) + \delta(T_j, \mu) < \theta^{-1},\ 2 \le j \le k; \\ C_k \text{ such that } C_k := T_k. \end{cases}$$

Since $\psi(S_{k-1}) + \delta(T_k, \mu) < \theta^{-1}$, $\theta^{-1} < 0 < 1$ and $C_k = T_k$ we obtain

$$\psi(S_{k-1}) + \delta(T_k, \mu) < 1$$

$$-1 + \psi(S_{k-1}) + \delta(C_k, \mu) < 0$$

$$S_k < 0.$$

Thus $\rho_L(G_k) < \mu$ because sign$(\Pi)(G_k, \mu) = (-, \ldots, -)$.

This procedure is random since we can choose different trees in each step and it will always work because $1 - \mu < \theta^{-1}$ so, in the worst case, we can choose $T_1 = [0]$ and the same is true for the next ones since the previous $S_{j-1} < \theta^{-1}$ implies that $\psi(S_{j-1}) < \theta^{-1}$, by construction.

We notice that necessarily $\mathbb{T}, C \in \mathcal{L}_\infty$ because for each choice of T_j or C_k we must have subgraphs of G_k which has spectral radius smaller than μ, a fixed number, this means that the width $\omega(T_j)$ or $\omega(C_k)$ can not exceed $\mu - 4$ (recall that $\rho_L(G_k) \geq \max\{\omega(T_j)+3, \omega(C_k)+2\}$).

From the way we built the G_k's, we always end up with a generalized Shearer sequence since C is a shift of $\mathbb{T}$. We then have $\rho_L(G_k) < \rho_L(G_{k+1})$ and, in particular, there exists

$$\lim_{k\to\infty} \rho_L(G_k) = \gamma(\mu) \leq \mu.$$

Definition 5.1. *We denote by $\Omega \subseteq [4.38+, \infty)$ the set of all Laplacian limit points produced by the Generalized random Shearer's method*

$$\Omega := \{\gamma \mid \lim_{k\to\infty} \rho_L(G_k(\mu)) = \gamma\}.$$

Remark 5.2. *We notice that, if in addition to the above requirements, we restrict our choices to $\max\{h(T_j), h(C_k)\} = 1$ and we maximize the value of each $S_j < \theta^{-1}$ we recover the caterpillars from the classic Shearer's paper. Thus $\Omega_1 \subset \Omega$.*

Example 5.3. *Consider $\mu = 5.4$ and a random Shearer process running up computationally producing a generalized Shearer sequence*

$$\mathbb{T} := ([1],[1,2],[1],[0],[0],[1],[1],[0],[1],[2],[2],[2],[1],[1],[2],[2],[2],[1],[2],[1],[2],$$
$$[1],[2],[2],[1],[2],[1],[2],[2],[2],[2],[2],[2],[2],[1],[1],[2],[1],[2],[2],[1],[2],[2],[2],[1],$$
$$[2],[2],[2],[2],[2],[1],[2],[2],[2],[2],[2],[2],[1],[2],[2],[1],[1],[2],[2],[2],[2],[2],[2],[2],$$
$$[1],[2],[2],[1],[2],[1],[1],[2],[2],[2],[1],[1],[1],[1],[1],[1],[2],[2],[1],[1],[2],[2],[1],[2],$$
$$[2],[1],[2],[1],[2],[2],[2],[0]^\infty).$$

By a direct computation we obtain
$G_1 := [[1]]$ *and* $\rho_L(G_1) = 2$*;*
$G_2 := [[1],[1,2]]$ *and* $\rho_L(G_2) = 4.302775+$*;*
$G_3 := [[1],[1,2],[1]]$ *and* $\rho_L(G_3) = 5.236067+$*;*
$G_4 := [[1],[1,2],[1],[0]]$ *and* $\rho_L(G_4) = 5.3817984+$*;*
$G_5 := [[1],[1,2],[1],[0],[0]]$ *and* $\rho_L(G_4) = 5.397488988+$
and so on. For instance $\rho_L(G_{100}) = 5.39999504+$, proving that by continuing the process we obtain some Laplacian limit point in the interval (μ_*, μ^*). This proves that the generalized random Shearer process generates Laplacian limit points which are not achieved by classical approach using caterpillars, that is, $\Omega_1 \subsetneq \Omega$.

It is worth mentioning that, for the same $\mu = 5.4$, we could have made different choices in each step, obtaining a very different sequence. We will compute the first terms using the maximum criteria (as the classical Shearer's method) and leave to the reader to check that these are optimal choices once we assume $h(T_j) \leq 2$. This, however, result that is worse in terms of approximation to $\mu = 5.4$:
Take $T_1 := [2,2,2]$ *because* $S_1 = 1 - \mu - \delta([2,2,2],\mu) = -0.8843+ < -0.3252+ = \theta^{-1}$*;*
Take $T_2 := [0]$ *because* $S_2 = \psi(S_1) - \delta([0],\mu) = -1.1994+ < -0.3252+ = \theta^{-1}$*;*
Take $T_3 := [2]$ *because* $S_3 = \psi(S_2) - \delta([2],\mu) = -1.2511+ < -0.3252+ = \theta^{-1}$*;*
Take $T_4 := [2]$ *because* $S_4 = \psi(S_3) - \delta([2],\mu) = -1.2855+ < -0.3252+ = \theta^{-1}$ *and so on.*
Thus we have the sequence $\mathbb{T} := ([2,2,2],[0],[2],[2],[2]^\infty)$. By a direct computation we obtain
$G_1 := [[2,2,2]]$ *and* $\rho_L(G_1) = 4.4142+$
$G_2 := [[2,2,2],[0]]$ *and* $\rho_L(G_2) = 5.2360+$
$G_3 := [[2,2,2],[0],[2]]$ *and* $\rho_L(G_3) = 5.3105+$
$G_4 := [[2,2,2],[0],[2],[2]]$ *and* $\rho_L(G_4) = 5.3325+$
$G_5 := [[2,2,2],[0],[2],[2],[2]]$ *and* $\rho_L(G_4) = 5.3423+$.
Finally, notice that $\rho_L(G_{100}) = 5.3554+ < \mu_*$ indicating that the limit does not reach the interval (μ_*, μ^*) as the previous one!

In some sense, a generalized Shearer sequence $G_k(\mathbb{T}, C)$ is always the realization of a generalized random Shearer process, for a given μ.

Lemma 5.4. *Consider the generalized Shearer sequence $G_k(\mathbb{T}, C)$*

$$\begin{cases} G_1(\mathbb{T}) := [T_1] \\ G_k(\mathbb{T}) := [T_1, T_2, \ldots, T_{k-1}, C_k],\ k \geq 2, \end{cases}$$

where $\mathbb{T} = (T_1, T_2, \ldots) \in \mathcal{L}_\infty$ *and* $C = (C_2, C_3, \ldots) \in \mathcal{L}_\infty$*. Let* $\mu := \lim_{k\to\infty} \rho_L(G_k)$ *and* $\theta := \frac{-(\mu-2)-\sqrt{(\mu-2)^2-4}}{2}$*, then*

$$\begin{cases} S_1 = 1 - \mu + \delta(T_1, \mu) < \theta^{-1}; \\ S_j = \psi(S_{j-1}) + \delta(T_j, \mu) < \theta^{-1}, \ 2 \le j \le k, \end{cases}$$

where $\Pi(G_k, \mu) = (S_1, \ldots, S_k)$ *is obtained from* Diagonalize$(G_k, -\mu)$*. In particular, if* C *is a shift of* $\mathbb{T}$ *then we have a generalized random Shearer process.*

Proof. We know that $\rho_L(G_k)$ is strictly increasing, thus $\rho_L(G_k) < \mu$. Applying Diagonalize$(G_k, -\mu)$ we should obtain only negative values S_j. Let us suppose, by contradiction, that $S_{j_0} > \theta^{-1}$, for some $j_0 \ge 1$, that is, $S_{j_0} \in (\theta^{-1}, 0)$, then

$$S_{j_0+1} = \psi(S_{j_0}) + \delta(T_{j_0+1}, \mu) \ge \psi(S_{j_0}) > S_{j_0}.$$

If $S_{j_0+1} > 0$ we have a contradiction, otherwise $S_{j_0+1} \in (\theta^{-1}, 0)$ and

$$S_{j_0+2} = \psi(S_{j_0+1}) + \delta(T_{j_0+2}, \mu) \ge \psi^2(S_{j_0}).$$

In this way, after m steps not reaching a contradiction we obtain

$$S_{j_0+m} \ge \psi^m(S_{j_0}).$$

As ψ is uniformly expanding in the interval $(\theta^{-1}, 0)$ (actually $\psi'(t) > \psi'(\theta^{-1}) = \frac{1}{(\theta^{-1})^2} = \theta^2 > 1$) we must have $S_{j_0+m} > 0$ for a large enough iterate m (see Figure 8), a contradiction with $S_j < 0, j \ge 1$.

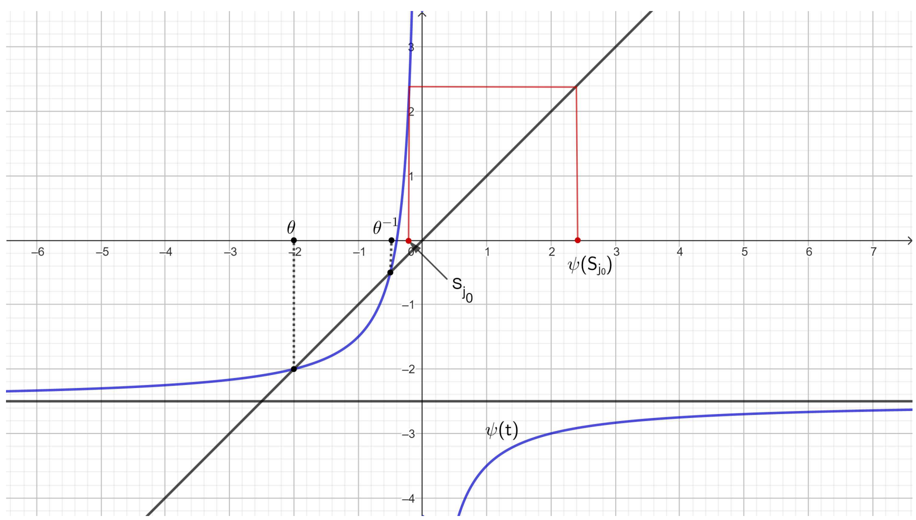

FIGURE 8. Representation of ψ. In this case $S_{j_0+m} > 0$ for $m = 1$.

