Modular Invariant Slow Roll Inflation

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Abstract. We propose new classes of inflation models based on the modular symmetry, where the modulus field τ serves as the inflaton. We establish a connection between modular inflation and modular stabilization, wherein the modulus field rolls towards a fixed point along the boundary of the fundamental domain. We find the modular symmetry strongly constrain the possible shape of the potential and identify some parameter space where the inflation predictions agree with cosmic microwave background observations. The tensor-to-scalar ratio is predicted to be smaller than 10^{-6} in our models, while the running of spectral index is of the the order of 10^{-4} .

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1 Introduction

Despite the Standard Model (SM) being tested in high-energy collider experiments with high precision, many convincing pieces of evidence suggest that the SM should not be considered the final theory. It is known that there are huge hierarchies among the masses of quarks and charged leptons, the mixing patterns of quarks and leptons are drastically different. The organizing principle of the fermion masses and flavor mixing is still elusive. Moreover, neutrino oscillation experiments show that neutrinos have tiny masses, the origin of neutrino mass remains to be explained. Another strong hint for new physics comes from recent observations of the cosmic microwave background (CMB). The Planck and BICEP/Keck experiments strongly favor an exponentially expanding period of early universe called inflation [1–3]. Its dynamics are also absent in the SM if higgs field is not the inflaton. The possibility of slow roll inflation was proposed in [4]. In this paradigm, inflation happens when a scalar field rolls over a flat plateau of its potential. The characteristic features of CMB, such as the spectral index n_s and tensor-to-scalar ratio r, are intricately linked to the shape of the inflationary potential.

Modular symmetry has been widely used to explain lepton masses and mixing angles [5], see Refs. [6, 7] for reviews. These are a class of supersymmetric models where chiral supermultiplets transform non-trivially under modular transformations, and the Yukawa couplings are modular forms, which are holomorphic functions of the complex modulus τ . The requirement of the whole Lagrangian to be modular invariant puts a strong constraint on the superpotential and leads to very predictive results. In bottom-up modular invariant models, the vacuum

expectation value (VEV) of the complex modulus τ is fixed by confronting the model predictions with experimental data, and different models favor different VEVs of τ . Remarkably, in some of the models, the values of the modulus field are close to the boundary of fundamental domain, the imaginary axis, and three fixed points $i, \omega = e^{i2\pi/3}, i\infty$ [8–12]. The stabilization of the modulus field around fixed points i, ω is favored by modular symmetry itself. They have attracted special attention when one needs to dynamically fix the VEV of the modulus field and have been extensively studied [13–16]. They successfully find the global minima at the boundary of the fundamental domain and the fixed points. By considering the effects of the dilaton field, one can achieve either Minkowski minima or De Sitter (dS) minima at the fixed points in certain region of parameter space [17–19]. Modular stabilization for multiple modulus fields is also discussed in [17, 19].

The scalar potential of the modulus field can not only dynamically fix its VEV, but can also accommodate inflation, which is used to explain the isotropy and homogeneity of the CMB. It has been noticed that realizing slow roll inflation in the moduli sector is closely related to admitting metastable dS vacua [20]. It has been argued that it is impossible to realize inflation with a single modulus field and the logarithmic Kähler potential [20, 21]. A stabilizer field X besides the modulus field τ was introduced in Refs. [22, 23] to build an inflation potential, and the Kähler potential modified by X was considered to flatten the scalar potential in the whole complex plane. Higher powers of τ are included in the logarithm of the Kähler potential to realize modular inflation in [24].

In this paper we intend to answer whether the potential from modular stabilization could also realize slow roll inflation. The modular stabilization merely cares about the minima of the potential while inflation cares about the shape of the potential. We find that the scalar potential is strongly constrained by modular symmetry and a class of inflation potentials can be generated in this approach. They are all compatible with current cosmological observations. We will focus on the slow roll along the boundary of the fundamental domain, where the property of modular invariance can be maximally used.

Throughout the paper, we adopt the framework of supergravity which can be considered as a 4 dimensional effective theory of super-string theory. In order to be more general, we do not use any concrete models of string theory and any specific compactification mechanisms. The remainder of this paper is organised as follows. In section 2, we provide a brief introduction to modular symmetry and its applications in supersymmetry and supergravity. In section 3, we first review the basic concepts of single-field inflation. Next, we study the inflation scalar potential from the perspective of modular stabilisation, emphasizing its special features due to modular invariance. Ultimately, we introduce the inflationary models based on modular symmetry. In section 4, we conclusion and discuss possible developments. In Appendix A, we present the numerical results for concrete inflationary models. Some relevant formulae about modular forms are listed in Appendix B.

2 Modular symmetry and modular invariance

The successful application of modular symmetry to describe the flavor structure serves a motivation to introduce modular symmetry into inflation. In this paper, the modular invariance constrains the inflaton superpotential and we name this model *modular invariant inflation*. Therefore, in this section, we will introduce some relevant facts about modular symmetry.

2.1 Introduction to modular symmetry

We start with a brief review of the modular symmetries. The full modular group Γ is defined as [5]

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}.$$
 (2.1)

Therefore the modular group is the group composed of two-dimensional matrices with integer entries and unit determinant. The complex modulus τ takes value in the upper half complex plane denoted as $\mathcal{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}$. The modular group Γ acts on the complex modulus $\tau \in \mathcal{H}$ as linear fraction transformation,

$$\tau \to \gamma \tau = \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad \text{Im} \tau > 0.$$
 (2.2)

One sees that γ and $-\gamma$ give the same linear fraction transformation, and consequently the group of the above modular transformation is isomorphic to the projective group $\overline{\Gamma} \equiv \Gamma/\{\pm \mathbb{1}\}$. If all elements of Γ acting on same point in \mathcal{H} and point sets generated are equivalent, \mathcal{H} can be decomposed as set of trajectories \mathcal{H}/Γ . \mathcal{H}/Γ , also called the fundamental domain of Γ , is given by

$$\mathcal{D} = \{ \tau \in \mathcal{H} \mid |\tau| > 1, -1/2 \le \text{Re}(\tau) < 1/2 \} \cup \{ |\tau| = 1, -1/2 \le \text{Re}(\tau) \le 0 \},$$
 (2.3)

where no two points in \mathcal{D} are related by modular transformations. The modular group Γ has an infinite number of elements and it can be generated by two generators \mathcal{S} and \mathcal{T} :

$$S: \tau \to -\frac{1}{\tau}, \qquad \mathcal{T}: \tau \to \tau + 1.$$
 (2.4)

The matrix representation of S and T are given by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.5}$$

which obey the relations $S^4 = (ST)^3 = 1$ and $S^2T = TS^2$. There is a class of complex functions over the plane \mathcal{H} called modular forms, which are holomorphic functions of τ transforming under the modular groups as

$$f(\gamma \tau) = (ck+d)^k f(\tau), \quad \gamma \in \Gamma,$$
 (2.6)

where the weight k is a generic non-negative integer.

2.2 Effective action and modular invariant potential in supergravity

As a effective theory of superstring theory, the spectrum of a N=1 supergravitry theory normally contains the dilaton, Kähler moduli, complex structure moduli, gauge fields, and twisted and untwisted matter fields after heterotic orbifold compactifications. Given the fact that a single Kähler moduli is not enough to realise inflation [20, 21], we find it is natural to include dilaton field into our analysis. This choice has been adopted in existing literature to study different phenomenon [17, 18]. In this paper, the Kähler modulus field τ plays the

role of inflaton field. The effective action is determined by the following modular-invariant full SUGRA Kähler potential [25, 26]

$$\mathcal{G}(\tau, \bar{\tau}, S, \bar{S}) = \kappa_4^2 \mathcal{K}(\tau, \bar{\tau}, S, \bar{S}) + \ln |\kappa_4^3 \mathcal{W}(\tau, S)|^2, \qquad (2.7)$$

where $\kappa_4 = \sqrt{8\pi G_N} = 1/M_{\rm Pl}$ is the gravitational coupling constant and $M_{\rm Pl} = 2.4 \times 10^{18}$ GeV denotes the reduced Planck scale. \mathcal{G} is a combination of the Kähler potential \mathcal{K} and the superpotential \mathcal{W} . The Kähler potential \mathcal{K} is often expressed as

$$\kappa_d^2 \mathcal{K}(\tau, \bar{\tau}, S, \bar{S}) = K(\tau, \bar{\tau}, S, \bar{S}) - h \ln\left[-i(\tau - \bar{\tau})\right], \qquad (2.8)$$

where the parameter h is a dimensionless constant which depends on the choice of the number of compactified complex dimensions [22]. The effective SUGRA description of the low-energy limit of superstring theory gives $h=3^1$. At the tree level, there is no τ -dependence in the dilaton Kähler potential, hence $K(\tau, \bar{\tau}, S, \bar{S}) = -\ln(S + \bar{S})$. When we consider the non-perturbative contributions, the corresponding dilaton Kähler potential is given by

$$K(\tau, \bar{\tau}, S, \bar{S}) = K(S, \bar{S}) = -\ln(S + \bar{S}) + \delta k(S, \bar{S}),$$
 (2.10)

where the additional term δk associated with Shenker-like effects in heterotic string theories [27]. It provides non-perturbative corrections to Kähler potential within the 4D low-energy effective field theory, of the order of $\mathcal{O}(e^{-1/g_s^2})$, where g_s^2 represents the closed string coupling constant.

The combination $\tau - \bar{\tau}$ transform as $(-i\tau + i\bar{\tau}) \to |c\tau + d|^{-2}(-i\tau + i\bar{\tau})$ under the action of modular symmetry. Therefore the moduli Kähler potential transforms as follow under modular transformations

$$-3\ln[-i(\tau - \bar{\tau})] \to -3\ln[-i(\tau - \bar{\tau})] + 3\ln(c\tau + d) + 3\ln(c\bar{\tau} + d). \tag{2.11}$$

These extra terms has to be cancelled by the transformation of superpotential W to keep the modular invariance of the SUGRA function \mathcal{G} . This requires that the superpotential W be a modular function of weight -h and its modular transformation is,

$$W \to e^{i\delta(\gamma)}(c\tau + d)^{-h}W$$
, (2.12)

where $\delta(\gamma)$ is a phase depending on the modular transformation γ , and it is the so-called multiplier system.

