

Amenable actions of real and p -adic algebraic groups

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May 6, 2024

Abstract

Let K be a locally compact field of characteristic 0. Let G be a linear algebraic group defined over K , acting algebraically on an algebraic variety V . We prove that the action of $G(K)$ (the group of K -rational points of G) on $V(K)$ is topologically amenable, if and only if all points stabilizers in $G(K)$ are solvable-by-compact. This follows by combining a result by Borel-Serre [BS64] with the following fact: let G be a second countable locally compact group acting continuously on a second countable locally compact space Y . If the action $G \curvearrowright Y$ is smooth (i.e. the Borel structure on $G \backslash Y$ is countably separated), then topological amenability of $G \curvearrowright Y$ is equivalent to amenability of all point stabilizers in G .

1 Introduction

Amenability of group actions is a far-reaching generalization of classical amenability for locally compact groups. It was proposed by R.J. Zimmer (Def. 4.3.1 in [Zi84]) in the measurable setting, and in the topological setting by J. Renault (measurewise amenability, see Def. 2.3.6 in [Re80]) and C. Anantharaman-Delaroche (topological amenability, see Def. 2.1 in [An02]). We recall the relevant definitions.

Definition 1.1 ¹Let G be a locally compact group acting measurably on a measure space (X, μ) , where μ is a quasi-invariant measure. We say that the action $G \curvearrowright X$ is **amenable in the sense of Zimmer** if there exists a G -equivariant conditional expectation $L^\infty(X \times G, \mu \times \lambda_G) \rightarrow L^\infty(X, \mu)$, where λ_G denotes Haar measure on G .

Definition 1.2 Let G be a locally compact group acting continuously on a locally compact space Y .

¹This is not Zimmer's original definition, but it is equivalent to it by Theorem A in [AEG94].

- 1) The action $G \curvearrowright Y$ is **measurewise amenable** if the action of G on (Y, μ) is amenable in the sense of Zimmer for every quasi-invariant measure μ on Y .
- 2) We denote by $\text{Prob}(G)$ the set of probability measures on G , equipped with the weak $*$ -topology. We say that the action $G \curvearrowright Y$ is **topologically amenable** if there exists a net $(m^i)_{i \in I}$ of continuous functions $m^i : Y \rightarrow \text{Prob}(G) : y \mapsto m_y^i$ such that $\lim_{i \rightarrow \infty} \|g_* m_y^i - m_y^i\| = 0$ uniformly on compact subsets of $Y \times G$ (where the norm corresponds to the total variation distance).

It follows from any of the three definitions above, that a locally compact group is amenable if and only if it acts amenably on a one-point space.

It was proved by Anantharaman-Delaroche and Renault that topological amenability implies measurewise amenability (Proposition 3.3.5 in [ADR00]), and that they are equivalent for actions of countable discrete groups. The equivalence for general group actions was a long-standing open question that was settled only recently through combined work by Buss-Echterhoff-Willett [BEW] and Bearden-Crann [BC22]: for a locally compact second countable group G acting continuously on a locally compact second countable space Y , the action $G \curvearrowright Y$ is topologically amenable if and only if it is measurewise amenable (see Corollary 3.29 in [BEW]).

The goal of this Note is to prove:

Theorem 1.3 *Let K be a locally compact field of characteristic 0. Let G be a linear algebraic group defined over K , acting algebraically on an algebraic variety V . The action of $G(K)$ (the group of K -rational points of G) on $V(K)$ is topologically amenable, if and only if all points stabilizers in $G(K)$ are solvable-by-compact.*

Recall that, for a locally compact second countable group G acting continuously on a locally compact second countable space Y , the action of G on Y is *smooth* if the quotient Borel structure on $G \backslash Y$ is countably separated. In section 2 we prove that, for smooth actions, topological amenability is equivalent to amenability of stabilizers (see Proposition 2.4). As explained in section 3, Theorem 1.3 follows then easily by combining this result with a deep result by Borel-Serre [BS64] asserting that the action $G(K) \curvearrowright V(K)$ is always smooth.

We end this Introduction with the example that motivated the present study.

Example 1.4 *For $n \geq 0$, let ρ_n denote the $(n + 1)$ -dimensional irreducible representation of $SL_2(K)$ on the space $P_n(K)$ of homogeneous polynomials of degree n with coefficients in K in the 2 variables X, Y . So, for $P(X, Y) \in$*

$P_n(K)$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(K)$, we define

$$(\rho_n(\begin{pmatrix} a & b \\ c & d \end{pmatrix}))P(X, Y) = P(aX + cY, bX + dY).$$

Going to the projective space of $P_n(K)$, we get an action of $SL_2(K)$ on the projective space $\mathbb{P}^n(K)$. In Proposition 3.5 of [JV24], the authors gave a direct but ad hoc proof of the fact that, for $K = \mathbb{R}, \mathbb{C}$, the action $SL_2(K) \curvearrowright \mathbb{P}^n(K)$ is topologically amenable. Now, for any local field K of characteristic 0, stabilizers of points in $\mathbb{P}^n(K)$ are proper algebraic subgroups of dimension at most 2, so they are solvable. By Theorem 1.3, the action $SL_2(K) \curvearrowright \mathbb{P}^n(K)$ is topologically amenable.