□

6. Density of Laplacian limit points

The main goal of this section is to investigate, for a generalized Shearer sequence $G_k(\mathbb{T}, C)$

$$\begin{cases} G_1(\mathbb{T}) := [T_1] \\ G_k(\mathbb{T}) := [T_1, T_2, ..., T_{k-1}, C_k], \ k \ge 2, \end{cases}$$

where $\mathbb{T} = (T_1, T_2, ...) \in \mathcal{L}_\infty$ and $C = (C_2, C_3, ...) \in \mathcal{L}_\infty$, the Laplacian limit points $\mu := \lim_{k\to\infty} \rho_L(G_k)$, which we denoted by $\mathcal{S}$ (see Definition 4.7).

To justify our choices, we start by pointing out that for a starlike $T_j \in \mathbb{T}$, its position, that is dependent on the index j, affects $\rho_L(G_k)$ in different ways. In fact, if $j = 1$ then $\deg(v_1) = 1 + \omega(T_j)$ and $\rho_L(G_k) \ge 2 + \omega(T_j)$. However, if $j > 1$ then $\deg(v_j) = 2 + \omega(T_j)$ and $\rho_L(G_k) \ge 3 + \omega(T_j)$. Thus, in order to stay within an interval $[m, m+1]$ we must control $\omega(T_j)$ across the entire sequence. For instance for $\mathbb{T} = ([1, 1], T_2, ...) \in \mathcal{L}_\infty$ we can have limit points in $[4, 5]$ but we need to require $\omega(T_j) \le 1$ for $j \ge 2$, otherwise all the limit points will be in $[5, \infty)$. In other words every G_k must be a quipu!

Another important remark is that if $\mathbb{T} = (T_1, T_2, ...) \in \mathcal{L}_\infty$ is constant or pre-periodic, we can always compute explicitly the limit point, often being an algebraic number. Recall that we already show that for $\mathbb{T} = ([1,1,1],[1]^\infty,\ldots) \in \mathcal{L}_\infty$ and $C = ([1,1],[1,1]^\infty,\ldots) \in \mathcal{L}_\infty$ we obtain

$$\lim_{k\to\infty} \rho_L(G_k) = \mu_*$$

where $\mu_* = \frac{5+\sqrt{33}}{2}(= 5.3722+)$ is the larger root of the polynomial $x^2 - 5x - 2$.

Other remarkable case is the sequence $\mathbb{T} = ([0]^\infty, \ldots) \in \mathcal{L}_\infty$ and

$$C = ([1,1],[1,2],[1,3],\ldots) \in \mathcal{L}_\infty.$$

A close inspection shows that
$G_1 := [[0]]$;
$G_2 := [[0],[1,1]] = [1,1,1]$;
$G_3 := [[0],[0],[1,2]] = [1,2,2]$
and so on. Thus $G_k := [[0]^{k-1},[1,k-1]] = [1,k-1,k-1]$. Obviously G_k satisfy the subgraph condition constituting a generalized Shearer sequence. By [10], Theorem 4.2, we know that

$$\lim_{k\to\infty} \rho_L(G_k) = 2 + \epsilon = 4.382975+$$

where ϵ is the real root of $x^3 - 4x - 4$. It is worth mentioning that the convergence is quite fast, for instance $\rho_L([[0],[0],[0],[0],[0],[0],[0],[0],[1,8]]) = 4.3829331+$ is already 10^{-4} close to the actual value!

Example 6.1. *Motivated by the above considerations we may take a closer look at the following family of generalized Shearer sequences:*

$$\mathcal{F}_1 := \{G_k(\mathbb{T},C) \mid T_1 = [1,1,1],\ T_j \in \{[0],[1],[1,1]\}, j \geq 2, \quad \textit{and } C \textit{ is a shift of } \mathbb{T}\} \tag{16}$$

As we observed before $\lim_{k\to\infty} \rho_L(G_k) > 5$ *for any* $G_k \in \mathcal{F}_1$*, thus the question is the distribution of these limit points in the interval* $[5, 5+m]$, $m \geq 1$.

To give some idea on that distribution we run a numerical experiment sampling 3000 sequences for $G_k \in \mathcal{F}_1$ *and plot, in the vertical coordinate the value* $\rho_L(G_{100})$ *in Figure 9.*

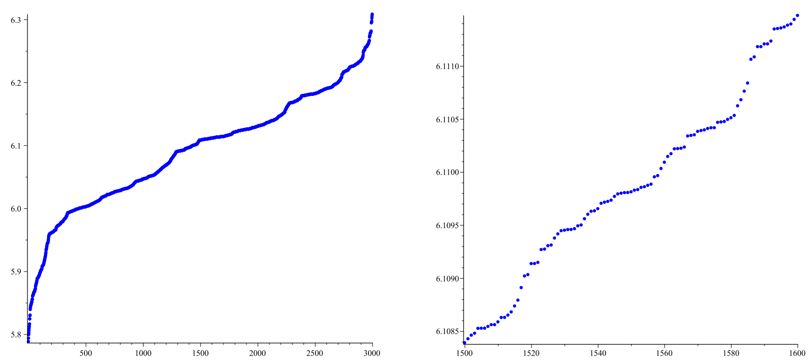

FIGURE 9. Limit points distribution (in the left) and a zoom of it (in the right).

Figure 9 suggests an abundance of limit points, but we need a rigorous study to quantify it. In the experiment $\{[0],[1],[1,1]\}$ *were sorted with equal probability, which may affect the variability (see the zoom on the right of Figure 9). The smallest gap is* 1.4×10^{-8} *while the largest is* $0.01+$. *We limit in 3000 the number of samples, because larger simulations are computationally prohibitive and pointless to establish density results.*

In order to compare different limit points, we start with some general properties regarding the perturbation of a generalized Shearer sequence. In the next result we will use a truncation process, where a sequence $\mathbb{T} = (T_1, T_2, T_3, \ldots)$ is replaced by $\tilde{\mathbb{T}} = (T_1, T_2, T_3, \ldots, T_{k_0}, [0]^\infty) \in \mathcal{L}_\infty$ always having a smaller limit point because, after k_0 there is no drift in the recurrence relation used to compute Diagonalize$(\hat{G}_k, -\mu_0)$ for $\lim_{k\to\infty} \rho_L(G_k) = \mu_0$. Then, $S_{k_0} < \theta^{-1}$ means that $S_{k_0+i} = \psi(S_{k_0}) < \psi(\theta^{-1}) = \theta^{-1}$, for all $i \geq 1$. Perhaps more interesting is the fact that this truncation process provides a set of algebraic limit points that is dense in a subset of $\mathcal{S}$ (see Definition 5.1).

Theorem 6.2. *Let $G_k(\mathbb{T}, C)$ be a generalized Shearer sequence where $C = (T_2, T_3, \ldots)$ is a shift of $\mathbb{T} = (T_1, T_2, T_3, \ldots)$ and $\lim_{k\to\infty} \rho_L(G_k) = \mu_0 \in \mathcal{S}$. Consider $\varepsilon > 0$ and $k_\varepsilon \geq 1$ such that $\rho_L(G_{k_\epsilon}) > \mu_0 - \varepsilon$.*

(a) *If $\hat{\mathbb{T}} = (T_1, T_2, T_3, \ldots, T_{j_0-1}, \hat{T}_{j_0}, T_{j_0+1}, \ldots)$, defines the sequence $\hat{G}_k := G_k(\hat{\mathbb{T}})$ and $\delta(\hat{T}_{j_0}, \mu_0) > \delta(T_{j_0}, \mu_0)$ then $\lim_{k\to\infty} \rho_L(\hat{G}_k) \geq \mu_0$;*

(b) *If $\tilde{\mathbb{T}} = (T_1, T_2, T_3, \ldots, T_{k_\varepsilon}, [0]^\infty)$, defines the sequence $\tilde{G}_k := G_k(\tilde{\mathbb{T}})$ and*
$$\lim_{k\to\infty} \rho_L(\tilde{G}_k) = \mu_1$$
then $\mu_1 \in (\mu_0 - \varepsilon, \mu_0]$;

(c) *The set $\mathcal{A}$ of Laplacian limit points of sequences defined by*
$$\tilde{\mathbb{T}} = (T_1, T_2, T_3, \ldots, T_{k_0}, [0]^\infty) \in \mathcal{L}_\infty$$
is algebraic;

(d) *The set $\mathcal{A}$ is dense in the set $\mathcal{B}$ of Laplacian limit points of sequences defined by $\mathbb{T} = (T_1, T_2, \ldots) \in \mathcal{L}_\infty$ and $C = (T_2, T_3, \ldots) \in \mathcal{L}_\infty$.*

Proof. (a) Since $\hat{G}_k = G_k$ for $k < j_0$, we may assume $k > j_0$. This time we will apply Diagonalize$(G_k, -\mu)$ and Diagonalize$(\hat{G}_k, \mu)$ with $\mu := \rho_L(G_k)$, take v_{j_0} as root, $f_{G_k}(v_{j_0}) = 0$ the final value in G_k, and $f_{\hat{G}_k}(v_{j_0})$ the final value in $\hat{G}_k$. By Lemma 2.1, we know that $\rho_L(\hat{G}_k) > \rho_L(G_k)$ if $f_{\hat{G}_k}(v_{j_0}) > 0$.

By direct computation, we obtain
$$0 = f_{G_k}(v_{j_0}) = 2 - \mu - \frac{1}{f_{G_k}(v_{j_0-1})} - \frac{1}{f_{G_k}(v_{j_0+1})} + \delta(T_{j_0}, \mu),$$
or
$$2 - \mu - \frac{1}{f_{G_k}(v_{j_0-1})} - \frac{1}{f_{G_k}(v_{j_0+1})} = -\delta(T_{j_0}, \mu).$$
On the other hand, using the above equation we get
$$f_{\hat{G}_k}(v_{j_0}) = 2 - \mu - \frac{1}{f_{G_k}(v_{j_0-1})} - \frac{1}{f_{G_k}(v_{j_0+1})} + \delta(\hat{T}_{j_0}, \mu) = \delta(\hat{T}_{j_0}, \mu) - \delta(T_{j_0}, \mu) > 0,$$
by hypothesis.

(b) The easy part is to prove that $\mu_1 > \mu_0 - \varepsilon$. We just need to observe that, by hypothesis, $\rho_L(G_{k_\varepsilon}) > \mu_0 - \varepsilon$ and for $k > k_\varepsilon$ we have $\tilde{G}_k = [T_1, T_2, T_3, \ldots, T_{k_\varepsilon}, [0]^{k-k_\varepsilon}]$, meaning that, G_{k_ε} is a subgraph of $\tilde{G}_k$. Thus, $\mu_1 = \lim_{k\to\infty} \rho_L(\tilde{G}_k) \geq \rho_L(G_{k_\varepsilon}) > \mu_0 - \varepsilon$.