The modular invariant scalar potential is given by the usual supergravity expression [25]

$$V(\tau, S) = e^{\kappa_4^2 \mathcal{K}} (\mathcal{K}^{\alpha \overline{\beta}} D_\alpha \mathcal{W} \overline{D_\beta \mathcal{W}} - 3\kappa_4^2 |\mathcal{W}|^2), \qquad (2.13)$$

$$\mathcal{K} = -\ln(S + \bar{S}) + \sum_{i=1}^{3} \ln[-i(\tau_i - \bar{\tau}_i)], \qquad (2.9)$$

for which K is completely symmetric under the exchange of the three τ_i . We consider the minimal case, the symmetric point with $\tau_1 = \tau_2 = \tau_3 = \tau$ that freezes all moduli fields except the single modulus τ . This gives h = 3.

¹In corresponding theory, the compactification of six dimensions will bring about three moduli τ_i (i = 1, 2, 3) that corresponds to the radii of the three two-tori of the internal space and its standard form of Kähler potential [17]

where the covariant derivative is defined by $D_{\alpha}W \equiv \partial_{\alpha}W + \kappa_4^2W(\partial_{\alpha}K)$ and $K^{\alpha\beta}$ is the inverse of the Kähler metric $K_{\alpha\beta} = \partial_{\alpha}\partial_{\beta}K$. The indices α , β run over all superfields for our discussion, α , $\beta = \tau$, S. Combined with the kinetic term, the total bosonic action is given by [28]:

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\rm Pl}^2}{2} \mathcal{R} - g^{\mu\nu} \mathcal{K}_{\alpha\overline{\beta}} \partial_{\mu} \phi^{\alpha} \partial_{\nu} \overline{\phi^{\beta}} - V(\phi) \right] , \qquad (2.14)$$

where \mathcal{R} is the Ricci scalar and field $\phi^{\alpha} = (\tau, S)^T$. Specific expressions for the Kähler matrix and the superpotential depends on the models of slow-roll inflation, which will be discussed in the next section.

3 Modular invariant inflation

3.1 A short introduction to slow roll inflation

Inflation has been widely employed to address many shortcomings of the standard big bang cosmological model, such as the horizon and flatness problems. It suggests a period of exponential expansion, during which quantum fluctuations seed the primordial perturbations in our universe. The simplest way to realize inflation is through the slow roll mechanism: The inflationary field, inflaton ϕ , evolves over a flat region of the potential $V(\phi)$ [29]. The potential energy sources the exponential expansion, and the flatness of the potential ensures that this expansion lasts for a sufficiently long duration. Conventionally, we use derivatives of the potential to measure its flatness, which are referred to as the first (second, third, fourth) slow roll parameters [30]:

$$\varepsilon_V = \frac{M_{\rm Pl}^2}{2} \left(\frac{V'}{V}\right)^2 \,, \tag{3.1}$$

$$\eta_V = M_{\rm Pl}^2 \left(\frac{V''}{V}\right) \,, \tag{3.2}$$

$$\xi_V = M_{\rm Pl}^2 \left(\frac{V'V'''}{V^2}\right)^{\frac{1}{2}},$$
(3.3)

$$\varpi_V = M_{\rm Pl}^2 \left(\frac{V'^2 V''''}{V^3} \right)^{\frac{1}{3}},$$
(3.4)

where V represents the potential, and the prime 'denotes the derivative of the potential with respect to the inflaton field ϕ . The duration of the expansion is reflected by the growth of the scale factor a. We define the number of e-folds for a given mode k_* as the logarithm of its growth:

$$N_e(\phi_*) = \ln\left(\frac{a_{\rm end}}{a_*}\right) = \int_{\phi_*}^{\phi_{\rm end}} \frac{\mathrm{d}\phi}{\sqrt{2\varepsilon_V}},$$
 (3.5)

where a_* represents the scale factor when the k_* mode first crosses out of the horizon, and $a_{\rm end}$ is the scale factor at the end of inflation. Here ϕ_* and $\phi_{\rm end}$ are their corresponding field values respectively. Successful inflation requires both the first and second slow-roll parameters to be small, ε_V , $|\eta_V| \ll 1$, and inflation terminates when they become of order one: $\varepsilon_V(\phi_{\rm end}) = 1$ or $|\eta_V(\phi_{\rm end})| = 1$.

The constraints on inflation phenomenology mainly arise from the cosmic microwave background (CMB). The overall isotropies of the CMB tell us that the number of e-folds

must be sufficiently large. Its small anisotropies characterize the primordial cosmological perturbations. For single-field inflation, the reduced spectrum of curvature perturbations is usually parameterized by a power law:

$$\mathcal{P}_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1} \,, \tag{3.6}$$

where k_* serves as a reference (or "pivot") scale. The spectral index, denoted by n_s , is given by $n_s - 1 = 2\eta_V(\phi_*) - 6\varepsilon_V(\phi_*)$ for single-field slow-roll inflation.

Tensor perturbations induce primordial gravitational waves, exhibiting a similar powerlaw behavior:

$$\mathcal{P}_t(k) = A_t \left(\frac{k}{k_*}\right)^{n_t}. \tag{3.7}$$

The tensor-to-scalar ratio r is defined as $r = A_t/A_s$. In single-field inflation, the tensor spectral index $n_t = -r/8 = -2\varepsilon_V(\phi_*)$, which is known as the consistency relation. The current bounds on the spectral index and tensor-to-scalar ratio are important for constraining inflationary models: [2, 30]:

$$\ln (10^{10} A_s) = 3.044 \pm 0.014 (68\% CL),$$

$$n_s = 0.9649 \pm 0.0042 (68\% CL),$$

$$r < 0.036 (95\% CL).$$
(3.8)

The detailed bound on the number of e-folds depends on the post-inflationary dynamics [31, 32], and it is mostly chosen to be $50 < N_e < 60$.

3.2 General properties of the modular invariant scalar potential

In this paper, our focus is on the following Kähler potential and superpotential [17]:

$$\mathcal{K}(\tau, \bar{\tau}, S, \bar{S}) = M_{\text{Pl}}^2 K(S, \bar{S}) - 3M_{\text{Pl}}^2 \ln(-i(\tau - \bar{\tau})),$$

$$\mathcal{W}(S, \tau) = \Lambda_W^3 \frac{\Omega(S) H(\tau)}{\eta^6(\tau)},$$
(3.9)

where S is the dilaton field, which is a modular invariant field, and Λ_W is the characteristic energy scale for this interaction. The function $\Omega(S)$ is technically arbitrary. It could take the form $\Omega(S) = h + e^{-S/b_a}$, which arises from gaugino condensation. Here h is a constant and b_a is related to the beta function of gauge group factor. However, we will assume the dilation field is stabilized as a premise. Thus we will not specify it in this work. τ is the modulus field which serves as the inflaton. We use the same parameterization of the superpotential as in [16]:

$$H(\tau) = (j(\tau) - 1728)^{m/2} j(\tau)^{n/3} \mathcal{P}(j(\tau)), \qquad (3.10)$$

where $j(\tau)$ is the modular invariant j function, and $\mathcal{P}(j(\tau))$ is an arbitrary polynomial function of $j(\tau)$. m and n are both non-negative integers. To simplify the analysis, we choose the polynomial to be second order in $j(\tau)$:

$$\mathcal{P}(j(\tau)) = 1 + \beta \left(1 - \frac{j(\tau)}{1728} \right) + \gamma \left(1 - \frac{j(\tau)}{1728} \right)^2, \tag{3.11}$$

where β, γ are free real parameters. An equivalent parameterization, as discussed in [18], is given by:

 $H(\tau) = \left(\frac{G_4(\tau)}{\eta^8(\tau)}\right)^n \left(\frac{G_6(\tau)}{\eta^{12}(\tau)}\right)^m \mathcal{P}(j(\tau)), \qquad (3.12)$

where G_4 and G_6 are Eisenstein series of weight 4 and 6, respectively. Their definition can be found in Appendix B. Both G_4/η^8 and G_6/η^{12} transform with a phase factor under modular transformation. Note Eq. (3.10) and Eq. (3.12) are equivalent up to a normalization constant². This superpotential was first proposed in [17] to avoid singularities inside the fundamental domain, based on a theorem in [33]. One can directly see that H vanishes at $\tau = \omega$ when $n \ge 1$ and at $\tau = i$ when $m \ge 1$. Using Eq. (2.13), we can straightforwardly obtain the scalar potential as follows:

$$V(\tau, S) = \Lambda^4 e^{K(S,\bar{S})} Z(\tau, \bar{\tau}) |\Omega(S)|^2 \left[(A(S,\bar{S}) - 3) |H(\tau)|^2 + \hat{V}(\tau, \bar{\tau}) \right], \tag{3.13}$$

where we have defined $\Lambda = (\Lambda_W^6/M_{\rm Pl}^2)^{1/4}$ with

$$A(S,\bar{S}) = \frac{K^{S\bar{S}}D_SWD_{\bar{S}}\bar{W}}{|W|^2} = \frac{K^{S\bar{S}}|\Omega_S + K_S\Omega|^2}{|\Omega|^2},$$

$$\hat{V}(\tau,\bar{\tau}) = \frac{-(\tau - \bar{\tau})^2}{3} \left| H_{\tau}(\tau) - \frac{3i}{2\pi}H(\tau)\hat{G}_2(\tau,\bar{\tau}) \right|^2,$$

$$Z(\tau,\bar{\tau}) = \frac{1}{i(\tau - \bar{\tau})^3|\eta(\tau)|^{12}},$$
(3.14)

where the subscript in $\Omega_S = \partial \Omega/\partial S$ and $H_\tau = \partial H/\partial \tau$ denotes the derivative with respect to the specific field. \hat{G}_2 is the non-holomorphic modular form of weight 2. Its definition can be found in the Appendix B. In addition, both the functions \hat{V} and Z are modular functions with weight 0. We have assumed that the dilaton sector is stabilized, thus $A(S, \bar{S})$, $e^{K(S, \bar{S})} |\Omega(S)|^2$ are constants. Only the τ dependence of the scalar potential is important, which is described by the following scalar potential:

$$V(\tau) = \frac{\Lambda_S^4}{i(\tau - \bar{\tau})^3 |\eta(\tau)|^{12}} \left[(A(S, \bar{S}) - 3) |H(\tau)|^2 + \hat{V}(\tau, \bar{\tau}) \right], \tag{3.15}$$

where $\Lambda_S^4 = \Lambda^4 e^{K(S,\bar{S})} |\Omega(S)|^2$ represents the overall scale of the potential, which will be fixed by the normalization of Gaussian curvature perturbations power spectrum. The $A(S,\bar{S})$ term is crucial for uplifting the potential. Once $A(S,\bar{S})>3$, we can guarantee that the potential in Eq. (3.13) is positive semi-definite. The vacuum structure of this potential at $\tau=i$ and at $\tau=\omega=e^{i2\pi/3}$ has been extensively studied in [18], where they find the following results based on the choice of (m,n) in Eq. (3.10):

- If m = n = 0, then both fixed points can have a de Sitter (dS) vacuum.
- If m > 1, n = 0, then $\tau = \omega$ is a dS minimum, while $\tau = i$ is Minkowski minimum.
- If m=0, n>1, then $\tau=i$ is a conditional dS minimum, which depends on the value of $A(S,\bar{S})$. $\tau=\omega$ is always a Minkowski minimum.

