

ZX Graphical Calculus for Continuous-Variable Quantum Processes

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Continuous-variable (CV) quantum information processing is a promising candidate for large-scale fault-tolerant quantum computation. However, analysis of CV quantum process relies mostly on direct computation of the evolution of operators in the Heisenberg picture, and the features of CV space has yet to be thoroughly investigated in an intuitive manner. One key ingredient for further exploration of CV quantum computing is the construction of a computational model that brings visual intuition and new tools for analysis. In this paper, we delve into a graphical computational model, inspired by a similar model for qubit-based systems called the ZX calculus, that enables the representation of arbitrary CV quantum process as a simple directed graph. We demonstrate the utility of our model as a graphical tool to comprehend CV processes intuitively by showing how equivalences between two distinct quantum processes can be proven as a sequence of diagrammatic transformations in certain cases. We also examine possible applications of our model, such as measurement-based quantum computing, characterization of Gaussian and non-Gaussian processes, and circuit optimization.

I. INTRODUCTION

Quantum computing is expected to become the next-generation technology by surpassing the performance of ordinary computers for certain tasks. Recent experiments have already demonstrated this in several platforms [1, 2], though practical applications remain yet to be demonstrated. Toward the ultimate goal of universal quantum computing, further investigation into the ability of quantum computation is needed, as well as construction of feasible computational architectures.

In the field of classical computational theory, various computational models have been proposed [3], such as the Turing machine, lambda calculus, and process algebra, to name a few. These models not only provide new perspectives from which to comprehend the theoretical nature of computation but also contribute to the designs of real computer architectures and programming languages. As the development of quantum devices is progressing rapidly, the necessity for a high-level quantum computational model has increased [4].

Most of the computational models for quantum computation proposed so far are natural generalization of conventional computational models, including the quantum Turing machine [5] and the quantum lambda calculus [6]. These models are based on a discrete-variable (DV) approach based on qubits. In the quantum circuit model—the most widely used quantum computa-

tional model—computing processes are represented as sequences of quantum gates [7], whose concept originates from the logic circuit of classical bit registers and has a natural interpretation as an array of physical unitary transformations evolving under specific Hamiltonians.

Another interesting DV quantum model that emerged from a different point of view is what is called a *graphical calculus* model. This idea derives from categorical quantum mechanics, a research field that investigates the general mathematical structure of quantum information theory, providing insights into the unique features of quantum theory such as entanglement and superposition [8]. One of the most useful variations is called the *ZX calculus* [9, 10], proposed by Coecke and Duncan. It provides a graphical language with which to visualize qubit processes and manipulate them in an intuitive manner. In the framework of the ZX calculus, any quantum process is represented as a simple diagram consisting of and undirected graph with parameters on the vertices. By following clearly defined procedures, one can easily verify the equivalence between two quantum processes via graphical transformations of corresponding diagrams. Since the ZX calculus cleverly exploits the algebraic properties of the qubit Hilbert space, it has various applications such as quantum circuit optimization [11, 12], analysis of measurement-based quantum computing [13, 14], graphical reasoning of the surface code [15], quantum machine learning [16], and others.

Aside from the DV approach, a continuous-variable (CV) approach [17] is attracting attentions in recent years. Unlike DV computing with qubits, a CV approach exploits the infinite-dimensional nature of bosonic systems. As the bosonic Hilbert space is much larger than

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two-dimensional qubit space, it is possible to build error resilience directly into the states used as the fundamental quantum information carriers. For instance, the CV approach is considered to be compatible with fault-tolerant quantum computation for various types of logical qubit encoding, such as the Gottesman-Kitaev-Preskill qubit [18] and the binomial code [19]. Experimentally, the CV approach has been intensively investigated in recent years in multiple platforms, including optics [20–23], trapped ions [24–26], and superconducting resonators [27–30].

However, the major difficulty with the CV approach is the same as its main feature: the infinite-dimensional Hilbert space at its core and the continuous nature of the quantum variables used for computation. As DV computing can be seen as natural expansion of classical computing, where bits are expanded to superposition of bits (qubits), the operational semantics of qubit computing is largely comprehensible. For example, some of the DV quantum gates have a natural interpretation as Boolean logic gates, such as the correspondences between the quantum X gate and classical NOT gate, the quantum CNOT gate and classical XOR gate, etc. The notion of quantum circuits originally arose from classical logic circuits [31], and nowadays it is used as the standard tool to describe DV quantum computing. In contrast, CV processes are often treated as purely physical operations, and their computational meanings are less obvious to translate to computational language. This is part of the reason that DV computing has been actively investigated at the level of abstract algorithms and multi-qubit quantum error correction, while most theoretical research on CV computing either remains at the operational level or includes a reduction into DV computing through logical qubit encoding.