In order to see that $\mu_1 \leq \mu_0$ we need a more sophisticated argument. Again we apply Diagonalize$(G_k, -\mu_0)$ and Diagonalize$(\hat{G}_k, -\mu_0)$ taking v_k as root in both graphs. Let $S_1, \ldots, S_{k-1}, S_k$ the output of Diagonalize$(G_k, -\mu_0)$ at the vertices $v_1, \ldots, v_{k-1}, v_k$ and $\tilde{S}_1, \ldots, \tilde{S}_{k-1}, \tilde{S}_k$ the output of Diagonalize$(\tilde{G}_k, -\mu_0)$ at the vertices $v_1, \ldots, v_{k-1}, v_k$.

From Lemma 5.4 we know that if $\lim_{k\to\infty} \rho_L(G_k) = \mu_0$ and $\theta_0 := \frac{-(\mu_0-2)-\sqrt{(\mu_0-2)^2-4}}{2}$, then
$$\begin{cases} S_1 = 1 - \mu_0 + \delta(T_1, \mu_0) < \theta_0^{-1}; \\ \psi(S_{j-1}) + \delta(T_j, \mu_0) < \theta_0^{-1},\ 2 \leq j \leq k. \end{cases}$$
Assuming $k > k_\varepsilon$ (if $k \leq k_\varepsilon$ then $\rho_L(\tilde{G}_k) = \rho_L(G_k)$), we have $\tilde{S}_j = S_j < \theta_0^{-1} < 0,\ 1 \leq j \leq k_\varepsilon$. For the next index, we obtain
$$S_{k_\varepsilon+1} = \psi(S_{k_\varepsilon}) + \delta(T_{k_\varepsilon+1}, \mu_0) < \theta_0^{-1}$$
and
$$\tilde{S}_{k_\varepsilon+1} = \psi(S_{k_\varepsilon}) + \delta([0], \mu_0) = \psi(S_{k_\varepsilon}) < \theta_0^{-1}.$$
Proceeding in this way we obtain
$$\tilde{S}_{k_\varepsilon+m} = \psi(S_{k_\varepsilon+m-1}) + \delta([0], \mu_0) = \psi^{m-1}(S_{k_\varepsilon}) < \theta_0^{-1}.$$
In particular, as $\psi(S_{k-1}) < \theta_0^{-1}$ implies that $\tilde{S}_k = -1 + \psi(S_{k-1}) < -1 + \theta_0^{-1} < 0$. Thus $\rho_L(\tilde{G}_k) < \mu_0$ for $k \geq k_\varepsilon$ and $\mu_1 = \lim_{k\to\infty} \rho_L(\tilde{G}_k) \leq \mu_0$.

(c) Take $\mu \in \mathcal{A}$, then there exists $\mu_k := \rho_L(\tilde{G}_k)$, defined by $\tilde{\mathbb{T}} = (T_1, T_2, T_3, ..., T_{k_0}, [0]^\infty) \in \mathcal{L}_\infty$ such that $\mu = \lim_{k\to\infty} \rho_L(\tilde{G}_k)$.

Consider $S_1,\dots,S_{k-1},S_k$ the output of Diagonalize$(\tilde{G}_k,-\mu_k)$ at the vertices $v_1,\dots,v_{k-1},v_k$. We know that $S_1,\dots,S_{k_0-1},S_{k_0}$ are negative rational functions with respect to the variable μ_k. Now, $S_{k_0+1}=\psi(S_{k_0})+\delta([0],\mu_k)=\psi(S_{k_0})$, $S_{k_0+2}=\psi(S_{k_0+1})+\delta([0],\mu_k)=\psi(S_{k_0+1})$ until we reach $S_k=-1+\psi(S_{k-1})+\delta([0],\mu_k)=-1+\psi(S_{k-1})=0$, by construction, or equivalently μ_k must solve the equation

$$\psi(S_{k-1})=1.$$

Defining $a_1=S_{k_0}$, $a_2=S_{k_0+1}=\psi(a_1)$, $\dots$, $a_{k-k_0}=S_{k_0+(k-k_0)}=\psi(S_{k-1})$, where $\psi(t)=2-\mu_k-\frac{1}{t}, t\neq 0$. From [10] we know that $\Delta=(2-\mu_k)^2-4>0$ thus

$$a_j:=\theta_k+\frac{\theta_k^{-1}-\theta_k}{\beta_k\theta_k^{2(j-1)}+1},\ 1\leq j,$$

where $\beta_k:=\frac{\theta_k^{-1}-\theta_k}{a_1-\theta_k}-1$ and $\theta_k=\frac{(2-\mu_k)-\sqrt{(2-\mu_k)^2-4}}{2}$.

Now we are able to rewrite the equation $\psi(S_{k-1})=1$ as $a_{k-k_0}=1$ or

$$\theta_k+\frac{\theta_k^{-1}-\theta_k}{\beta_k\theta_k^{2(k-k_0-1)}+1}=1$$

$$\theta_k^{-1}-\theta_k=(1-\theta_k)\left(\beta_k\theta_k^{2(k-k_0-1)}+1\right)$$

$$\theta_k^{-1}-\theta_k=(1-\theta_k)\left(\left(\frac{\theta_k^{-1}-\theta_k}{S_{k_0}-\theta_k}-1\right)\theta_k^{2(k-k_0-1)}+1\right)$$

$$(\theta_k^{-1}-\theta_k)(S_{k_0}-\theta_k)=(1-\theta_k)\left(\left(\theta_k^{-1}-S_{k_0}\right)\theta_k^{2(k-k_0-1)}+S_{k_0}-\theta_k\right)$$

$$\frac{(\theta_k^{-1}-\theta_k)(S_{k_0}-\theta_k)}{\theta_k^{2(k-k_0-1)}}=(1-\theta_k)\left(\left(\theta_k^{-1}-S_{k_0}\right)+\frac{S_{k_0}-\theta_k}{\theta_k^{2(k-k_0-1)}}\right).$$

Notice that $(\theta_k^{-1}-\theta_k)=\sqrt{(2-\mu_k)^2-4}\not\to 0$ when $k\to\infty$. Thus, or $S_{k_0}-\theta_k\to 0$ when $k\to\infty$ or taking the limit in the above equation we obtain $\theta_k^{-1}-S_{k_0}\to 0$ when $k\to\infty$. Thus $\mu=\lim\limits_{k\to\infty}\rho_L(\tilde{G}_k)$ must solve one of the equations

$$\theta^{-1}=S_{k_0} \text{ or } \theta=S_{k_0}.$$

All in all, as $S_{k_0}=\frac{P(\mu)}{Q(\mu)}$ we conclude that μ must solve

$$\frac{(2-\mu)\pm\sqrt{(2-\mu)^2-4}}{2}=\frac{P(\mu)}{Q(\mu)}$$

$$\pm\sqrt{(2-\mu)^2-4}=\frac{2P(\mu)}{Q(\mu)}-(2-\mu)$$

$$(2-\mu)^2-4=\left(\frac{2P(\mu)}{Q(\mu)}-(2-\mu)\right)^2$$

$$(2-\mu)^2-4=4\left(\frac{P(\mu)}{Q(\mu)}\right)^2-4(2-\mu)\left(\frac{P(\mu)}{Q(\mu)}\right)+(2-\mu)^2$$

$$-4=4\left(\frac{P(\mu)}{Q(\mu)}\right)^2-4(2-\mu)\left(\frac{P(\mu)}{Q(\mu)}\right)$$

$$\left(\frac{P(\mu)}{Q(\mu)}\right)^2-(2-\mu)\left(\frac{P(\mu)}{Q(\mu)}\right)+1=0$$

$$P^2(\mu)-(2-\mu)\left(P(\mu)Q(\mu)\right)+Q^2(\mu)=0.$$

After a simplification we conclude that μ is a real root of the polynomial at the numerator, meaning that μ is an algebraic number.

(d) Consider $\mu_0\in\mathcal{B}$ and $\varepsilon>0$ arbitrarily small. By (b) we can find k_ε such that $\tilde{\mathbb{T}}_\varepsilon=(T_1,T_2,T_3,...,T_{k_\varepsilon},[0]^\infty)$ defines a sequence $\tilde{G}_k:=G_k(\tilde{\mathbb{T}}_\varepsilon)$ such that $\lim\limits_{k\to\infty}\rho_L(\tilde{G}_k)=\mu_1\in(\mu_0-\varepsilon,\mu_0]$. As $\mu_1\in\mathcal{A}$, this proves the density. $\square$

The next general result is on the set of possible limit points *dominated* by μ.

Definition 6.3. *Let $\mu \geq 4.38+$ be a fixed real number. A generalized Shearer sequence $G_k(\mathbb{T}, C)$ is dominated by μ if $\lim_{k\to\infty} \rho_L(G_k) \leq \mu$. We denote by $\Gamma(\mu)$ the set of all generalized Shearer sequences $G_k(\mathbb{T}, C)$ which are dominated by μ.*

We notice that if $\Gamma(\mu)$ is known, this means we completely understand the approximation of μ by Laplacian limit points originated by linear trees. In other words, if

$$\sup_{G_k\in\Gamma(\mu)} \lim_{k\to\infty} \rho_L(G_k) < \mu$$

then either μ is not a Laplacian limit point or the approximation must be taken through a wider class such as general trees or general graphs. On the other hand if

$$\sup_{G_k\in\Gamma(\mu)} \lim_{k\to\infty} \rho_L(G_k) = \mu$$

then μ is certainly a Laplacian limit point, but we do not know whether $\mu \in \mathcal{S}$ that is, whether the supremum is attained by a sequence of linear trees.