²The equivalence of the two expressions of the H function in Eq. (3.10) and Eq. (3.12) can be seen from Eqs. (B.24) and (B.25).

- If m = 1, n > 0 or n = 1, m > 0, the vacuum is unstable.
- If m > 1, n > 1, then we always have Minkowski extrema in these two fixed points.

The scalar potential in Eq. (3.13) is modular invariant. Its derivatives, $\partial_{\tau}V$, $\partial_{\bar{\tau}}V$, are a weight (2,0)/(0,2) non-holomorphic modular forms. Hence, they vanish at the fixed points:

$$\frac{\partial V}{\partial \tau}\Big|_{\tau=i,\omega} = \frac{\partial V}{\partial \bar{\tau}}\Big|_{\bar{\tau}=i,\omega} = 0.$$
 (3.16)

Moreover, the scalar potential is also invariant under $\tau \to -\bar{\tau}$. This can be proven by noticing the following transformation properties³:

$$\eta(\tau) \to \eta(\tau)^*, \quad H(\tau) \to H(\tau)^*,
H_{\tau} \to -H_{\tau}^*, \quad \hat{G}_2 \to \hat{G}_2^*.$$
(3.17)

This fact comes from the reality of the scalar potential. Together with the modular transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$, they ensure the first derivative along certain directions at the boundary of the fundamental domain vanishes [17]⁴:

$$\frac{\partial V}{\partial Re(\tau)}\Big|_{Re(\tau)=\pm 1/2,0} = 0, \qquad \frac{\partial V}{\partial \rho}\Big|_{\rho=1} = 0,$$
 (3.18)

where we have used $\tau = \text{Re}(\tau) + i \text{Im}(\tau)$ in the first equality and $\tau = \rho e^{i\theta}$ in the second equality. To simplify our analysis during inflation, we will assume $d\text{Re}(\tau)/dt = 0$ or $d\rho/dt = 0$, where t is the cosmological time, i.e. We neglect the motion of $\text{Re}(\tau)$ and ρ at the corresponding boundaries.

Based on the observation we made above, we consider two different trajectories of inflation along the boundary of fundamental domain:

- $m = 0, n \ge 2$, we consider slow roll along the lower boundary (arc) from one fixed point i to another fixed point ω .
- $m \ge 2, n \ge 2$, we consider slow roll along the left boundary from $i\infty$ to the fixed point ω .

We illustrate these two trajectories in figure 1. We will show some concrete examples below where the scalar potential is flat enough to accommodate inflation.

$$\frac{\partial V(x)}{\partial x}|_{x=-0.5} = \frac{\partial V(-x-1)}{\partial x}|_{x=-0.5} = -\frac{\partial V(u)}{\partial u}|_{u=-0.5}$$

here, we have defined u = -x - 1. Hence, the derivative vanishes at the fixed points $x = \pm 0.5$.

For the second case, we can express $\tau = \rho e^{i\theta}$. The combined transformations of $\tau \to -1/\tau$ and $\tau \to -\bar{\tau}$ indicate that the potential is invariant under $\rho \to 1/\rho, \theta \to \theta$. Hence $V(\rho) = V(1/\rho)$ and $\rho = 1$ is a fixed point of the transformation. Taking derivative of ρ on both sides tells us that the derivative has to vanish.

$$\frac{\partial V(\rho)}{\partial \rho}|_{\rho=1} = \frac{\partial V(1/\rho)}{\partial \rho}|_{\rho=1} = -u^2 \frac{\partial V(u)}{\partial u}|_{u=1} = -\frac{\partial V(u)}{\partial u}|_{u=1}.$$

where we have defined $u = 1/\rho$.

³We have assumed that the coefficients of the polynomial $\mathcal{P}(j)$ to be real.

⁴Let's write $\tau = x + iy$, where x and y are the real and imaginary parts of τ , respectively. The combined modular symmetry $\tau \to \tau + 1$ and reality condition $\tau \to -\bar{\tau}$ tell us that the potential is invariant under $x \to -x - 1, y \to y$ and x = -0.5 is a fixed point of the symmetry. Hence we have V(x) = V(-x - 1). Taking derivative with respect to x on both sides yields:

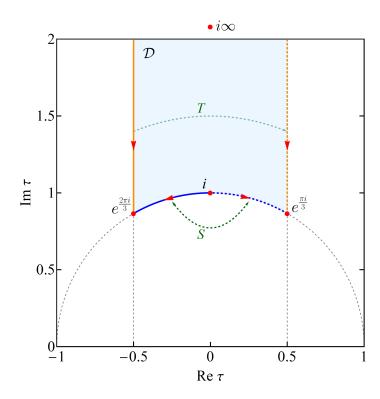


Figure 1: The light blue region represents the fundamental domain \mathcal{D} of the modular group, while the blue line denotes the inflationary trajectory from maxima $\tau = i$ to minima $\tau = \omega = e^{i2\pi/3}$. Additionally, the blue dashed line depicts the inflaton slowly rolls from i to $-\omega^2 = e^{i\pi/3}$. Meanwhile, the orange line signifies the occurrence of accidental inflation to the point ω .

3.3 Constraints on the scalar potential from inflation and modular stabilization

Let's first focus on the case where inflation occurs at the lower boundary. Before going to details of inflation, we would like to first emphasize the difference between the scalar potential suitable for inflation and previous scalar potentials used to study modular stabilization [17, 18]. The most distinct point is the property of scalar potential at the fixed points $\tau = i, \omega$. In the context of modular stabilization, these fixed points are chosen to be the minimum of the scalar potential. In the context of inflation, we choose one of them to be a saddle point of the potential. Hence we have a very different parameter space compared with the existing results. Modular stabilization also doesn't care how the fixed points are connected. There are also similarities among two type of works. We both need a stabilized dilaton sector, hence we have to exclude the possibility for m = 1 or n = 1. We also need the potential non-negative in the fundamental domain, hence we always require $A(S, \bar{S}) \geq 3$.

The modulus field, as a complex scalar, has two degrees of freedom, while the single field slow roll inflation only needs one dynamical field. Hence we would like to separate the modulus field to one inflaton field and another one perpendicular to inflation direction. We further require that the scalar potential has a maximum at one fixed point along the inflation direction such that inflaton can smoothly roll down from it. To stabilize the trajectory, it needs to be a minimum in the perpendicular direction. We also need a minimum to be the destination of the inflaton field. Therefore one of the fixed points has to be a saddle point and

another one remains to be a minimum. This can not be achieved for $m \geq 2, n = 0$ since both fixed points are local minima of the scalar potential. In principle, there exist some parameter space for m = n = 0 where $\tau = i$ is a saddle point and $\tau = \omega$ is a dS minimum. We may refer V(i) to inflation scale and $V(\omega) \approx 10^{-122}$ to cosmological constant today⁵. However, the ratio of their value reads:

$$\frac{V(i)}{V(\omega)} = \frac{\Gamma^{18}(1/3)}{\Gamma^{12}(1/4)} \frac{|\mathcal{P}(1728)|^2}{|\mathcal{P}(0)|^2}.$$
 (3.19)

Thus for a TeV scale inflation where $V(i) \approx 10^{-30}$, We need $|\mathcal{P}(1728)|^2/|\mathcal{P}(0)|^2 \approx 10^{90}$. Given the fact that $j(\tau)$ only varies by 10^3 from $\tau = i$ to $\tau = \omega$, it seems very unnatural to consider such a huge difference in the polynomial $\mathcal{P}(j)$. Hence we do not consider the choice m = n = 0 in this work.

Another possibility is $n \geq 2, m = 0$ where $\tau = i$ is a saddle point and $\tau = \omega$ is a global minimum with $V(\omega) = 0$. To explicitly discuss such a possibility, we first calculate the Hessian matrix at the fix points. The first derivative of the scalar potential vanishes and the second derivatives at $\tau = i$ read:

$$V(i) = \Lambda^4 e^{K(S,\bar{S})} |\Omega(S)|^2 (A-3) 12^{2n} \frac{(2\pi)^9}{\Gamma^{12}(1/4)} |\mathcal{P}(1728)|^2 ,$$

$$\partial_{\tau}^2 V(i) = -\mathcal{C}_n (A-1) \mathcal{B}_n , \quad \partial_{\tau} \partial_{\bar{\tau}} V(i) = \mathcal{C}_n (A-2+|\mathcal{B}_n|^2) .$$
(3.20)

where the functions \mathcal{B}_n and \mathcal{C}_n are respectively defined as

$$\mathcal{B}_{n} \equiv \frac{\Gamma^{8}(1/4)}{192\pi^{4}} \left(1 + 8n + 41472 \frac{\mathcal{P}'(1728)}{\mathcal{P}(1728)} \right) ,$$

$$\mathcal{C}_{n} \equiv \Lambda^{4} e^{K(S,\bar{S})} |\Omega(S)|^{2} \frac{(2\pi)^{9} 4^{2n-1} 3^{2n+1}}{\Gamma^{12}(1/4)} |\mathcal{P}(1728)|^{2} .$$
(3.21)

For inflation along the arc, it is convenient to calculate the Hessian matrix **H** in terms of the polar coordinates $\tau = \rho e^{i\theta}$. Their derivative can be calculated from:

$$\begin{split} \frac{\partial V}{\partial \rho} &= 2 \mathrm{Re} \left[\tau \frac{\partial V}{\partial \tau} \right], \qquad \frac{\partial V}{\partial \theta} = -2 \mathrm{Im} \left[\tau \frac{\partial V}{\partial \tau} \right], \\ \frac{\partial^2 V}{\partial \rho^2} &= 2 \mathrm{Re} \left[\tau^2 \frac{\partial^2 V}{\partial \tau^2} \right] + 2 \frac{\partial^2 V}{\partial \tau \partial \overline{\tau}}, \\ \frac{\partial^2 V}{\partial \theta^2} &= -2 \mathrm{Re} \left[\tau^2 \frac{\partial^2 V}{\partial \tau^2} \right] - 2 \mathrm{Re} \left[\tau \frac{\partial V}{\partial \tau} \right] + 2 \frac{\partial^2 V}{\partial \tau \partial \overline{\tau}}, \\ \frac{\partial^2 V}{\partial \theta \partial \rho} &= -2 \mathrm{Im} \left[\tau^2 \frac{\partial^2 V}{\partial \tau^2} \right] - 2 \mathrm{Im} \left[\tau \frac{\partial V}{\partial \tau} \right]. \end{split}$$
(3.22)

Utilizing Eqs. (3.20), elements of the Hessian matrix **H** at $\tau = i$ read:

$$\frac{\partial^2 V}{\partial \rho^2} = 2C_n \left[A - 2 + |\mathcal{B}_n|^2 + (A - 1) \operatorname{Re}(\mathcal{B}_n) \right] ,$$

$$\frac{\partial^2 V}{\partial \theta^2} = 2C_n \left[A - 2 + |\mathcal{B}_n|^2 - (A - 1) \operatorname{Re}(\mathcal{B}_n) \right] ,$$

$$\frac{\partial^2 V}{\partial \theta} = -2C_n (A - 1) \operatorname{Im}(\mathcal{B}_n) .$$
(3.23)

⁵We use Planck units here.