2 Amenable actions vs. amenable equivalence relations

The next lemma is probably well-known.

Lemma 2.1 *Let the locally compact group G act continuously on a locally compact space Y . If the action of G on Y is topologically amenable, then for every $y \in Y$ the stabilizer G_y is amenable.*

Proof: As the action $G \curvearrowright Y$ is topologically amenable, by Proposition 2.5 in [An02] there exists a net $(h_i)_{i \in I}$ of continuous, compactly supported, positive type functions² on $Y \times G$ that converges to 1 uniformly on compact subsets on $Y \times G$. For $t \in G_y$, set $g_i(t) = h_i(y, t)$. Then $(g_i)_{i \in I}$ is a net of continuous, compactly supported, positive type functions on G_y that converges to 1 on compact subsets of G_y : the existence of such a net characterizes amenability of G_y . ■

We must now discuss amenable equivalence relations as introduced in [CFW81]. So let X be a standard Borel space and $\mathcal{R} \subset X \times X$ be a Borel equivalence relation. A transverse measure Λ for \mathcal{R} is defined as in Definition 1 in Chapter II in [Co79]; our aim is to define the amenability of the pair $(\mathcal{R}, \text{class of } \Lambda)$. Denote by $p_1 : X \times X \rightarrow X : (x, y) \mapsto x$ the first projection map. We choose an auxiliary *transverse function* ν for \mathcal{R} , i.e. a map $x \mapsto \nu^x$ from X to the space of positive measures on \mathcal{R} , such that:

- For every $x \in X$, the measure ν^x is non-zero and supported on $[x]_{\mathcal{R}} =: p_1^{-1}(x) \cap \mathcal{R}$. Moreover ν is invariant in the sense that $\gamma\nu^x = \nu^y$ for every $\gamma = (y, x) \in \mathcal{R}$.

²Recall that a function h on $Y \times G$ is positive type if, for every $n \geq 1, y \in Y, t_1, \dots, t_n \in G, \lambda_1, \dots, \lambda_n \in \mathbb{C}$, we have $\sum_{i,j} \overline{\lambda_i} \lambda_j h(t_i^{-1}y, t_i^{-1}t_j) \geq 0$.

- For every Borel set A in \mathcal{R} , the map $X \rightarrow [0, +\infty] : x \mapsto \nu^x(A)$ is measurable.
- ν is proper in the sense that \mathcal{R} is a countable union of Borel sets $(A_n)_{n \in \mathbf{N}}$ such that, for every $n \in \mathbf{N}$, the function $x \mapsto \nu^x(A_n)$ is bounded.

Theorem 3 in Chapter II of [Co79] then provides a bijection between transverse measures on \mathcal{R} and ordinary measures μ on X such that the measure $m =: \int_X \nu^x d\mu(x)$ on \mathcal{R} is invariant under the flip on \mathcal{R} .

Definition 2.2 [See p. 446 in [CFW81]] *The pair $(\mathcal{R}, \text{class of } \Lambda)$ is amenable if, for almost every $x \in X$ there exists a state p_x on $L^\infty([x]_{\mathcal{R}}, \nu^x)$ such that:*

- The family $(p_x)_{x \in X}$ is invariant in the sense that $\gamma p_x = p_y$ for every $\gamma = (y, x) \in \mathcal{R}$.*
- With $m = \int_X \nu^x d\mu(x)$ as above, for every $f \in L^\infty(\mathcal{R}, m)$ the function $x \mapsto p_x(f)$ is μ -measurable.*

It is proved in lemma 15 of [CFW81] that this definition is in fact independent of the choice of the transverse function ν .

With X, \mathcal{R} as above, we say that the equivalence relation \mathcal{R} is *smooth* if the quotient Borel structure on $\mathcal{R} \backslash X$ is countably separated (see Definition 2.1.9 in [Zi84]). Extending the first part of Definition 1.2, we say that \mathcal{R} is *measurewise amenable* if $(\mathcal{R}, \text{class of } \Lambda)$ is amenable for every transverse measure Λ .

Lemma 2.3 *Any smooth Borel equivalence relation is measurewise amenable.*

Proof: Let \mathcal{R} be a smooth equivalence relation on the standard Borel space X .