Lemma 6.4. *Let $\mu \geq 4.38+$ be a fixed real number and let $G_k(\mathbb{T}, C)$ be a generalized Shearer sequence where $C = (T_2, T_3, \ldots)$ is a shift of $\mathbb{T} = (T_1, T_2, T_3, \ldots)$.*

(a) *If $G_k(\mathbb{T}, C) \in \Gamma(\mu)$ then*
$$\omega(T_j) \leq \delta(T_j, \mu) < \mu,$$
for all $j \geq 1$. The same is true for C, that is, $\omega(C_k) \leq \delta(C_k, \mu) < \mu$;

(b) *Consider $G_k(\mathbb{T}, C) \in \Gamma(\mu)$ and a new sequence*
$$\tilde{\mathbb{T}} = (T_1, T_2, T_3, \ldots, T_{j_0-1}, \tilde{T}_{j_0}, T_{j_0+1}, \ldots)$$
defining $\tilde{G}_k := G_k(\tilde{\mathbb{T}})$. If $\delta(\tilde{T}_{j_0}, \mu) < \delta(T_{j_0}, \mu)$ then $\tilde{G}_k \in \Gamma(\mu)$;

(c) *Consider $G_k(\mathbb{T}, C) \in \Gamma(\mu)$ and starlike trees $\hat{T}_{j_0}, \hat{T}_{j_0+1}$ such that $\delta(\hat{T}_{j_0}, \mu) \leq \delta(T_{j_0}, \mu)$ and $\theta^{-1} - \psi(\hat{S}_{j_0}) > \delta(\hat{T}_{j_0+1}, \mu) \geq \delta(T_{j_0+1}, \mu)$. There exist some $m \in \mathbb{N}$ and starlike trees $\hat{T}_{j_0+2}, \ldots, \hat{T}_{j_0+m}$ such that, defining*
$$\hat{\mathbb{T}} = (T_1, T_2, T_3, \ldots, T_{j_0-1}, \hat{T}_{j_0}, \hat{T}_{j_0+1}, \ldots, \hat{T}_{j_0+m}, T_{j_0+m+1}, \ldots),$$
we obtain a generalized Shearer sequence $\hat{G}_k := G_k(\hat{\mathbb{T}}, C)$ with $\hat{G}_k \in \Gamma(\mu)$.

Proof. (a) Recall that $\Pi(G_k, \mu) = (S_1, S_2, \ldots, S_k)$ where

$$\begin{cases} S_1 = 1 - \mu + \delta(T_1, \mu), \\ S_j = \psi(S_{j-1}) + \delta(T_j, \mu),\ 2 \leq j \leq k-1, \\ S_k = -1 + \psi(S_{k-1}) + \delta(C_k, \mu). \end{cases}$$

As $G_k(\mathbb{T}, C) \in \Gamma(\mu)$, we have from Lemma 5.4 that

$$1 - \mu + \delta(T_1, \mu) < \theta^{-1},$$

but

$$\theta^{-1} - (1-\mu) = \frac{2 - \mu + \sqrt{(2-\mu)^2 - 4}}{2} - (1-\mu) =$$
$$= \frac{2 - \mu - 2(1-\mu) + \sqrt{(2-\mu)^2 - 4}}{2} = \frac{\mu + \sqrt{(2-\mu)^2 - 4}}{2}$$

thus

$$\omega(T_1) \leq \delta(T_1, \mu) < \frac{\mu + \sqrt{(2-\mu)^2 - 4}}{2}.$$

By construction, $S_j > S_1 > 1 - \mu$ for all j and $S_j < \theta^{-1} \Rightarrow \psi(S_{j-1}) < \theta^{-1}$ thus

$$1 - \mu + \delta(T_j, \mu) < S_j = \psi(S_{j-1}) + \delta(T_j, \mu) < \theta^{-1}$$

$$\omega(T_j) \leq \delta(T_j, \mu) < \theta^{-1} - (1-\mu) = \frac{\mu + \sqrt{(2-\mu)^2 - 4}}{2}.$$

Notice that

$$\frac{\mu + \sqrt{(2-\mu)^2 - 4}}{2} = \frac{\mu}{2}\left(1 + \sqrt{\left(\frac{2}{\mu} - 1\right)^2 - \left(\frac{2}{\mu}\right)^2}\right) =$$

$$= \frac{\mu}{2}\left(1+\sqrt{\left(\frac{2}{\mu}-1+\frac{2}{\mu}\right)\left(\frac{2}{\mu}-1-\frac{2}{\mu}\right)}\right) = \frac{\mu}{2}\left(1+\sqrt{\left(1-\frac{4}{\mu}\right)}\right) \leq \frac{\mu}{2}2 = \mu,$$

because $\frac{4}{\mu} < 1$.

Finally, $S_k = -1+\psi(S_{k-1})+\delta(C_k,\mu) < 0$ and $1-\mu < S_{k-1} < \theta^{-1} \Rightarrow 1-\mu < \psi(S_{k-1}) < \theta^{-1}$ thus $-\mu < -1+\psi(S_{k-1}) < -1+\theta^{-1}$. So,

$$-1+\psi(S_{k-1})+\delta(C_k,\mu) < 0$$
$$\delta(C_k,\mu) < -(-1+\psi(S_{k-1})) < \mu$$
$$\omega(C_k) \leq \delta(C_k,\mu) < -(-1+\psi(S_{k-1})) < \mu.$$

(b) Consider $\Pi(G_k,\mu) = (S_1, S_2, \ldots, S_k)$ where (by Lemma 5.4)

$$\begin{cases} S_1 = 1-\mu+\delta(T_1,\mu) < \theta^{-1}, \\ S_j = \psi(S_{j-1})+\delta(T_j,\mu) < \theta^{-1},\ 2 \leq j \leq k-1, \\ S_k = -1+\psi(S_{k-1})+\delta(C_k,\mu) < 0. \end{cases}$$

Now we compute $\Pi := (\tilde{G}_k,\mu) \mapsto \left(\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_k\right)$. We know that $\tilde{S}_j = S_j < \theta^{-1} < 0$ for $1 \leq j_0 - 1$ and

$$\tilde{S}_{j_0} = \psi(S_{j_0-1})+\delta(\tilde{T}_{j_0},\mu)-\delta(T_{j_0},\mu)+\delta(T_{j_0},\mu) =$$
$$= S_{j_0}+\delta(\tilde{T}_{j_0},\mu)-\delta(T_{j_0},\mu) < S_{j_0}+0 < \theta^{-1} < 0.$$

Since $\tilde{S}_{j_0} < S_{j_0}$, $\tilde{T}_{j_0+m} = T_{j_0+m}$, $\forall m \geq 1$ and because ψ is order preserving, we have $\tilde{S}_{j_0+m} < S_{j_0+m} < \theta^{-1} < 0$,

$$\tilde{S}_{j_0+m} < S_{j_0+m} < \theta^{-1}$$
$$\psi(\tilde{S}_{j_0+m}) < \psi(S_{j_0+m}) < \psi(\theta^{-1})$$
$$\psi(\tilde{S}_{j_0+m})+0 < \psi(S_{j_0+m})+0 < \theta^{-1}$$
$$\tilde{S}_{j_0+(m+1)} < S_{j_0+(m+1)} < \theta^{-1}.$$

In particular $\tilde{S}_{j_0+1} < S_{j_0+1} < \theta^{-1} < 0$. This concludes the proof because $\mathrm{sign}(\Pi) := (\tilde{G}_k,\mu) = (-,-,\ldots,-)$ thus $\rho_L(\tilde{G}_k) < \mu$ and $\lim_{k\to\infty} \rho_L(\tilde{G}_k) \leq \mu$ or $\tilde{G}_k \in \Gamma(\mu)$.

(c) We have $\Pi(G_k,\mu) = (S_1, S_2, \ldots, S_k)$ where

$$\begin{cases} S_1 = 1-\mu+\delta(T_1,\mu) < \theta^{-1}, \\ S_j = \psi(S_{j-1})+\delta(T_j,\mu) < \theta^{-1},\ 2 \leq j \leq k-1, \\ S_k = -1+\psi(S_{k-1})+\delta(C_k,\mu) < 0. \end{cases}$$

At the same time we have $\Pi(\hat{G}_k,\mu) = \left(\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_k\right)$. For $1 \leq j_0 - 1$ we have $\hat{S}_j = S_j < \theta^{-1}$ and for $\hat{S}_{j_0}$ we have

$$S_{j_0} = \psi(S_{j_0-1})+\delta(T_{j_0},\mu) < \theta^{-1}$$

and by hypothesis

$$\delta(\hat{T}_{j_0},\mu) < \delta(T_{j_0},\mu) < \theta^{-1}-\psi(S_{j_0-1}) \tag{17}$$

On the other hand

$$\hat{S}_{j_0} = \psi(S_{j_0-1})+\delta(\hat{T}_{j_0},\mu) < S_{j_0}$$
$$\psi(\hat{S}_{j_0}) < \psi(S_{j_0})$$
$$\theta^{-1}-\psi(\hat{S}_{j_0}) > \theta^{-1}-\psi(S_{j_0}).$$

Using this inequality in Equation (17) we obtain

$$\delta(\hat{T}_{j_0},\mu) < \delta(T_{j_0},\mu) < \theta^{-1}-\psi(S_{j_0-1}) < \theta^{-1}-\psi(\hat{S}_{j_0})$$
$$\hat{S}_{j_0} = \psi(\hat{S}_{j_0})+\delta(\hat{T}_{j_0},\mu) < \theta^{-1}.$$

The next is $\hat{S}_{j_0+1} = \psi(\hat{S}_{j_0})+\delta(\hat{T}_{j_0+1},\mu)$. In order to have $\hat{S}_{j_0+1} < \theta^{-1}$ we need $\delta(\hat{T}_{j_0+1},\mu) < \theta^{-1}-\psi(\hat{S}_{j_0})$.

Notice that

$$\theta^{-1}-\psi(\hat{S}_{j_0}) > \theta^{-1}-\psi(S_{j_0}) > \delta(T_{j_0+1},\mu).$$

It is always possible $\theta^{-1}-\psi(\hat{S}_{j_0}) > \delta(\hat{T}_{j_0+1},\mu)$ (or equivalently $\hat{S}_{j_0+1} < \theta^{-1}$) provided that

$$\theta^{-1}-\psi(\hat{S}_{j_0}) > \delta(\hat{T}_{j_0+1},\mu) \geq \delta(T_{j_0+1},\mu),$$

in the worst case one may take $\hat{T}_{j_0+1} = T_{j_0+1}$.

Assuming our hypothesis, we know that $\hat{S}_{j_0+1} < \theta^{-1}$ we may take, in the worst case, $\hat{T}_{j_0+2} = [0]$, $\hat{T}_{j_0+3} = [0]$, ..., $\hat{T}_{j_0+m+1} = [0]$. This means that $\hat{S}_{j_0+2} = \psi(\hat{S}_{j_0+1}) < \theta^{-1}$, $\hat{S}_{j_0+3} = \psi^2(\hat{S}_{j_0+1}) < \theta^{-1}$, ..., $\hat{S}_{j_0+m+1} = \psi^m(\hat{S}_{j_0+1}) < \theta^{-1}$. As m increases $\hat{S}_{j_0+m+1} = \psi^m(\hat{S}_{j_0+1}) \to \theta < \theta^{-1}$. Thus, unless the original sequence is formed only by $[0]$'s after T_{j_0+2} will exist such a m in the way that $\hat{S}_{j_0+m+1} < S_{j_0+m+1} < \theta^{-1}$ for all the next indices. Finally we conclude that $\operatorname{sign}(\Pi)(\hat{G}_k, \mu) = (-, -, \ldots, -)$ for all k thus $\hat{G}_k \in \Gamma(\mu)$.