The fixed point $\tau = i$ can be maximum for angular direction θ and minimum for the radial direction ρ . In Eqs. (3.23), C_n is a positive number. The imaginary part of \mathcal{B}_n also vanishes at $\tau = i$ (the coefficients of $\mathcal{P}(j)$ are real in our setting, hence \mathcal{P} is always real at $\tau = i$) which means the cross derivative is zero. Our combined requirements:

$$\frac{\partial^2 V}{\partial \rho^2}\bigg|_{\tau=i} > 0, \quad \frac{\partial^2 V}{\partial \theta^2}\bigg|_{\tau=i} < 0, \quad V(i) > 0, \tag{3.24}$$

leads to:

$$A > 2 + \mathcal{B}_n \,, \qquad \mathcal{B}_n > 1 \,. \tag{3.25}$$

Note that $\mathcal{B}_n > 1$ restricts the polynomial parameter in Eq. (3.11):

$$\beta < \frac{1}{24} \left(8n + 1 - \frac{192\pi^4}{\Gamma^8(1/4)} \right) . \tag{3.26}$$

And $A > 2 + \mathcal{B}_n$ can be written into a more suggestive form:

$$A > \frac{\Gamma^8(1/4)}{192\pi^4} \left(1 + 8n - 24\beta\right) + 2. \tag{3.27}$$

In particular, when $\beta = 0$ and n = 2, the parameter A should greater than 29.132. In terms of the dilaton sector, there exist a solution where the dilaton is stabilized ($\partial_S V = 0$):

$$A(S,\bar{S}) = \exp\left[-\int dS \frac{\partial_S \Omega + \Omega \partial_S K}{\Omega(S)} + C(\bar{S})\right] + 3, \qquad (3.28)$$

where C may be a function of \bar{S} . Besides the local property of the scalar potential at the fix point, we would also like to impose an additional constraint along the inflationary trajectory $\pi/2 < \theta < 2\pi/3$:

$$\frac{\partial V}{\partial \theta} < 0. \tag{3.29}$$

This ensures that the inflaton smoothly rolls down to the minimum at $\tau = \omega$. The algebraic expression for this condition is rather complicated and we solve it numerically. This constraint is demonstrated in figure 2.

The potential at the fixed point $\tau = \omega$ or $\tau = -\omega^2$ is much simpler, we have:

$$V(\omega) = 0$$
, $\partial_{\tau}^{2}V(\omega) = 0$, $\partial_{\tau}\partial_{\bar{\tau}}V(\omega) \ge 0$. (3.30)

From Eq. (3.13), as long as $A \ge 3$ the potential is non-negative in the whole complex space. Hence $V(\omega) = 0$ is enough to ensure $\tau = \omega$ is a global minimum of the scalar potential.

3.4 Slow roll along the unit arc

In this scenario, it might be useful to rewrite the scalar potential in terms of radial and angular component, $\tau = \rho e^{i\theta}$. The kinetic term reads [34]:

$$\mathcal{L}_{\rm kin} = -\frac{\partial^2 \mathcal{K}}{\partial \tau \partial \bar{\tau}} \partial_{\mu} \tau \partial^{\mu} \bar{\tau} = 3 \frac{M_{\rm Pl}^2}{(-i\tau + i\bar{\tau})^2} \partial_{\mu} \tau \partial^{\mu} \bar{\tau} = \frac{3M_{\rm Pl}^2}{4\sin^2(\theta)} \left(\frac{1}{\rho^2} \partial_{\mu} \rho \partial^{\mu} \rho + \partial_{\mu} \theta \partial^{\mu} \theta \right) . \quad (3.31)$$

As shown in Eq. (3.18), the modular invariance of the scalar potential requires $\partial V/\partial \rho|_{\rho=1}=0$. Here and hereafter we will always set $\rho=1$ and keep θ as the only degree of freedom. To normalize the kinetic term of θ , we further introduce the canonical field $\phi=\sqrt{3/2}M_{\rm Pl}\ln(\tan(\theta/2))$. As an example, we have $\phi=0$ when $\theta=\pi/2$.

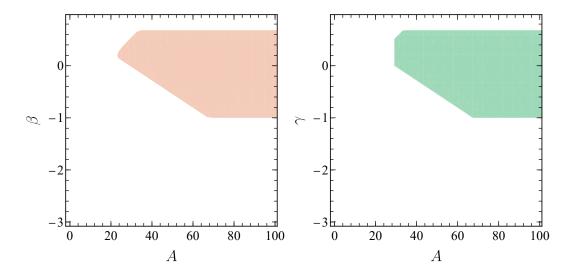


Figure 2: In the left panel, we examine the parameter space (A, β) with $\gamma = 0$, constrained by the condition $\partial V/\partial \theta < 0$. Similarly, the right panel explores the parameter space (A, γ) with $\beta = 0$. Here, we illustrate this concept using the example n = 2.

In this section, we consider the case where $m=0, n\geq 2$ in Eq. (3.10). This potential has a local maximum at $\tau=i$ and a local Minkowski minimum at $\tau=\omega$. As we said before, the modular symmetry ensures that the first derivative of the potential vanishes at $\tau=i$, which motivates us to investigate the inflation near it. The inflation trajectory is shown in figure 1. The inflation phenomenology can be approximated by its Taylor expansion near $\tau=i$ ($\phi=0$). The full potential in Eq. (3.15) can be approximated by the following term during inflation:

$$V(\phi) = V_0(1 - \sum_{k=1}^{\infty} C_{2k} \phi^{2k}), \qquad (3.32)$$

where each coefficient depends on the choice of $A(S, \bar{S})$, (m, n) and the parameterization of the polynomial function $\mathcal{P}(j)$. Note that the potential is an even function of ϕ , which comes from the S symmetry of the modular group. Along the arc, the S symmetry $\tau \to -1/\tau$ indicates a Z_2 symmetry in terms of the canonically normalised field $\phi \to -\phi$. We will focus on the case where $\phi > 0$. During inflation, we find the potential is mostly dominated by $C_2\phi^2$ and $C_{2p}\phi^{2p}$ terms where p is a specific integer. This means $0 < |C_{2p'}| \ll C_{2p}$ for all the p' < p. Let's first investigate this simplified potential:

$$V(\phi) = V_0(1 - C_2\phi^2 - C_{2p}\phi^{2p}), \qquad (3.33)$$

The slow-roll parameters read:

$$\varepsilon_{V} = \frac{1}{2} \left(\frac{V'}{V} \right)^{2} = \frac{1}{2} \left(\frac{-2C_{2}\phi - 2pC_{2p}\phi^{2p-1}}{1 - C_{2}\phi^{2} - C_{2p}\phi^{2p}} \right)^{2},
\eta_{V} = \frac{V''}{V} = \frac{-2C_{2} - 2p(2p-1)C_{2p}\phi^{2p-2}}{1 - C_{2}\phi^{2} - C_{2p}\phi^{2p}},$$
(3.34)

where $C_2 \ll C_{2p}$ and $C_{2p} \gg 1$. We are interested in the region where $\phi \ll 1$, which means $\varepsilon_V \approx (\eta_V \phi)^2 \ll \eta_V$. Note the start point of the observable inflation ϕ_* can be calculated

from:

$$n_s = 1 - 6\epsilon(\phi_*) + 2\eta(\phi_*) \approx 1 + 2\eta(\phi_*) \approx 1 - 4C_2 - \mathcal{O}(\phi_*^2),$$
 (3.35)

and the CMB observation suggests $n_s \approx 0.9649$ [30]. For our setup we would like to require $1-4C_2 > n_s$, otherwise the spectral index will be smaller than the observed value of CMB. This roughly requires $0 < C_2 < 0.008$.

The starting point ϕ_* and end point ϕ_e of the observable inflation is controlled by η and corresponding field value reads:

$$\phi_* \approx \left(-\frac{\eta_* + 2C_2}{2p(2p-1)C_{2p}} \right)^{\frac{1}{2p-2}},$$

$$\phi_e \approx \left(-\frac{\eta_e + 2C_2}{2p(2p-1)C_{2p}} \right)^{\frac{1}{2p-2}},$$
(3.36)

where we have defined $\eta_* = (n_s - 1)/2$, $\eta_e = -1$ and used $1 - C_2\phi^2 - C_{2p}\phi^{2p} \approx 1$. This can be verified by plugging in the above solution to the expression, which is suppressed by large C_{2p} . The number of e-folds is given by:

$$N_{e} = \int_{\phi_{*}}^{\phi_{e}} \frac{1}{\sqrt{2\epsilon(\phi)}} d\phi$$

$$\approx \int_{\phi_{*}}^{\phi_{e}} \frac{1}{2C_{2}\phi + 2pC_{2p}\phi^{2p-1}} d\phi$$

$$= \frac{\ln(2pC_{2p} + 2C_{2}\phi^{2-2p})}{2C_{2}(2p-2)} \Big|_{\phi_{*}}^{\phi_{e}}$$

$$= \frac{1}{2C_{2}(2p-2)} \left(\ln \frac{\eta_{e} + (4-4p)C_{2}}{\eta_{e} + 2C_{2}} - \ln \frac{\eta_{*} + (4-4p)C_{2}}{\eta_{*} + 2C_{2}} \right),$$
(3.37)

which first decrease with C_2 and then increase with it in the region where $0 < C_2 < 0.008$. The minimum of N_e is $N_{e,\text{min}} \approx 77$ for $n_s = 0.9649$, p = 2 and $N_{e,\text{min}} \approx 50$ for $n_s = 0.9649$, p = 3. Thus p = 2 always generates too many e-folds while $p \ge 3$ could be consistent with our current observation. It is thus necessary to ensure C_2 , $C_4 \ll C_6$ in the expansion.