- We denote by $\pi : X \rightarrow Y =: \mathcal{R} \backslash X$ the quotient map. By the discussion on page 47 of [Co79], there is a bijection between transverse measures for \mathcal{R} and ordinary measures on Y , so we choose a measure λ on Y , which we may assume to be a probability measure.
- By the von Neumann selection theorem (Theorem A.9 in [Zi84]), there exists a co-null standard Borel set $Z \subset Y$ and a Borel section $s : Z \rightarrow X$ such that $\pi \circ s = Id_Z$. Replacing Y by Z and X by $\pi^{-1}(Z)$, we then define the Borel map $\iota : X \rightarrow \mathcal{R} : x \mapsto (x, s(\pi(x)))$ which is a section of $p_1|_{\mathcal{R}}$; observe that, for $\gamma = (y, x) \in \mathcal{R}$, we have $\gamma \iota(x) = \iota(y)$. We then define the transverse function ν as $\nu^x =: \delta_{\iota(x)}$ (the Dirac mass at $\iota(x)$, viewed as a probability measure on \mathcal{R}); we will use ν to check the conditions in Definition 2.2. Observe that $\nu^{s(y)} = \delta_{\iota(s(y))} = \delta_{(s(y), s(y))}$ for $y \in Y$.

- Set $\mu =: s_*\lambda$, form the measure $m = \int_X \nu^x d\mu(x)$ on \mathcal{R} . Since

$$m = \int_Y \nu^{s(y)} d\lambda(y) = \int_Y \delta_{(s(y), s(y))} d\lambda(y),$$

the measure m is supported on the diagonal, hence invariant under the flip on \mathcal{R} .

- Finally, since $L^\infty([x]_{\mathcal{R}}, \nu^x)$ is one-dimensional, the state p_x is tautological: it is given by $p_x(\varphi) = \varphi(\iota(x))$ for $\varphi \in L^\infty([x]_{\mathcal{R}}, \nu^x)$. The invariance of the family $(p_x)_{x \in X}$ is then clear. Then for $f \in L^\infty(\mathcal{R}, m)$, the function $x \mapsto p_x(f) = f(\iota(x))$ is clearly μ -measurable. This concludes the proof. ■

Let G be a locally compact second countable group acting continuously on a locally compact second countable space Y . The action of G on Y is *smooth* if the orbital equivalence relation \mathcal{R}_G is smooth.

Proposition 2.4 *Let G act continuously on Y as above, and assume that G acts smoothly on Y . The following are then equivalent:*

- (i) *The action $G \curvearrowright Y$ is topologically amenable.*
- (ii) *All point stabilizers in G are amenable.*

Proof: The implication (i) \Rightarrow (ii) follows immediately from lemma 2.1. For the converse implication (ii) \Rightarrow (i): in view of the discussion following Definition 1.2, it is enough to prove that the action $G \curvearrowright Y$ is measurewise amenable. So fix a quasi-invariant measure μ on Y , we must prove that $G \curvearrowright (Y, \mu)$ is amenable in the sense of Zimmer. We appeal to a result by S. Adams, G. Elliott and T. Giordano (Theorem 5.1 in [AEG94]): *Assume that the second countable locally compact group G acts measurably on a standard measure space (X, ν) with ν a quasi-invariant probability measure. If stabilizers are amenable ν -almost everywhere, and if the orbital equivalence relation \mathcal{R}_G is amenable, then the action of G on (X, ν) is amenable in the sense of Zimmer.* In view of the Adams-Elliott-Giordano result, since stabilizers are assumed to be amenable, the result follows immediately from lemma 2.3. ■

3 Proof of Theorem 1.3

We will appeal to a deep result of A. Borel and J.-P. Serre ([BS64], Proposition 4.9 and Corollaire 6.4; see also Theorem 3.1.3 in [Zi84]): all orbits of $G(K)$ in $V(K)$ are locally closed. By the Glimm-Effros theorem (see Theorem 2.1.14 in [Zi84]), having all orbits locally closed is equivalent to

smoothness of the action, so we are in the assumptions of our Proposition 2.4, to the effect that topological amenability of $G(K) \curvearrowright V(K)$ is equivalent to amenability of stabilizers. Since solvable-by-compact groups are amenable, one implication in Theorem 1.3 becomes clear.

For the converse, assume that the action $G(K) \curvearrowright V(K)$ is amenable. Fix $y \in V(K)$, and denote by H the stabilizer of y in $G(K)$: by lemma 2.1, the subgroup H is amenable, and we must prove that H is solvable-by-compact. Since H is the group of K -points of a linear algebraic group, factoring out the solvable radical we may assume that H is semi-simple and must prove that H is compact. For algebraic groups over local fields of characteristic 0, being compact is equivalent to being anisotropic. So assume that H is isotropic. Then by the Jacobson-Morozov theorem H contains a copy either of $SL_2(K)$ or of $PGL_2(K)$, contradicting amenability of H . ■

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