□

Remark 6.5. *In item (c) of Lemma 6.4 we can take $\hat{T}_{j_0} = T_{j_0}$ and the proof would still be valid. At the same time, taking $\delta(\hat{T}_{j_0}, \mu) < \delta(T_{j_0}, \mu)$, we will be able to choose $\hat{T}_{j_0+1}$ in a wider range. Moreover, one can take $\hat{T}_{j_0+1}$ in such a way that we maximize the quantity $\delta(\hat{T}_{j_0+1}, \mu)$ in the range*

$$[\delta(T_{j_0+1}, \mu),\ \theta^{-1} - \psi(\hat{S}_{j_0})).$$

Finally, instead of taking the easier choice $\hat{T}_{j_0+2} = [0]$, $\hat{T}_{j_0+3} = [0]$, $\ldots$, $\hat{T}_{j_0+m+1} = [0]$, one can simply take

$$\delta(\hat{T}_{j_0+m+1}, \mu) < \delta(T_{j_0+m+1}, \mu),\ m \geq 1,$$

with m such that $\hat{S}_{j_0+m+1} < S_{j_0+m+1}$ may be reached much later.

7. Variational technique

The goal of this section is to investigate, for a generalized Shearer sequence $G_k(\mathbb{T}, C)$

$$\begin{cases} G_1(\mathbb{T}) := [T_1] \\ G_k(\mathbb{T}) := [T_1, T_2, ..., T_{k-1}, C_k],\ k \geq 2, \end{cases}$$

where $\mathbb{T} = (T_1, T_2, ...) \in \mathcal{L}_\infty$ and $C = (C_2, C_3, ...) \in \mathcal{L}_\infty$, when the Laplacian limit points $\mu_0 := \lim\limits_{k\to\infty} \rho_L(G_k)$ is close to a given number $\mu \geq 4.38+$.

In order to do that, we assume $G_k \in \Gamma(\mu)$ and consider a variation $\mu_\varepsilon := \mu - \varepsilon$. We ask whether $\mu_0 > \mu_\varepsilon$. We already know that $\mu_0 > \mu_\varepsilon$ if, and only if, there exists some $k \in \mathbb{N}$ such that the vector

$$\operatorname{sign}(\Pi)(G_k, \mu_\varepsilon)$$

has some positive entry, where $\Pi(G_k, \mu_\varepsilon) = (S_1(\varepsilon), S_2(\varepsilon), \ldots, S_k(\varepsilon))$ is given by

$$\begin{cases} S_1(\varepsilon) = 1 - \mu_\varepsilon + \delta(T_1, \mu_\varepsilon), \\ S_j(\varepsilon) = \psi_\varepsilon(S_{j-1}(\varepsilon)) + \delta(T_j, \mu_\varepsilon),\ 2 \leq j \leq k-1, \\ S_k(\varepsilon) = -1 + \psi_\varepsilon(S_{k-1}(\varepsilon)) + \delta(C_k, \mu_\varepsilon). \end{cases}$$

We need to take a closer look at some terms at the above formula:

- The function ψ_ε is given by $\psi_\varepsilon(t) = 2 - \mu_\varepsilon - \frac{1}{t} = 2 - \mu - \frac{1}{t} + \varepsilon = \psi(t) + \varepsilon$. Of course, $\psi_0(t) = \psi(t)$.
- The processing of each path in T_i satisfies the formula

$$\begin{cases} b_1(\varepsilon) = 1 - \mu_\varepsilon \\ b_{j+1}(\varepsilon) = \psi_\varepsilon(b_j(\varepsilon)), \quad j > 1. \end{cases} \tag{18}$$

By a reasoning analogous to the case $\varepsilon = 0$, one can prove that for a small ε, the iterates $b_j(\varepsilon)$ obeying Equation (18) has the following properties. Let θ_ε and θ'_ε be fixed points of $\psi_\varepsilon(t) = t$. They are given by

$$\theta_\varepsilon := \frac{-(\mu_\varepsilon - 2) - \sqrt{(\mu_\varepsilon - 2)^2 - 4}}{2} \text{ and } \theta'_\varepsilon := \theta_\varepsilon^{-1} > \theta_\varepsilon. \tag{19}$$

They are attracting and repelling points, respectively, meaning that for $b_1(\varepsilon) \leq \theta'_\varepsilon$, we have $b_j(\varepsilon)$ converging to θ_ε. Since $\mu > 4.38+$ we have $\mu_\varepsilon > 4$, so that

$$1 - \mu_\varepsilon < b_1(\varepsilon) \leq b_j(\varepsilon) < \theta_\varepsilon < 0$$

and, in particular, that all $b_j(\varepsilon) < 0$.

- The function $\varepsilon \to b_j(\varepsilon)$ is differentiable and $\frac{d\, b_j(\varepsilon)}{d\,\varepsilon} \geq 1$. In particular, $\varepsilon \to b_j(\varepsilon)$ is strictly increasing. Define $b'_j(\varepsilon) := \frac{d\, b_j(\varepsilon)}{d\,\varepsilon}$. Then, the sequence $(b'_j(\varepsilon))$ satisfy the recurrence relation

$$\begin{cases} b'_1(\varepsilon) = 1 \\ b'_j(\varepsilon) = 1 + \frac{1}{(b_{j-1}(\varepsilon))^2} b'_{j-1}(\varepsilon),\ j \geq 2. \end{cases} \tag{20}$$

Indeed, $b'_1(\varepsilon) = \frac{d\,(1-\mu_\varepsilon)}{d\,\varepsilon} = 1 > 0$. For $j > 1$ we have $b'_j(\varepsilon) = \frac{d\,\psi_\varepsilon(b_{j-1}(\varepsilon))}{d\,\varepsilon} = 1 + \frac{1}{b_{j-1}(\varepsilon)^2} b'_{j-1}(\varepsilon)$. By induction we get $b'_j(\varepsilon) \geq 1$, for $j \geq 1$.
Additionally, we can also set a upper bound to $b'_j(\varepsilon)$:

$$b_j(\varepsilon) < \theta_\varepsilon < 0 \Rightarrow 0 < \frac{1}{-b_j(\varepsilon)} < \frac{1}{-\theta_\varepsilon} \Rightarrow \frac{1}{(b_j(\varepsilon))^2} < \theta_\varepsilon^{-2}.$$

Thus

$$b'_j(\varepsilon) = 1 + \frac{1}{(b_{j-1}(\varepsilon))^2} b'_{j-1}(\varepsilon) \leq 1 + \theta_\varepsilon^{-2} b'_{j-1}(\varepsilon),$$

which means that $b'_j(\varepsilon) \leq 1 + \theta_\varepsilon^{-2} + \ldots + \theta_\varepsilon^{-2j} \leq \frac{1}{1-\theta_\varepsilon^{-2}} < \infty$.

- The function $\varepsilon \to b_j(\varepsilon)$ is twice differentiable and $\frac{d^2\, b_j(\varepsilon)}{d\,\varepsilon^2} \geq 0$. In particular, if $j > 1$ then $\varepsilon \to b_j(\varepsilon)$ is strictly concave, because $\frac{d\, b_j(\varepsilon)}{d\,\varepsilon}$ is not constant. Define $b''_j(\varepsilon) := \frac{d^2\, b_j(\varepsilon)}{d\,\varepsilon^2}$. Then, the sequence $(b''_j(\varepsilon))$ satisfies the recurrence relation

$$\begin{cases} b''_1(\varepsilon) = 0 \\ b''_j(\varepsilon) = -\frac{2}{b_{j-1}(\varepsilon)^3} \left(b'_j(\varepsilon)\right)^2 + \frac{1}{b_{j-1}(\varepsilon)^2}\, b''_{j-1}(\varepsilon),\ j \geq 2. \end{cases} \tag{21}$$

Indeed, we already computed $b'_1(\varepsilon) = 1$ thus $b''_1(\varepsilon) = \frac{d}{d\,\varepsilon}(1) = 0$.
For $j > 1$ we have

$$b''_j(\varepsilon) = \frac{d}{d\,\varepsilon}\left(1 + \frac{1}{b_{j-1}(\varepsilon)^2} b'_{j-1}(\varepsilon)\right) =$$

$$= -\frac{2}{b_{j-1}(\varepsilon)^3} \left(b'_j(\varepsilon)\right)^2 + \frac{1}{b_{j-1}(\varepsilon)^2}\, b''_{j-1}(\varepsilon).$$

By induction we get $b''_j(\varepsilon) \geq 0$, for $j \geq 1$.

- Let T_j be one starlike tree attached to G_k. Recall that $\omega(T_j)$ denote the number of paths of $T_j = [q_1^j, \ldots, q_{\omega(T_j)}]$ and, for $i = 1, \ldots, \omega(T_j)$, by $b_{q_i^j}$ the last value output by the algorithm Diagonalize$(G_k, -\mu_\varepsilon)$ at the i-th path of T_j. We denote the net result of the starlike T_j by

$$\delta(T_j, \mu_\varepsilon) := \begin{cases} \sum_{i=1}^{\omega(T_j)} \left(1 - \frac{1}{b_{q_i^j}(\varepsilon)}\right), \text{ for non-empty } T_j \\ 0 \text{ if } T_j \text{ is empty.} \end{cases} \tag{22}$$

From the previous items we can define the differential drift, $\delta'(T_j, \mu_\varepsilon) := \frac{d}{d\,\varepsilon}\delta(T_j, \mu_\varepsilon)$ which satisfies the formula for $T_j \neq [0]$

$$\delta'(T_j, \mu_\varepsilon) = \sum_{i=1}^{\omega(T_j)} \frac{1}{\left(b_{q_i^j}(\varepsilon)\right)^2} b'_{q_i^j}(\varepsilon) \geq 0. \tag{23}$$

We can also find an upper bound for $\delta'(T_j, \mu_\varepsilon)$. Recall that $\frac{1}{(b_j(\varepsilon))^2} < \theta_\varepsilon^{-2}$ and $b'_j(\varepsilon) < \frac{1}{1-\theta_\varepsilon^{-2}}$, thus

$$\delta'(T_j, \mu_\varepsilon) = \sum_{i=1}^{\omega(T_j)} \frac{1}{\left(b_{q_i^j}(\varepsilon)\right)^2} b'_{q_i^j}(\varepsilon) \leq \sum_{i=1}^{\omega(T_j)} \theta_\varepsilon^{-2} \frac{1}{1-\theta_\varepsilon^{-2}} = \frac{\omega(T_j)}{\theta_\varepsilon^2 - 1} < \infty$$

because $\omega(T_j) < \infty$ (see Lemma 4.2).