We will work on the coefficients C_{2n} in the following section. To do this, we need the local expansion of the j invariant and η functions in terms of the canonically normalised field ϕ :

$$j(\phi) \approx 1728(1 - 9.579\phi^2 + 40.142\phi^4 - 102.618\phi^6) + \mathcal{O}(\phi^8) ,$$

$$|\eta(\phi)| \approx 0.768 + 0.083\phi^2 - 0.028\phi^4 + 0.005\phi^6 + \mathcal{O}(\phi^8) ,$$
 (3.38)

and the explicit form of the polynomial $\mathcal{P}(j)$. For convenience, we will parameterize it using:

$$\mathcal{P}(j(\tau)) = 1 + \beta \left(1 - \frac{j}{1728}\right) + \gamma \left(1 - \frac{j}{1728}\right)^2 + \dots$$
 (3.39)

As one can see from Eq. (3.38), β controls the ϕ^2 term and γ controls the ϕ^4 term in the polynomial $\mathcal{P}(j)$. The overall scaling of the potential does not depend on the specific form of $\mathcal{P}(j)$, it reads:

$$V_0 = \frac{1728^{2n/3} \Lambda_S^4 \left[A(S, \bar{S}) - 3) \right]}{8|\eta(i)|^{12}}, \tag{3.40}$$

while the coefficients do, thus we will discuss them separately.

3.4.1
$$\mathcal{P}(j) = 1$$

We start our study by considering the simplest case, P(j) = 1. In this scenario, the shape of the potential is completely determined by the dilaton contribution $A(S, \bar{S})$ and we can approximate the potential by:

$$C_{2} = 0.298 + 6.386n - \frac{0.178 + (7.617 + 81.554n)n}{A - 3},$$

$$C_{4} = -0.438 + 1.917n - 20.389n^{2} + \frac{0.992 + 14.387n - 122.250n^{2} + 520.779n^{3}}{A - 3},$$

$$C_{6} = 0.234 + 9.024n - 18.321n^{2} + 43.398n^{3} + \frac{-1.998 - 23.195n - 718.051n^{2} + 1247.593n^{3} - 1662.767n^{4}}{A - 3}.$$

$$(3.41)$$

The requirement for $0 < C_2 < 0.008$ becomes an algebraic constraint on A:

$$3.596 + 12.771n < A < 3.612 + 12.771n, (3.42)$$

which suggests that A is tightly constrained. In this case, the corresponding C_4 and C_6 reads:

$$C_4 \approx 1.228 + n(-9.571 + 20.389n),$$

 $C_6 \approx 1.289 - \frac{0.206}{0.047 + n} + n[-52.076 + (85.482 - 86.797n)n].$ (3.43)

For n=2 this roughly means $A\approx 29.142, C_4\approx 63.640, C_6\approx -455.408$ and one can extend the results for general n easily. Even though $|C_6|>C_4$, we have verified that ϕ^4 contribution is larger than ϕ^6 during inflation and the end of the inflation is controlled by ϕ^4 term. It is also interesting to consider the cases with large n. Note C_4 scales as n^2 while C_6 scales as n^3 and one might be worried about the later term will eventually overgrows the former one. This is not the case. We can exam their relative contribution at the end of inflation, where $C_6\phi^6$ contributes the most. Using Eq. (3.36) with p=2 and keep the assumption that $C_4\phi^4$ is much larger than $C_6\phi^6$, we have:

$$\phi_e^2 = \frac{1}{12C_4}, \qquad \frac{C_6\phi^6}{C_4\phi^4} = \frac{C_6}{12C_4^2} \propto \frac{1}{n},$$
(3.44)

which tells us the relative contribution of $C_6\phi^6$ term is suppressed by large n. This results perfectly align with our assumption and higher order terms never dominate the potential. Meanwhile, the relative large coefficients C_4 leads to an overproduction of the number of e-folds if we choose n_s to be around its central value. i.e. To have $50 < N_e < 60$, n_s has to be smaller than the CMB measurement. Theoretical prediction of this case can be found in figure 3. We use solid and dashed lines to represent the results for $N_e = 50$ and $N_e = 60$, respectively. Different colors are used to label different choices of n. Increasing n does not change the preferred region of the spectral index. It decrease the tensor to scalar ratio, though. The spectral index has to sit between 0.942 - 0.955, which is excluded by the Planck2018 results. Thus we conclude that the simplest choice of $\mathcal{P}(j)$ can have inflation, however, it will not be able to fit to our observation in our framework.

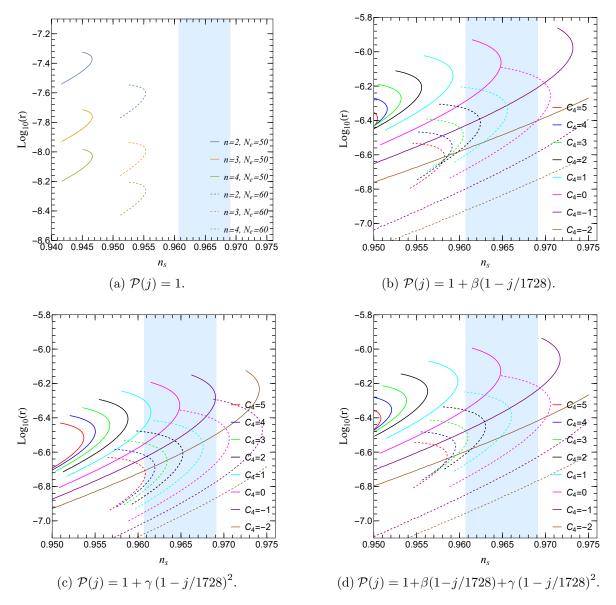


Figure 3: We present theoretical predictions of modular invariant inflation with different choices of the polynomial $\mathcal{P}(j)$. When $\mathcal{P}(j)=1$, the parameter A varies within the region 3.596+12.771n < A < 3.612+12.771n given by Eq. (3.42). Conversely, when $\mathcal{P}(j) \neq 1$, we fix n=2 for plotting purposes. Additionally, we select C_2 and C_4 as the physical parameters and plot the lines by varying C_2 while holding C_4 constant. In the last panel, we set $A(S, \bar{S}) = 25$. The x-axis represents the spectral index of the CMB power spectrum n_S , while the y axis is the tensor-to-scalar ratio r on a logarithm scale. Solid lines indicate predictions for $N_e = 50$, whereas dashed lines are the predictions for $N_e = 60$. Different colors denotes varying choices of C_4 .

3.4.2
$$\mathcal{P}(i) = 1 + \beta (1 - i/1728)$$

The next simplest choice would be let $\beta \neq 0$. In this case, we have two free parameters A and β , with two algebraic constraints: $0 < C_2 < 0.008$ and $0 < C_4 \ll C_6$. The additional β

dependent terms in C_2, C_4, C_6 reads:

$$C_{2} = C_{2}|_{\beta=0} + C_{2,\beta}$$

$$= C_{2}|_{\beta=0} - 19.157\beta + \frac{-733.987\beta^{2} + 22.851\beta + 489.325\beta n}{A - 3},$$

$$C_{4} = C_{4}|_{\beta=0} + C_{4,\beta}$$

$$= C_{4}|_{\beta=0} - 91.748\beta^{2} + \beta(85.997 + 122.331n)$$

$$+ \frac{\beta^{2}(12741.904 + 9374.025n) + \beta(-262.043 - 3953.512n - 4687.013n^{2})}{A - 3},$$

$$C_{6} = C_{6}|_{\beta=0} + C_{6,\beta}$$

$$= C_{6}|_{\beta=0} + \beta^{2}(796.369 + 585.877n) - \beta(237.566 + 475.949n + 390.584n^{2})$$

$$+ \frac{\beta^{2}(106147.206 + 133975.980n + 52377.172n^{2})}{A - 3}$$

$$- \frac{\beta(1383.980 + 16226.188n + 26183.928n^{2} + 19953.208n^{3})}{A - 3},$$

where we have separated the coefficients into β dependent and independent terms. $C_{2,4,6}|_{\beta=0}$ are the same as Eqs. (3.41). The solution for $0 < C_2 < 0.008$ is:

$$3.596 - 38.314\beta + 12.771n < A < 3.612 - 38.314\beta + 12.771n$$
. (3.46)

We further choose $-2 < C_4 < 5$, such that $|C_4| \ll C_6$. If $C_4 \gg 1$, inflation potential will be dominated by ϕ^4 term. We have argued such a case is ruled out by the small spectral index n_s . When $C_4 < -2$, it will create additional maximum along the inflation trajectory. The solution reads⁶:

$$n=2, \quad 0.1261 < \beta < 0.1268,$$

 $n=3, \quad 0.2450 < \beta < 0.2454,$
 $n=4, \quad 0.3752 < \beta < 0.3755.$ (3.47)

Plugging the above solution into C_6 , we find $C_6 \approx 350$ for n=2. As in this case we have $C_2, C_4 \ll C_6$, we find this set up can generate inflation potential which agrees with CMB observation. We show a possible shape of the potential and its cross-section in figure 4. Theoretical prediction of this case can be found in figure 3 where we take n=2 to be an example. It is straightforward to extend our results to n>2. We use different color to label different choice of C_4 . Solid and dashed lines represent results for $N_e=50$ and $N_e=60$, respectively. The latter cases are slightly shifted towards a smaller tensor to scalar ratio and lager spectral index. Increasing C_4 will decrease the spectral index and vice versa, as we expected. As one can see from the figure, the spectral index n_s can be in the 1σ region constrained by the CMB observation, while the tensor to scalar ratio r is of order $\mathcal{O}(10^{-6})$. This is well below the current sensitivity. We also show the parameter space spanned by $A(S, \bar{S})$, β in figure 5. The left and right segment are results for $N_e=50$ and $N_e=60$, respectively. We use different color to represent different values of spectral index n_s .

⁶There are other solutions, however, they introduce additional barrier between inflation point and minimum of the potential. Thus we will ignore them in this work.