Theorem 7.1. *Let $G_k := G_k(\mathbb{T}, C)$ be a generalized Shearer sequence such that $(G_k) \in \Gamma(\mu)$. Then $\mu := \lim_{k\to\infty} \rho_L(G_k)$ if and only if the sequence (ε_j) converges to zero, where ε_j is the smallest positive solution of $S_j(\varepsilon) = 0$, for $\Pi(G_k, \mu_\varepsilon) = (S_1(\varepsilon), S_2(\varepsilon), \ldots, S_k(\varepsilon))$.*

Proof. First, we will investigate the sign of $S_1(\varepsilon) = 1 - \mu_\varepsilon + \delta(T_1, \mu_\varepsilon)$. In this case,

$$\operatorname{sign}(S_1(\varepsilon)) = + \Leftrightarrow 1 - \mu_\varepsilon + \delta(T_1, \mu_\varepsilon) > 0 \Leftrightarrow \varepsilon > \mu - 1 - \delta(T_1, \mu_\varepsilon).$$

To study the above inequality we define the function $g_1 : [0, \infty) \to \mathbb{R}$ given by

$$g_1(\varepsilon) = S_1(\varepsilon)) = 1 - \mu_\varepsilon + \delta(T_1, \mu_\varepsilon).$$

It is easy to see that $g_1(0) = 1 - \mu + \delta(T_1, \mu) < \theta^{-1} < 0$ $(G_k) \in \Gamma(\mu)$. Also, $g_1(\varepsilon)$ is differentiable in some neighborhood of $\varepsilon = 0$. Indeed, for $T_1 = [q^1_1, \ldots, q^1_{\omega(T_1)}]$ we get

$$\frac{d\, g_1(\varepsilon)}{d\, \varepsilon} = 1 + \delta'(T_1, \mu_\varepsilon) \geq 1.$$

Since $\frac{d\, g_1(\varepsilon)}{d\, \varepsilon} \geq 1$ and $g_1(0) < 0$ we obtain an unique root ε_1 such that $g_1(\varepsilon) < 0$ for $\varepsilon < \varepsilon_1$ and $g_1(\varepsilon) > 0$ for $\varepsilon > \varepsilon_1$. In this way, we conclude that $\operatorname{sign}(S_1(\varepsilon)) = +$ for $\varepsilon > \varepsilon_1$, in other words, $\operatorname{sign}(\Pi)(G_k, \mu_\varepsilon) = (+, ?, \ldots, ?)$, meaning that, $\rho_L(G_k) > \mu_\varepsilon$.

In order to get a better approximation of μ we need to look for the possibility of having $\operatorname{sign}(S_2(\varepsilon)) = +$ for $\varepsilon < \varepsilon_1$. Analogously to the previous case, we define

$$g_2(\varepsilon) = S_2(\varepsilon) = \psi_\varepsilon(S_1(\varepsilon)) + \delta(T_2, \mu_\varepsilon),\ 0 \geq \varepsilon < \varepsilon_1.$$

It is easy to see that $g_2(0) = S_2(0)) = S_2 < \theta^{-1} < 0$ because $(G_k) \in \Gamma(\mu)$. For $T_2 = [q^2_1, \ldots, q^2_{\omega(T_2)}]$ we get

$$\frac{d\, g_2(\varepsilon)}{d\, \varepsilon} = 1 + \frac{1}{\left(g_1(\varepsilon)\right)^2} \frac{d\, g_1(\varepsilon)}{d\, \varepsilon} + \delta'(T_2, \mu_\varepsilon) \geq 1.$$

Again, as $\frac{d\, g_2(\varepsilon)}{d\, \varepsilon} \geq 1$ and $g_2(0) < 0$ we obtain a unique root ε_2 such that $g_2(\varepsilon) < 0$ for $\varepsilon < \varepsilon_2$ and $g_2(\varepsilon) > 0$ for $\varepsilon > \varepsilon_2$.
We claim that $\varepsilon_2 < \varepsilon_1$. To see that, we analyze

$$\lim_{\varepsilon \to \varepsilon_1^-} g_2(\varepsilon) = \lim_{\varepsilon \to \varepsilon_1^-} 2 - \mu - \frac{1}{g_1(\varepsilon)} + \varepsilon + \delta(T_2, \mu_\varepsilon) = +\infty$$

because $g_1(\varepsilon) < 0$ for $\varepsilon < \varepsilon_1$ and $\lim_{\varepsilon \to \varepsilon_1^-} g_1(\varepsilon) = 0$. Thus, $g_2(\varepsilon) > 0$ in some left neighborhood of ε_1, that is, the root $\varepsilon_2 < \varepsilon_1$.
In this way, we conclude that $\operatorname{sign}(S_2(\varepsilon)) = +$ for $\varepsilon_2 < \varepsilon < \varepsilon_1$, in other words, $\operatorname{sign}(\Pi)(G_k, \mu_\varepsilon) = (-, +, ?, \ldots, ?)$, meaning that, $\rho_L(G_k) > \mu_\varepsilon$.

If we continue this procedure for $2 \leq j \leq k-1$ we obtain a sequence $0 < \varepsilon_j < \ldots \varepsilon_2 < \varepsilon_1$ such that, for $\varepsilon_j < \varepsilon < \varepsilon_{j-1}$ we have $\operatorname{sign}(\Pi)(G_k, \mu_\varepsilon) = (-, \ldots, -, +, ?, \ldots, ?)$ (obviously, by construction, $\operatorname{sign}(\Pi)(G_k, \mu_\varepsilon) = (-, \ldots, -, -, ?, \ldots, ?)$ for $\varepsilon < \varepsilon_j$), meaning that, $\rho_L(G_k) > \mu_\varepsilon$.

The last step is to look for the possibility of having $\operatorname{sign}(S_k(\varepsilon)) = +$ for $\varepsilon < \varepsilon_{k-1}$. Analogously to the previous case, we define

$$g_k(\varepsilon) = S_k(\varepsilon) = -1 + \psi_\varepsilon(S_{k-1}(\varepsilon)) + \delta(C_k, \mu_\varepsilon),\ 0 \geq \varepsilon < \varepsilon_{k-1}.$$

It is easy to see that $g_k(0) = S_k(0)) = S_k < 0$ because $(G_k) \in \Gamma(\mu)$. For $C_k = [p^k_1, \ldots, p^k_{\omega(C_k)}]$ we get

$$\frac{d\, g_k(\varepsilon)}{d\, \varepsilon} = 1 + \frac{1}{\left(g_{k-1}(\varepsilon)\right)^2} \frac{d\, g_{k-1}(\varepsilon)}{d\, \varepsilon} + \delta'(C_k, \mu_\varepsilon) \geq 1.$$

Again, as $\frac{d\, g_k(\varepsilon)}{d\, \varepsilon} \geq 1$ and $g_k(0) < 0$ we obtain a unique root ε_k such that $g_k(\varepsilon) < 0$ for $\varepsilon < \varepsilon_k$ and $g_k(\varepsilon) > 0$ for $\varepsilon > \varepsilon_k$.
We claim that $\varepsilon_k < \varepsilon_{k-1}$. To see that, we analyze

$$\lim_{\varepsilon \to \varepsilon_{k-1}^-} g_k(\varepsilon) = \lim_{\varepsilon \to \varepsilon_{k-1}^-} 2 - \mu - \frac{1}{g_{k-1}(\varepsilon)} + \varepsilon + \delta(C_k, \mu_\varepsilon) = +\infty$$

because $g_{k-1}(\varepsilon) < 0$ for $\varepsilon < \varepsilon_{k-1}$ and $\lim_{\varepsilon \to \varepsilon_{k-1}^-} g_{k-1}(\varepsilon) = 0$. Thus, $g_k(\varepsilon) > 0$ in some left neighborhood of ε_1, that is, the root $\varepsilon_k < \varepsilon_{k-1}$.
In this way, we conclude that $\operatorname{sign}(S_k(\varepsilon)) = +$ for $\varepsilon_k < \varepsilon < \varepsilon_{k-1}$, in other words, $\operatorname{sign}(\Pi)(G_k, \mu_\varepsilon) = (-, \ldots, -, +)$, meaning that, $\rho_L(G_k) > \mu_\varepsilon$.

The conclusion is that, $\lim_{k \to \infty} \rho_L(G_k) = \mu$ if, and only if, the sequence (ε_j) converges to zero. Otherwise, if $\varepsilon_j \geq \varepsilon_\infty > 0$ we get $\rho_L(G_k) < \mu - \varepsilon_\infty$ thus $\mu_0 = \lim_{k \to \infty} \rho_L(G_k) < \mu$. □

Remark 7.2. *Notice that ε_k is the solution of*

$$S_k(\varepsilon_k) = -1 + \psi_\varepsilon(S_{k-1}(\varepsilon_k)) + \delta(C_k, \mu_{\varepsilon_k}) = 0$$

meaning that

$$-\varepsilon_k := -1 + \psi(S_{k-1}(\varepsilon_k)) + \delta(C_k, \mu_{\varepsilon_k})$$

If $\varepsilon_j \to 0$ then $\mu_{\varepsilon_k} = \mu - \varepsilon_k \to \mu$ and $S_k = -1 + \psi(S_{k-1}) + \delta(C_k, \mu) = [-1 + \psi(S_{k-1}) + \delta(C_k, \mu) - (-1 + \psi(S_{k-1}(\varepsilon_k)) + \delta(C_k, \mu_{\varepsilon_k}))] - \varepsilon_k \to 0$.

For practical purposes, it is feasible, from a theoretical point of view, but computationally very hard, to obtain the numbers (ε_j), because each function $g_j(\varepsilon)$ is obtained recursively from the previous one. We notice, however, that we can compute another sequence (α_j) such that $\varepsilon_j < \alpha_j$. Thus, if (α_j) converges to zero then (ε_j) converges to zero, providing a sufficient condition to ensure that $\lim_{k\to\infty} \rho_L(G_k) = \mu$. The numbers (α_j) will be the root of the linear approximation $h_j(\varepsilon) := g_j(0) + \varepsilon \frac{d\,g_j}{d\,\varepsilon}(0)$ of $g_j(\varepsilon)$ in $\varepsilon = 0$, see Figure 10.