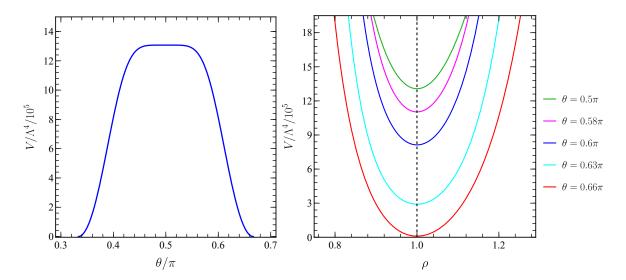


Figure 4: When the potential has parameters m=0, n=2, A=24.3091, and $\beta=0.126425$, the left panel demonstrates the cross-section of the scalar potential via $\rho=1$ during inflation. Similarly, the right panel displays the cross-sections of scalar potential via various θ values throughout the inflationary process.

3.4.3
$$\mathcal{P}(j) = 1 + \gamma (1 - j/1728)^2$$

The next simple choice would be turn on the ϕ^4 contribution in $\mathcal{P}(j)$ by introducing non-zero value of γ . In this case, C_2 remains unchanged while C_4 , C_6 get additional contribution, which we denote by $C_{4,\gamma}$ and $C_{6,\gamma}$:

$$\begin{split} C_2 = & C_2|_{\gamma=0} \,, \\ C_4 = & C_4|_{\gamma=0} + C_{4,\gamma} \\ = & C_4|_{\gamma=0} - 183.497\gamma + \frac{(437.766 + 9374.025n)\gamma}{A-3} \,, \\ C_6 = & C_6|_{\gamma=0} + C_{6,\gamma} \\ = & C_6|_{\gamma=0} + \gamma (1171.753n + 1592.738) \\ & + \frac{-269368.311\gamma^2 - 6821.952\gamma - 113625.201n\gamma - 74824.531n^2\gamma}{A-3} \,. \end{split}$$

Since C_2 is unchanged, the solution for $0 < C_2 < 0.008$ remains the same as the $\mathcal{P}(j) = 1$ case:

$$3.596 + 12.771n < A < 3.612 + 12.771n. (3.49)$$

Again requiring $-2 < C_4 < 5$, γ is constrained to be:

$$n = 2, \quad -0.119 < \gamma < -0.106,$$

 $n = 3, \quad -0.287 < \gamma < -0.274,$
 $n = 4, \quad -0.529 < \gamma < -0.516.$ (3.50)

These conditions ensure $C_2, C_4 \ll C_6$ so that the ϕ^2, ϕ^6 terms dominate during inflation. We demonstrate the theoretical prediction for n=2 in figure 3. There is no surprise that the prediction is similar to the previous case where we choose $\beta \neq 0$. They have similar

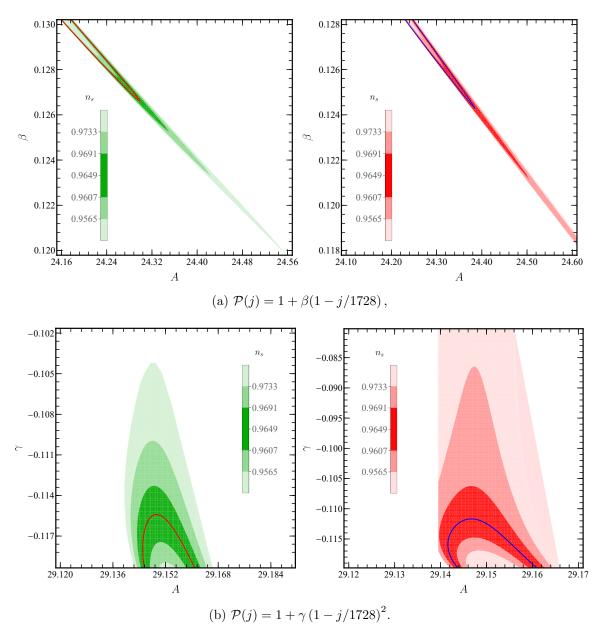


Figure 5: These panels illustrate contour plot of the spectral index n_s across the parameter planes (A, β) (upper sector) and (A, γ) (lower sector). In the left segment, the deep green region displays the 68% CL region, with the red line indicating the contour of central value of n_s for 50 e-folds. Conversely, The right segment demonstrates the distribution of n_s , with the deep red region and blue line representing 68% CL region and its isopleth of central value for 60 e-folds, as documented in [30]. Furthermore, it's important to note that these panels adhere entirely to the constraint specified in Eq. (3.27).

local expansion and inflation trajectory. The spectral index n_s can again lie in the 1σ region constrained by the CMB observation, while the tensor to scalar ratio r is of order $\mathcal{O}(10^{-6})$. We also show the parameter space spanned by $A(S,\bar{S}), \gamma$ in figure 5. The left and right segment are results for $N_e = 50$ and $N_e = 60$, respectively. We use different color to represent different values of spectral index n_s .

3.4.4
$$\mathcal{P}(i) = 1 + \beta (1 - i/1728) + \gamma (1 - i/1728)^2$$

In this case we can choose A as a free parameter and deduce the value for β and γ accordingly. The expression for C_{2n} now include the mixing term between β and γ , they read:

$$C_{2} = C_{2}|_{\beta=\gamma=0} + C_{2,\beta},$$

$$C_{4} = C_{4}|_{\beta=\gamma=0} + C_{4,\beta} + C_{4,\gamma} - \beta\gamma \frac{28122.076}{A-3},$$

$$C_{6} = C_{6}|_{\beta=\gamma=0} + C_{4,\beta} + C_{4,\gamma} + \beta\gamma \left(-1757.630 + \frac{314263.030n + 603954.174}{A-3}\right).$$
(3.51)

For arbitrary A there are always two possible values of β which make the quadratic coefficient C_2 sufficiently small:

$$\beta_1 \approx 0.016 + 0.333n$$
,
 $\beta_2 \approx 0.094 - 0.026A + 0.333n$. (3.52)

The second solution accommodates the previous case where $n=2,\ \beta=0,\ A=29.142$. One can then find the correct γ value which ensures $C_4\ll C_6$. We find the most favored region is $C_2\approx 0.004,\ C_4<2,\ C_6\approx 900$. We show the theoretical prediction with $A=25,\ n=2$ in figure 3. The physics of inflation remains almost the same as the previous cases and yields a similar prediction. The tensor to scalar ratio r is tiny and the spectral index n_s stays in the 1σ region of the constraints. We choose $A=25,\ n=2$ as an example and list different parameter setups β, γ required for inflation in table 1.

One may wonder if we could achieve successful inflation in the opposite direction, where modular field slow roll from $\tau = \omega$ to $\tau = i$. This will require the potential to have a maximum at $\tau = \omega$ and a minimum at $\tau = i$. The latter condition is easy to fulfill, we just need to choose $m \neq 0$. The former one requires n = 0. However, in this case $\tau = \omega$ is always a local minimum of the potential if A > 3, which is irrelevant with the form of $\mathcal{P}(j)$. This can be seen from their local expansion:

$$j(\phi) \approx 1728(-9.36(\phi - \phi_{\omega})^{3} - 1.56(\theta - \phi_{\omega})^{5}) + \mathcal{O}\left((\phi - \phi_{\omega})^{6}\right),$$

$$V(\phi) \approx V_{0}\left(1 + \frac{1}{2}\frac{A - 2}{A - 3}(\phi - \phi_{\omega})^{2} + C_{3}(\phi - \phi_{\omega})^{3}\right) + \mathcal{O}\left((\phi - \phi_{\omega})^{4}\right),$$
(3.53)

where $V_0 = \frac{1728^m \Lambda_S^4 \left[A(S,\bar{S})-3\right]}{8\sin(2\pi/3)^3 |\eta(\omega)|^{12}}$, $\phi_\omega \approx 0.673$ is the canonical field when $\tau = \omega$. Thus as long as $A(S,\bar{S}) > 3$, which comes from the requirement that the potential should be positive during inflation, $\tau = \omega$ is always a local minimum of the potential. We will not consider slow roll inflation in this case.

3.5 Slow roll in the left (or right) boundary

We would extend our discussion to the case where $m, n \geq 2$. In this case, $\tau = i, \omega$ are both minima of the potential and we consider an inflation trajectory at the left boundary

(or the imaginary axis) of the fundamental domain. Unlike the previous case where we have good understanding of both inflation point and minimum of the potential, we can not make any assumption where inflation will happen. Thus it is generally difficult to give analytic expressions so we will only show an example here. However, the treatment here is using the rich vacuum structure of the modular potential and it should be very generic.

Let's first find the canonical field at the left boundary. The kinetic term for imaginary part of τ reads:

 $\mathcal{L}_{kin} = \frac{1}{2} \frac{3}{2(\text{Im}\tau)^2} \left(\partial_{\mu} \text{Im}\tau\right)^2. \tag{3.54}$

One can make a field redefinition, $\text{Im}\tau = \exp\left(\sqrt{2/3}\,\phi\right)$, to introduce the canonical field ϕ . The minimum of $\text{Im}\tau$ is $\sqrt{3}/2$, which corresponds to $\phi = \sqrt{3/2}\ln\left(\sqrt{3}/2\right) \approx -0.17$.

It has been noticed that there exist multiple local minima along the left boundary of the fundamental domain [18]. In the specific case where $m=2,\ n=2,\ A(S,\bar{S})=0,\ \mathcal{P}(j)=1+10^{-3}j$, they find an additional AdS minimum at the left boundary. This minimum can be uplifted into a dS minimum by turning on the $A(S,\bar{S})$ term. This was called Accidental Inflation in String Theory [35], where the up-lifting of adjacent minimum leads to inflation.

We exploit this idea and show how the potential are shaped by different $A(S,\bar{S})$ values. One can see from the figure 6, when A is small, this additional minimum located at $\phi\approx 0.13$ is a (A)dS minimum, which is separated from the minimum at $\phi=\phi_\omega$ by a barrier. If one increases the value of A, the potential becomes flat and an inflection point will emerge. This very flat region in the potential may break up the normal slow roll inflation approximation and lead to so called "ultra slow roll" inflation [36–38]. It might enhance the curvature perturbation and leads to production of primordial black holes [39–41]. Let's briefly clarify about the difference between slow roll inflation and ultra slow roll inflation. The dynamic of a spatially homogeneous inflaton field ϕ is governed by the Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi} \,, \tag{3.55}$$

where dot refers the derivative w.r.t. cosmic time and $H = \sqrt{(\dot{\phi}^2/2 + V)/3}$ is the Hubble parameter during inflation. In the slow roll approximation one neglects the $\ddot{\phi}$ term. In contrast, during ultra slow roll field derivative $\partial V/\partial \phi$ is neglected, which means $3H|\dot{\phi}| \gg |\partial V/\partial \phi|$. The existence of slow roll region requires both the first and second derivative of the potential to be small:

$$\varepsilon_V = \frac{1}{2} \left(\frac{V'}{V} \right)^2 \ll 1, \quad |\eta_V| = \left| \frac{V''}{V} \right| \ll 1,$$
 (3.56)

where prime means the derivative w.r.t. the inflaton ϕ . The ultra slow roll region is achieved by requiring [42]:

$$\varepsilon_V \ll \frac{9}{4} \left(\frac{V'}{3H\dot{\phi}} \right)^2 \Rightarrow V \gg 3H\dot{\phi} \approx 3\sqrt{V}\dot{\phi} \gg V',$$
 (3.57)

which depends on the kinetic energy of inflaton field and does not constraint on the second derivative η_V . The stability of Ultra slow roll requires $\eta_V > \sqrt{\varepsilon_V}$ or $\eta_V < -\sqrt{\varepsilon_V}$, though. Phenomenologyically, it is more interesting when ultra slow roll happens near the end of inflation, which can enhance the scalar power spectrum in small scales. This requires $\phi_* >$

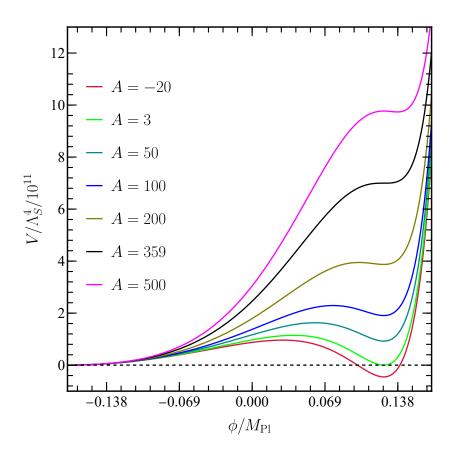


Figure 6: When m=2, n=2 and $\beta=-0.633431$, we show the scalar potential at the left boundary of the fundamental domain across different values of the parameter A. Notably, we observe the emergence of almost flat potential when $A \sim \mathcal{O}(100)$. The flat plateau occurs around $\tau_I \approx 1.11$ or $\phi \approx 0.13$. In addition, the corresponding potential attains its minimum at $\tau = \omega$ when A > 3. Particularly noteworthy is the case when A = 3, where the vacuum becomes degenerate. Conversely, for A < 3, such as A = -20, the AdS global minimum is located at $\tau = 1.10714i$, while the fixed point $\tau = \omega$ serves as a Minkowski local minimum.

 $\phi_0 > \phi_e$, where ϕ_* is the field value corresponding to CMB scale, ϕ_0 is the field value where V' vanishes and ϕ_e is the field value at the end of inflation. We find a narrow region where above inequation holds:

$$357.85 < A < 358.75. \tag{3.58}$$

The situation here needs a very careful treatment and we leave it for further works.

4 Summary and conclusion

In this paper we tried to combine the ideas of modular stabilization and modular inflation. Inspired by the vacuum structure of the modular potential, we successful found two different trajectories which can accommodate inflation phenomenology and agrees with CMB observations. Those trajectories follow the boundaries of the fundamental domain and the property of modular symmetry plays a significant role in our construction. We also find that the modular symmetry is a very strong constraint which prevents us from approximating it as a simple hilltop inflation.

We have three different sets of parameters in our model. The first one is $A(S, \bar{S})$ in Eq. (3.14), which determines the relative contribution from the dilaton sector. It has to be non-zero for successful inflation to happen. It is also the reason why we could evade some no-go theorem stated in previous studies on logarithmic Kähler potential inflation [20]. The second set of parameters are a pair of integers (m, n) from the H function in Eq. (3.10), which determine the vacuum structure of the inflation potentials. We have constructed different trajectories based on that. We keep them as free parameters. They are relevant to reheating process after inflation as they also affect how the inflaton oscillates at the minimum. This is not discussed in our paper, we leave it for further exploration. The third set of parameters (β, γ) are coefficients of the j-invariant polynomial in Eq. (3.11). They actually shape the potential and are essential for constructing inflation potentials.

In summary, the special properties of fixed points under modular symmetry motivate us to consider the following inflation scenario: when $n \geq 2$, m = 0, we find the scalar potential near the fixed point i can be flat enough to accommodate inflation. In this scenario, inflation occurs along the unit arc of the fundamental domain. It is necessary to include one of the parameters in (β, γ) , otherwise, the predicted spectral index n_s is smaller than observed. When n = 2, m = 2, we find the possibility of realizing (ultra) slow-roll inflation by uplifting the adjacent minima of the potential. In this scenario, inflation occurs along the left boundary of the fundamental domain. The contribution from the dilaton sector $A(S, \bar{S})$ is important for such an uplifting. We do not consider the case when $m \geq 2$, n = 0, as both fixed points are local minima.

The Fine-tuning problem still exists in our model. Once we fix the parameter (m, n), $A(S, \bar{S})$ and (β, γ) have to be fixed accordingly. There is no theoretical evidence, besides the argument for accidental inflation, suggesting that they should take these specific values. In this sense, we can only answer the question how a modular invariant inflation would look like. We can not ensure that modular stabilization naturally leads to inflation. In particular we have followed a bottom-up approach to inflation instead of a top-down approach. Here we have assumed that the dilaton is stabilized. The stabilisation of the dilaton sector and its dynamics deserve further research. It is also interesting to consider a scenario in which the dilaton field serves as the inflaton field.

In this paper, we have focused on the single-field inflationary scenario, selecting one degree of the modulus field to be the inflaton after normalisation. We leave the consideration of multi-field inflation scenario for possible future works. Moreover, string compactifications will lead to the appearance of abundant moduli fields. It is worth the effort to investigate how to construct the multiple moduli inflation model.

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A Concrete examples of modular inflation

In this section we list some concrete examples where inflation is realized with different choices of parameters and their predictions, as shown in table 1.

N_e	50			
\overline{A}	29.1470	24.3060	29.1495	25
β	0	0.126434	0	0.108376
γ	0	0	-0.115590	-0.016932
n_s	0.9462	0.9637	0.9643	0.9639
$\log_{10} r$	-7.39	-6.03	-6.30	-6.04
$arepsilon_V$	2.5×10^{-9}	6.7×10^{-8}	3.1×10^{-8}	5.6×10^{-8}
η_V	-0.0269	-0.019	-0.018	-0.018
ξ_V^2	0.00054	0.00028	0.00035	0.00033
ϖ_V^2	-7.7×10^{-6}	-1.2×10^{-5}	-1.2×10^{-6}	-1.2×10^{-5}
N_e			55	
\overline{A}	29.1470	24.3091	29.1441	25
β	0	0.126425	0	0.108152
γ	0	0	-0.115640	-0.017168
n_s	0.9510	0.9649	0.9649	0.9650
$\log_{10} r$	-7.50	-6.04	-6.60	-6.42
$arepsilon_V$	1.9×10^{-9}	5.7×10^{-8}	1.6×10^{-8}	1.5×10^{-8}
η_V	-0.024	-0.018	-0.018	-0.018
ξ_V^2	0.00045	0.00021	0.00038	0.00037
ϖ_V^2	-5.9×10^{-6}	-8.6×10^{-6}	-8.5×10^{-6}	-7.8×10^{-6}
N_e			60	
\overline{A}	29.1470	24.3108	29.1548	25
β	0	0.126420	0	0.108104
γ	0	0	-0.115567	-0.017219
n_s	0.9551	0.9649	0.9649	0.9654
$\log_{10} r$	-7.61	-6.09	-6.34	-6.61
$arepsilon_V$	1.5×10^{-9}	5.1×10^{-8}	2.8×10^{-8}	1.54×10^{-8}
η_V	-0.0225	-0.018	-0.018	-0.017
ξ_V^2	0.00037	0.00013	0.00012	0.00033
$arpi_V^2$	-4.7×10^{-6}	-5.9×10^{-6}	-5.8×10^{-6}	-5.4×10^{-6}

Table 1: We present numerical results of the slow-roll parameters $\{\varepsilon_V, \eta_V, \xi_V^2, \varpi_V^3\}$ and inflationary predictions n_s and r for various combinations of A, β and γ in the inflation potential. Notably, we highlight the results for distinguished values of A. It's worth noting that the spectral index n_s is a bit small in the cases of $\beta = \gamma = 0$, as indicated by data plotted in red.

B Relevant modular forms

• Dedekind eta function

The Dedekind eta function is a modular function of "weight 1/2" defined as

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \qquad q \equiv e^{2\pi i \tau},$$
(B.1)

which satisfies the identities $\eta(\tau+1) = e^{i\pi/12}\eta(\tau)$ and $\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$. The q-expansion of eta function is given by

$$\eta = q^{1/24} [1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \mathcal{O}(q^{22})]. \tag{B.2}$$

At the modular symmetry fixed points $\tau = \omega, i$, the eta function takes the following values,

$$\eta(i) = \frac{\Gamma(1/4)}{2\pi^{3/4}}, \quad \eta(\omega) = e^{-\frac{i\pi}{24}} \frac{\sqrt[8]{3}}{2\pi} \frac{\Gamma^{3/2}(1/3)}{2\pi}.$$
(B.3)

• Eisenstein series

The Eisenstein series $G_{2k}(\tau)$ of weight 2k for integer k > 1 is defined as [43]

$$G_{2k}(\tau) = \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1, n_2 \neq (0, 0)}} (n_1 + n_2 \tau)^{-2k},$$
(B.4)

and the Fourier series of Eisenstein series [44] take the following form

$$G_{2k}(q) = 2\zeta(2k) \left(1 + c_{2k} \sum_{i=1}^{\infty} \sigma_{2k-1}(i)q^i \right),$$
 (B.5)

where the coefficients c_{2k} are given by

$$c_{2k} = \frac{(2\pi i)^{2k}}{(2k-1)!\zeta(2k)} = -\frac{-4k}{B_{2k}} = \frac{2}{\zeta(1-2k)}.$$
 (B.6)