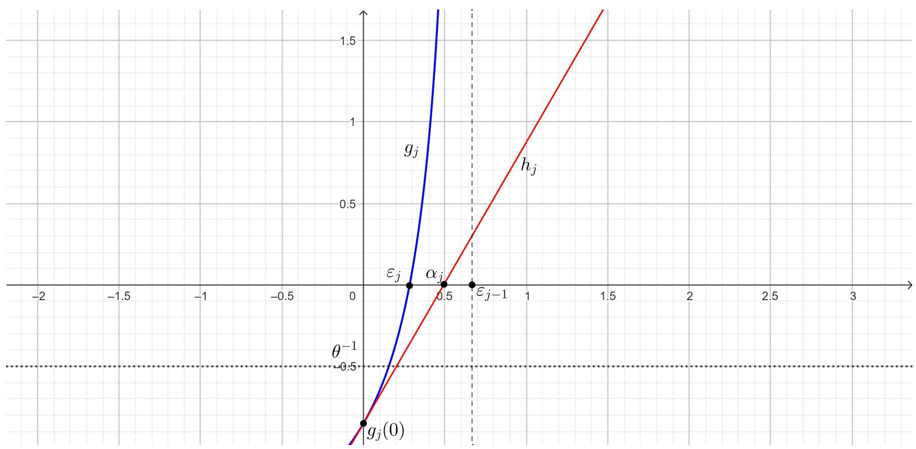

FIGURE 10. Representation of α_j.

Corollary 7.3. *Under the hypothesis of Theorem 7.1, consider the sequence (ε_j). Let (α_j) be the sequence of roots of the functions*

$$h_j(\varepsilon) := g_j(0) + \varepsilon \frac{d\,g_j}{d\,\varepsilon}(0).$$

Then, $\varepsilon_j < \alpha_j$ and, if (α_j) converges to zero, then $\lim_{k\to\infty} \rho_L(G_k) = \mu$. Moreover, $\alpha_1 := \dfrac{-S_1}{1 + \delta'(T_1, \mu_\varepsilon)(0)}$ and, for $2 \le j \le k$,

$$\alpha_j = \frac{-S_j}{\beta_j - \frac{1}{S_{j-1}} \frac{1}{\alpha_{j-1}}}, \tag{24}$$

where $\beta_j := 1 + \delta'(T_j, \mu_\varepsilon)(0)$, and for $2 \le j \le k-1$, $\beta_k := 1 + \delta'(C_k, \mu_\varepsilon)(0)$.

Proof. By definition, α_j is the solution of $h_j(\alpha_j) = 0$, which is given by

$$\alpha_j = \frac{-g_j(0)}{\frac{d\,g_j}{d\,\varepsilon}(0)}.$$

We recall that $g_j(0) = S_j$,

$$\frac{d\,b_j}{d\,\varepsilon}(0) = 1 + \frac{1}{b_{j-1}^2} \frac{d\,b_{j-1}}{d\,\varepsilon}(0)$$

and

$$\frac{d\,g_j}{d\,\varepsilon}(\varepsilon) = \begin{cases} 1 + \delta'(T_1, \mu_\varepsilon)(0), & j = 1,\ T_1 = [q_1^1, \ldots, q_{\omega(T_1)}^1] \\ 1 + \frac{1}{(g_{j-1}(\varepsilon))^2} \frac{d\,g_{j-1}(\varepsilon)}{d\,\varepsilon} + \delta'(T_j, \mu_\varepsilon)(0), & 2 \le j \le k-1,\ T_j = [q_1^j, \ldots, q_{\omega(T_j)}^j] \\ 1 + \frac{1}{(g_{k-1}(\varepsilon))^2} \frac{d\,g_{k-1}(\varepsilon)}{d\,\varepsilon} + \delta'(C_k, \mu_\varepsilon)(0), & j = k,\ C_k = [p_1^k, \ldots, p_{\omega(C_k)}^k] \end{cases}$$

so

$$\frac{d\,g_j}{d\,\varepsilon}(0) = 1 + \frac{1}{(S_{j-1})^2}\frac{d\,g_{j-1}}{d\,\varepsilon}(0) + \delta'(T_j,\mu_\varepsilon)(0). \tag{25}$$

Substituting the above formulas, for $2 \le j \le k-1$, we obtain

$$\alpha_j = \frac{-S_j}{1 + \frac{1}{(S_{j-1})^2}\frac{d\,g_{j-1}}{d\,\varepsilon}(0) + \delta'(T_j,\mu_\varepsilon)(0)}.$$

Notice that $\alpha_{j-1} = \dfrac{-g_{j-1}(0)}{\frac{d\,g_{j-1}}{d\,\varepsilon}(0)} = \dfrac{-S_{j-1}}{\frac{d\,g_{j-1}}{d\,\varepsilon}(0)}$ so

$$\alpha_j = \frac{-S_j}{\beta_j - \frac{1}{S_{j-1}}\frac{1}{\alpha_{j-1}}}, \tag{26}$$

where $\beta_j := 1 + \delta'(T_j,\mu_\varepsilon)(0)$. The above formula provides a way to obtain α_j from α_{j-1} without compositions, only the computation of S_j, which is easy to perform, and the knowledge of the derivatives $b'_j(0)$, which is a very standard recursion problem (see Equation (20), for $\varepsilon = 0$), independent from this particular construction.

In order to conclude our proof we just need to show that $\varepsilon_j < \alpha_j$. We claim that it is true if $g_j(\varepsilon)$ is a concave function ($\frac{d^2}{d\,\varepsilon^2}(g_j) \ge 0$). Indeed, $h_j(0) = g_j(0)$ and for $\varepsilon > 0$ we have

$$\frac{d}{d\,\varepsilon}(g_j(\varepsilon) - h_j(\varepsilon)) = \frac{d\,g_j}{d\,\varepsilon} - \frac{d\,g_j}{d\,\varepsilon}(0) \ge 0.$$

Thus $g_j(\alpha_j) - h_j(\alpha_j) = g_j(\alpha_j) \ge g_j(0) - h_j(0) = 0$, so that $g_j(\alpha_j) \ge 0$. Since g_j is increasing we get $\varepsilon_j < \alpha_j$.

In this way, we only need to show that $\frac{d^2}{d\,\varepsilon^2}(g_j) \ge 0$:

$$\frac{d^2\,g_j(\varepsilon)}{d\,\varepsilon^2} = \frac{d}{d\,\varepsilon}\left(1 + \frac{1}{(g_{j-1}(\varepsilon))^2}\frac{d\,g_{j-1}(\varepsilon)}{d\,\varepsilon} + \sum_{i=1}^{\omega(T_j)} \frac{1}{\left(b_{q_i^j}(\varepsilon)\right)^2}\frac{d\,b_{q_i^j}(\varepsilon)}{d\,\varepsilon}\right) =$$

$$= \frac{-2}{(g_{j-1}(\varepsilon))^3}\left(\frac{d\,g_{j-1}(\varepsilon)}{d\,\varepsilon}\right)^2 + \frac{1}{(g_{j-1}(\varepsilon))^2}\frac{d^2\,g_{j-1}(\varepsilon)}{d^2\,\varepsilon} +$$

$$\sum_{i=1}^{\omega(T_j)}\left(\frac{-2}{\left(b_{q_i^j}(\varepsilon)\right)^3}\left(\frac{d\,b_{q_i^j}(\varepsilon)}{d\,\varepsilon}\right)^2 + \frac{1}{\left(b_{q_i^j}(\varepsilon)\right)^2}\frac{d^2\,b_{q_i^j}(\varepsilon)}{d\,\varepsilon^2}\right) \ge 0,$$

because $b_j < 0$ for all j, $\frac{d^2}{d\,\varepsilon^2}(g_1) = 0$ and we already proved that $b''_j(\varepsilon) \ge 0$. □

Example 7.4. *For $\mu = \frac{5+\sqrt{33}}{2}$ and the generalized Shearer sequence $\mathbb{T} = ([1,1,1],[1],\ldots)$ and $C = ([1,1],\ldots)$ we know that $\lim_{k\to\infty} \rho_L(G_k) = \mu$. We can check, from the formula (24) that*

$\alpha_1 = 0.5930703308$

$\alpha_{10} = 0.0003726377$

$\ldots$

$\alpha_{100} = 1.33485599 \times 10^{-33}$

$\alpha_{150} = 5.84453701 \times 10^{-50}$

$\alpha_{190} = 4.78412668 \times 10^{-63}$. *Thus,* $|\rho_L(G_{190}) - \mu| \le 10^{-63}$.

On the other hand, using for $\mu = 5.4$ the same sequence (we can do that since $G_k \in \Gamma(5.4)$) we get $\alpha_1 = 0.62182468$, $\alpha_{10} = 0.670219903$, $\alpha_{50} = 0.807268557543450193961813121639$, $\alpha_{100} = 0.807268557543452357547180905158$,

$\alpha_{150} = 0.807268557543452357547180905157724166918214571702802706320003$ *and so on, as expected, since* $\lim_{k\to\infty} \rho_L(G_k) = \dfrac{5+\sqrt{33}}{2} = 5.37+ < 5.4$.

Finally, we consider a sequence selected from a genetic algorithm to fit to $\mu = 5.4$:

$$\mathbb{T} = ([0],[1,1],[1],[7],[5],[6],[7],[7],[2],[3],[5],[6],[2],[5],[4],[4],[6],$$
$$[6],[6],[0],[6],[1,1],[0],[6],[0],[3],[4],[4],[7],[1,1],\ldots)$$

and C a shift of $\mathbb{T}$. In this case $G_k \in \Gamma(5.4)$ and $\alpha_{29} = 0.0001005914$, suggesting that $\rho_L(G_{29})$is at least 10^{-4} close to $\mu = 5.4$. Actually, a direct computation shows that $\rho_L(G_{29}) = 5.399999999963451$, which is 3.65×10^{-11} close to $\mu = 5.4$. That is natural since the convergence of $\alpha_j \to 0$ is only a sufficient condition and ε_j could be much smaller than α_j.

An alternative way to prove that $\alpha_j \to 0$ and so $\lim\limits_{k\to\infty} \rho_L(G_k) = \mu$, without a direct computation, is the following:

Theorem 7.5. *If*

$$\sum_{m=1}^{j} \frac{1+\delta'(T_m,\mu_\varepsilon)(0)}{\left(\prod_{n=m}^{j-1} S_n\right)^2} \tag{27}$$

is unbounded, then $\lim\limits_{k\to\infty} \rho_L(G_k) = \mu$.

Proof. Let α_j be the sequence given by Corollary 7.3. By definition, α_j is given by

$$\alpha_j = \frac{-S_j}{\frac{d\,g_j}{d\,\varepsilon}(0)}.$$

From Equation (25) we get

$$\frac{d\,g_j}{d\,\varepsilon}(0) = 1 + \frac{1}{(S_{j-1})^2}\frac{d\,g_{j-1}}{d\,\varepsilon}(0) + \delta'(T_j,\mu_\varepsilon)(0),$$

and $\frac{d\,g_1}{d\,\varepsilon}(0) = 1 + \delta'(T_1,\mu_\varepsilon)(0)$.
Let us introduce the following auxiliary sequences

- $A_j := 1 + \delta'(T_j,\mu_\varepsilon)(0)$;
- $B_j := \frac{1}{(S_j)^2}$;
- $X_j := \frac{d\,g_j}{d\,\varepsilon}(0)$.