Here B_n are the Bernoulli numbers and $\zeta(z)$ denotes Riemann's zeta function. Finally $\sigma_p(n)$ in Eq.(B.5) is the divisor sum function,

$$\sigma_p(n) = \sum_{d|n} d^p, \tag{B.7}$$

where d|n is shorthand for "d divides n". For certain values of Riemann's zeta function, we have

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$
 (B.8)

In particular, the Lambert series of G_2 in terms of q reads

$$G_2(\tau) = \frac{\pi^2}{3} \left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \right).$$
 (B.9)

Note the modular function $G_2(\tau)$ and the Dedekind eta are related by [18]

$$\frac{\partial_{\tau}\eta(\tau)}{\eta(\tau)} = \frac{i}{4\pi}G_2(\tau). \tag{B.10}$$

The modular transformation of $G_2(\tau)$ is given by

$$G_2(\gamma \cdot \tau) = (c\tau + d)^2 G_2(\tau) - 2\pi i c(c\tau + d),$$
 (B.11)

which shows G_2 is not a modular form. The modified weight 2 Eisenstein series \hat{G}_2 is defined by

$$\hat{G}_2(\tau) = G_2(\tau) + \frac{2\pi}{i(\tau - \bar{\tau})},$$
 (B.12)

which is a non-holomorphic function but preserves modularity. It vanishes at the fixed points:

$$\widehat{G}_2(i) = \widehat{G}_2(\omega) = 0. \tag{B.13}$$

Sometimes it is much convenient to work in alternative representations of Eisenstein series $E_{2k}(\tau)$, which is different with G_{2k} up to a normalization constant [43]:

$$E_{2k}(\tau) = \frac{G_{2k}(\tau)}{2\zeta(2k)} = 1 - \frac{4k}{B_{2k}} \sum_{d,n\geq 1} n^{2k-1} q^{nd}.$$
 (B.14)

We provide some values of $E_{2k}(\tau)$ relevant to this paper:

$$E_2(i) = \frac{3}{\pi}, \qquad E_2(\omega) = \frac{2\sqrt{3}}{\pi},$$
 (B.15)

$$E_4(i) = \frac{3\Gamma^8(1/4)}{(2\pi)^6}, \quad E_4(\omega) = 0,$$
 (B.16)

$$E_6(i) = 0,$$
 $E_6(\omega) = \frac{6^3 \Gamma^{18}(1/3)}{(2\pi)^{12}}.$ (B.17)

For derivatives of the Eisenstein series, Ramanujan-Shen's differential equation [45] is useful:

$$qE'_{2k-2} = \frac{k-1}{2\pi^2\zeta(2k-2)} \sum_{n=1}^{k-1} \zeta(2n)\zeta(2k-2n)(E_{2n}E_{2k-2n} - E_{2k}),$$
 (B.18)

where $E'_{2k-2} = dE_{2k-2}/dq$. The first several relations read:

$$qE'_{2}(q) = \frac{1}{12}(E_{2}^{2} - E_{4}),$$

$$qE'_{4}(q) = \frac{1}{3}(E_{2}E_{4} - E_{6}),$$

$$qE'_{6}(q) = \frac{1}{2}(E_{2}E_{6} - E_{4}^{2}).$$
(B.19)

Using $\partial/\partial\tau = 2\pi i\partial/\partial q$ and first row of Eq.(B.19), we can express the derivative of \hat{G}_2 as follows:

$$\partial_{\tau}\widehat{G}_{2}(\tau) = \frac{i\pi^{3}}{18}(E_{2}^{2} - E_{4}) - \frac{2\pi}{i(\tau - \bar{\tau})^{2}},$$
(B.20)

and the numerical values of $\partial_{ au} \widehat{G}_2(au)$ at the fixed points are

$$\partial_{\tau} \hat{G}_2(i) = -i \frac{\Gamma^8(1/4)}{384\pi^3}, \quad \partial_{\tau} \hat{G}_2(\omega) = 0.$$
 (B.21)

The anti-holomorphic derivative of $\widehat{G}_2(\tau, \bar{\tau})$ is

$$\partial_{\bar{\tau}}\widehat{G}_2(\tau) = -\partial_{\tau}\overline{\widehat{G}}_2(\tau) = \frac{2\pi}{i(\tau - \bar{\tau})^2},$$
 (B.22)

the corresponding numerical values at the fixed points are

$$\partial_{\bar{\tau}}\widehat{G}_{2}(i) = -\partial_{\tau}\overline{\widehat{G}}_{2}(i) = i\frac{\pi}{2},
\partial_{\bar{\tau}}\widehat{G}_{2}(\omega) = -\partial_{\tau}\overline{\widehat{G}}_{2}(\omega) = i\frac{2\pi}{3}.$$
(B.23)

These relations are useful for calculating the derivatives of the scalar potential.

• Klein *j*-invariant function

The Klein j-invariant function is a modular form of weight zero, defined in terms of Dedekind eta function and Eisenstein series as follows [16, 17]

$$j(\tau) \equiv \frac{3^6 5^3}{\pi^{12}} \frac{G_4^3(\tau)}{\eta^{24}(\tau)} = \frac{3^6 5^3}{\pi^{12}} \frac{G_4^3(\tau)}{\Delta(\tau)} \,, \quad \Delta(\tau) \equiv \eta^{24}(\tau) \,, \tag{B.24}$$

which implies

$$j(\tau) - 1728 = \left(\frac{945}{2\pi^6}\right)^2 \left(\frac{G_6(\tau)}{\eta^{12}(\tau)}\right)^2 = \left(\frac{945}{2\pi^6}\right)^2 \frac{G_6^2(\tau)}{\Delta(\tau)}.$$
 (B.25)

From Eqs. (B.24, B.25), we can see that the two expressions of H function in Eq. (3.10) and Eq. (3.12) are equivalent. Given the identity of Eq. (B.14), j and j - 1728 can also be compactly written as

$$j - 1728 = \left(\frac{E_6}{\eta^{12}}\right)^2, \quad j = \left(\frac{E_4}{\eta^8}\right)^3.$$
 (B.26)

This j-function is a one-to-one map between points in the fundamental domain and the whole complex plane. At the fixed points, one has

$$j(i\infty) = +\infty$$
, $j(\omega) = 0$, $j(i) = 1728 = 12^3$. (B.27)

For convenience, the q-expansion of j-function is given by

$$j(\tau) = 744 + \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4$$
$$+333202640600q^5 + 4252023300096q^6 + 44656994071935q^7 + \mathcal{O}(q^8) \text{(B.28)}$$

The derivatives of j function read:

$$\frac{\partial j}{\partial \tau} = -2\pi i \frac{E_6(\tau) E_4^2(\tau)}{\eta^{24}(\tau)},
\frac{\partial^2 j}{\partial \tau^2} = \frac{(2\pi)^2}{\eta^{24}(\tau)} \left(\frac{1}{6} E_2(\tau) E_4^2(\tau) E_6(\tau) - \frac{1}{2} E_4^4(\tau) - \frac{2}{3} E_6^2(\tau) E_4(\tau) \right),$$
(B.29)

which helps to calculate the second derivative of scalar potential. The derivatives of j function also vanish at the fixed points:

$$\frac{\partial j}{\partial \tau}\Big|_{\tau=i} = \frac{\partial j}{\partial \tau}\Big|_{\tau=\omega} = 0.$$
 (B.30)

• *H* function

For convenience, we choose the following representation of H function in Eq. (3.10)

$$H(\tau) = \left(\frac{E_6}{\eta^{12}}\right)^m \left(\frac{E_4}{\eta^8}\right)^n \mathcal{P}(j(\tau)), \qquad (B.31)$$

and the relevant numerical values at the fixed points are

$$H(i) = \begin{cases} 0 & m > 0 \\ 12^{n} \mathcal{P}(1728) & m = 0 \end{cases},$$

$$H(\omega) = \begin{cases} 0 & n > 0 \\ i^{m} 2^{3m} 3^{\frac{3m}{2}} \mathcal{P}(0) & n = 0 \end{cases}.$$
(B.32)

The derivative of H function can be written as:

$$\partial_{\tau}H = -i\pi H \left(m \frac{E_4^2}{E_6} + \frac{2n}{3} \frac{E_6}{E_4} + \frac{i}{\pi} \frac{\mathrm{d}\ln \mathcal{P}}{\mathrm{d}j} \frac{\mathrm{d}j}{\mathrm{d}\tau} \right), \tag{B.33}$$

and the corresponding numerical values at the fixed points are:

$$\partial_{\tau}H(i) = \begin{cases} 0 & m = 0 \text{ or } m > 1\\ -i2^{2n+2}3^{2+n}\frac{\Gamma^{4}(1/4)}{(2\pi)^{2}}\mathcal{P}(1728) & m = 1 \end{cases},$$

$$\partial_{\tau}H(\omega) = \begin{cases} 0 & n = 0 \text{ or } n > 1\\ -i^{m+1}2^{3m+3}3^{3m/2+1}e^{i\frac{\pi}{3}}(2\pi)^{-3}\Gamma^{6}(1/3)\mathcal{P}(0) & n = 1 \end{cases}.$$
(B.34)

We are also interested in the second order derivative of H function. $\partial_{\tau}^{2}H$ takes the following value at the fixed points:

$$\partial_{\tau}^{2}H(i) = \begin{cases} 0 & m > 2 \\ -3^{n+4}2^{5+2n}\frac{\Gamma^{8}(1/4)}{(2\pi)^{4}}\mathcal{P}(1728) & m = 2 \\ 3^{n+2}2^{2n+2}\frac{\Gamma^{4}(1/4)}{(2\pi)^{2}}\mathcal{P}(1728) & m = 1 \end{cases}, \\ -2^{2n-1}3^{n}\frac{\Gamma^{8}(1/4)}{(2\pi)^{4}}\mathcal{P}(1728)\left(n + 3^{4}2^{6}\frac{\mathcal{P}'(1728)}{\mathcal{P}(1728)}\right) & m = 0 \end{cases}$$
(B.35)
$$\partial_{\tau}^{2}H(\omega) = \begin{cases} 0 & n = 0 \text{ or } n > 2 \\ -i^{m}e^{i\frac{2\pi}{3}}2^{3m+7}3^{\frac{3m}{2}+2}\frac{\Gamma^{12}(1/3)}{(2\pi)^{6}}\mathcal{P}(0) & n = 2 \\ i^{m}e^{i\frac{\pi}{3}}2^{3m+4}3^{\frac{3}{2}m+\frac{1}{2}}\frac{\Gamma^{6}(1/3)}{(2\pi)^{3}}\mathcal{P}(0) & n = 1 \end{cases}$$

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