Then, we obtain the standard difference equation problem

$$\begin{cases} X_j = A_j + B_{j-1}X_{j-1} \\ X_1 = A_1. \end{cases} \tag{28}$$

This equation has an explicit solution:

$$\begin{aligned} X_2 &= A_2 + B_1X_1 = A_2 + B_1A_1, \\ X_3 &= A_3 + B_2(A_2 + B_1A_1) = A_3 + A_2B_2 + A_1B_1B_2, \\ X_4 &= A_4 + A_3B_3 + A_2B_2B_3 + A_1B_1B_2B_3, \end{aligned}$$

and so on, obtaining

$$\frac{d\,g_j}{d\,\varepsilon}(0) = X_j = \sum_{m=1}^{j} A_m \prod_{n=m}^{j-1} B_n = \sum_{m=1}^{j} (1+\delta'(T_m,\mu_\varepsilon)(0)) \prod_{n=m}^{j-1} \frac{1}{(S_n)^2}.$$

Notice that $S_1 < S_j < \theta^{-1}$ so $-S_1 > -S_j > -\theta^{-1}$. Thus, if $\frac{d\,g_j}{d\,\varepsilon}(0)$ is unbounded when $j \to \infty$, then $\alpha_j \to 0$. By Corollary 7.3 we obtain $\lim\limits_{k\to\infty} \rho_L(G_k) = \mu$. □

Remark 7.6. *The formula*

$$\frac{d\,g_j}{d\,\varepsilon}(0) = \sum_{m=1}^{j} \frac{1+\delta'(T_m,\mu_\varepsilon)(0)}{\left(\prod_{n=m}^{j-1} S_n\right)^2},$$

has an analogous for adjacency limit points, and the fact that it is unbounded proves Shearer's result in [13]. Although the approach is a little different, the basis for Shearer's argument is to show that the product $\prod_{n=m}^{j-1}(S_n)$ has absolute values smaller than 1. This is somehow hidden in the proof of the main theorem (actually he proves that $R_{j-1}R_j < 1, R_{j-2}R_{j+1} < 1, \ldots$ thus $R_{j-2}R_{j-1}R_jR_{j+1} < 1$, etc, where the $R_j's$ are the values obtained in the adjacency case). In our Laplacian case, if this was true, we would obtain $\frac{d\,g_j}{d\,\varepsilon}(0) \to \infty$ and $\lim\limits_{k\to\infty} \rho_L(G_k) = \mu$ according to Theorem 7.5. Since it is not true that

Equation (27) *is unbounded for any* μ, *this explains why the classical Laplacian Shearer fails to prove convergence in general.*

8. Final considerations and future work

In this section we discuss possible consequences of the previous results and additional investigations one could carry on in the future, towards the proof of the conjecture that all the points in the interval $[4.38+, \infty)$ are Laplacian limit points.

We mention here an ongoing research work [11], where we study limit points for the spectral radius of A_α-matrix of a graph G, where $A_\alpha(G) := \alpha D(G) + (1-\alpha)A(G)$, for $0 \leq \alpha \leq 1$. It is easy to see that $A_0(G) = A(G)$, the adjacency matrix of G and $A_1(G) = D(G)$ the degree matrix of G. Also, $A_{1/2}(G) = 1/2(D(G) + A(G)) = 1/2Q(G)$ where Q is the signless Laplacian matrix of G. Our main result is that for any $\alpha \in [0, 1/2)$ there exists a positive number $\tau_2(\alpha) > 2$ such that any value $\lambda > \tau_2(\alpha)$ is an A_α-limit point. Notice that this generalization of the Shearer's work for the adjacency matrix seems to stretch full power (which is $\alpha < \frac{1}{2}$). Beyond that, the technique used does not work. As for $\alpha = \frac{1}{2}$ we have the signless Laplacian matrix, these two manuscripts show how difficult it is to study Laplacian limit points .

Question 1

Can one use Theorem 7.1, or Corollary 7.3, or Theorem 7.5 to prove that

$$\sup_{G_k \in \Gamma(\mu)} \lim_{k\to\infty} \rho_L(G_k) = \mu$$

at least for some class of points $\mu \geq 4.38$?
The answer of this question is definitive for approximation by linear trees (which include caterpillars), using increasing sequences of spectral radii.

Question 2

Are there are any points where $\sup\limits_{G_k \in \Gamma(\mu)} \lim\limits_{k\to\infty} \rho_L(G_k) < \mu$?
If yes, then μ is not a Laplacian limit point or at least it can not be limit of increasing sequences of Laplacian spectral radii of linear trees, because $\Gamma(\mu)$ contains all linear trees with spectral radius smaller than μ.

Question 3

In Question 2, can one find (from the previous lemmas and theorems) a maximum, where the supremum is attained, despite the fact that $\sup\limits_{G_k \in \Gamma(\mu)} \lim\limits_{k\to\infty} \rho_L(G_k) < \mu$?
If yes, this would be a special number in some way!

Question 4

Is there some particular subinterval of $[4.38+, \infty)$ regarding properties of the product $S_j S_{j+1} < 1$, provided that $S_j > -1$ and $S_{j+1} < -1$?
If the answer is yes, then one may be able to use Theorem 7.5 to prove that $\sup\limits_{G_k \in \Gamma(\mu)} \lim\limits_{k\to\infty} \rho_L(G_k) = \mu$.

Question 5

A natural future problem would be to investigate the application of Theorem 7.5. Can one prove that $X_k > m_k \to \infty$?
In order to do that, one needs to control weather $S_j S_{j+1} < 1$, at least for $j \geq k$ because (for $k \geq 2$)

$$X_4 = A_4 + A_3 B_3 + A_2 B_2 B_3 + A_1 B_1 B_2 B_3,$$

we start with $A_2 B_2 B_3$ where $A_2 = 1 + \delta'(T_2, \mu_\varepsilon)(0) > 1$ and $B_2 B_3 = \frac{1}{S_2^2 S_3^2} > 1$, provided that $S_2 S_3 < 1$. This does not depend on S_1, same as for the next additives, except for $A_1 B_1 B_2 B_3$. Recall that the goal is to show that $X_k \to \infty$, in this case one may have $X_4 > 3$ if $B_3 > 1$ because $A_4 > 1$, $A_2 B_2 B_3 > 1$, but we do not have control of $A_1B_1B_2B_3$ since S_1 could be negative and as large as $1-\mu$. In the worst case $X_4 > 2$.

Question 6

Despite the success of the Shearer's method for adjacency limit points, one may ask if part of our difficulties when we translate to the Laplacian limit points, comes from the choice of an iterative process that builds sequences of increasing spectral radii, as we did. One possible line of research is to consider Lapalacian limit points arbitrarily constructed, but still using treatable trees like caterpillars, linear trees and others. If one succeeds, then one may expect a proof of the conjecture or to conclude that one should look at a more general class of trees or general graphs.

Acknowledgments

This work is partially supported by MATH-AMSUD under project GSA, brazilian team financed by CAPES under project 88881.694479/2022-01. It was executed while F. Belardo was visiting UFRGS, under the financial of CAPES - PRINT, project 88887.716953/2022-00. The research of Francesco Belardo and Vilmar Trevisan was supported by a grant from the group GNSAGA of INdAM (Italy). Elismar Oliveira and Vilmar Trevisan acknowledge the support of CNPq grants 408180/2023-4 and 310827/2020-5.

References

[1] Anderson, William N.; Morley, T. D. Eigenvalues of the laplacian of a graph. *Linear and Multilinear Algebra 1985-oct vol. 18 iss. 2 18* (oct 1985).

[2] Estes, D. R. Eigenvalues of symmetric integer matrices. *J. Number Theory 42*, 3 (1992), 292–296.

[3] Guo, J.-M. On limit points of Laplacian spectral radii of graphs. *Linear Algebra Appl. 429*, 7 (2008), 1705–1718.

[4] Hoffman, A. J. On limit points of spectral radii of non-negative symmetric integral matrices. In *Graph theory and applications (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1972; dedicated to the memory of J. W. T. Youngs)* (1972), Lecture Notes in Math., Vol. 303, Springer, Berlin, pp. 165–172.

[5] Jacobs, D. P., and Trevisan, V. Locating the eigenvalues of trees. *Linear Algebra Appl. 434*, 1 (2011), 81–88.

[6] Johnson, C. R., Li, A. A., and Walker, A. J. Ordered multiplicity lists for eigenvalues of symmetric matrices whose graph is a linear tree. *Discrete Math. 333* (2014), 39–55.

[7] Johnson, C. R., and Saiago, C. M. *Eigenvalues, multiplicities and graphs*, vol. 211 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2018.

[8] Kirkland, S. A note on limit points for algebraic connectivity. *Linear Algebra Appl. 373* (2003), 5–11. Combinatorial Matrix Theory Conference (Pohang, 2002).

[9] Kirkland, S. Limit points for normalized Laplacian eigenvalues. *Electron. J. Linear Algebra 15* (2006), 337–344.

[10] Oliveira, E. R., and Trevisan, V. Applications of rational difference equations to spectral graph theory. *J. Difference Equ. Appl. 27*, 7 (2021), 1024–1051.

[11] Oliveira, E. R., and Trevisan, V. Limit points of A_α-matrices of graphs. *arXiv preprint arXiv:2404.10953* (2024).

[12] Salez, J. Every totally real algebraic integer is a tree eigenvalue. *J. Combin. Theory Ser. B 111* (2015), 249–256.

[13] Shearer, J. B. On the distribution of the maximum eigenvalue of graphs. *Linear Algebra Appl. 114/115* (1989), 17–20.

[14] Wang, J., Huang, Q., An, X., and Belardo, F. Some results on the signless Laplacians of graphs. *Appl. Math. Lett. 23*, 9 (2010), 1045–1049.

[15] Wang, J., Wang, J., and Brunetti, M. The hoffman program of graphs: old and new. *arXiv preprint arXiv:2012.13079* (2020).

[16] Zhang, F., and Chen, Z. Limit points of eigenvalues of (di)graphs. *Czechoslovak Math. J. 56(131)*, 3 (2006), 895–902.

[17] Zhang, X.-D. The laplacian eigenvalues of graphs: a survey. *arXiv preprint arXiv:1111.2897* (2011).

Department of Mathematics and Applications, University of Naples Federico II, Italy
Email address: fbelardo@unina.it

Instituto de Matemática e Estatística, UFRGS, Porto Alegre, Brazil
Email address: elismar.oliveira@ufrgs.br

Instituto de Matemática e Estatística, UFRGS, Porto Alegre, Brazil
Email address: trevisan@mat.ufrgs.br