One of the greatest difficulties in studying the CV approach is the absence of useful tools for physicists to describe CV quantum processes. There is an established way to represent Gaussian quantum states as graphs, and Gaussian operations appear as well-defined graph transformations [32]. We would like a graphical representation, though, that goes beyond Gaussian to include all CV states and operations. Previous research has also shown that the methodology of the ZX calculus can be extended into three and higher Hilbert-space dimensions [33]. However, these attempts are confined only to finite-dimensional spaces, partially due to complicated properties of the CV space involving infinities.

Despite these obstacles, here we present the key algebraic structure for a ZX calculus in the case of CV systems. We give definitions for CV diagrams and rewrite rules and demonstrate how various quantum protocols can be expressed intuitively through diagram transformations. Moreover, we discuss the possibility of applications of our graphical model, such as CV quantum circuit optimization, characterization of Gaussian processes, and graphical representation of measurement-based quantum computing. We expect that our model may serve as

education material for those not familiar with CV approaches, as well as a convenient calculation tool for experienced physicists and a high-level language for computer scientists.

II. PRELIMINARIES

In this chapter we review the basics of CV quantum computing [34] and the original ZX calculus [10]. Throughout this paper, we assume a natural unit system where $\hbar = 1$.

A. Continuous-variable system and quadrature eigenstates

In the CV (bosonic) quantum system, the quadrature operators $\hat{q}$ and $\hat{p}$ are defined as

$$\hat{q} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}), \hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}) \quad (1)$$

where $\hat{a}$ and $\hat{a}^\dagger$ represent the annihilation and creation operators of a bosonic mode, respectively. By definition, these quadrature operators are canonical variables that satisfy the canonical commutation relation $[\hat{q}, \hat{p}] = i$. Quadrature operators are Hermitian operators, and their eigenvalues ranges over all real numbers $\mathbb{R}$. We shall denote corresponding eigenstates as $|s\rangle_q$ and $|t\rangle_p$, respectively, satisfying

$$\hat{q}|s\rangle_q = s|s\rangle_q, \quad \hat{p}|t\rangle_p = t|t\rangle_p. \quad (2)$$

Using the Fock basis, $|s\rangle_q$ and $|t\rangle_p$ can be expanded as

$$|s\rangle_q = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!2^n\sqrt{\pi}}} H_n(s) \exp\left(-\frac{1}{2}s^2\right) |n\rangle \quad (3)$$

$$|t\rangle_p = \sum_{n=0}^{\infty} \frac{i^n}{\sqrt{n!2^n\sqrt{\pi}}} H_n(t) \exp\left(-\frac{1}{2}t^2\right) |n\rangle \quad (4)$$

where $H_n(x)$ denotes the n -th Hermite polynomial. These Fock expansions are associated with the definitions of quadrature operators (1) through the following recurrence relation:

$$2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x). \quad (5)$$

Note that quadrature eigenstates are unphysical because their norms are infinite. A Dirac δ normalization is thus chosen for these states:

$${}_q\langle s|s'\rangle_q = \delta(s - s'), \quad {}_p\langle t|t'\rangle_p = \delta(t - t'). \quad (6)$$

Also, each family of quadrature eigenstates forms a complete orthonormal basis, satisfying

$$\int_{\mathbb{R}} ds |s\rangle_q {}_q\langle s| = \int_{\mathbb{R}} dt |t\rangle_p {}_p\langle t| = \hat{I}. \quad (7)$$

For an arbitrary pure state $|\psi\rangle$, its wave functions $\psi(s)$ and $\tilde{\psi}(t)$ are defined as follows:

$$\psi(s) = {}_q\langle s|\psi\rangle, \quad \tilde{\psi}(t) = {}_p\langle t|\psi\rangle \quad (8)$$

These definitions, along with (7), offer two representations of CV quantum states:

$$|\psi\rangle = \int_{\mathbb{R}} \psi(s) |s\rangle_q = \int_{\mathbb{R}} \tilde{\psi}(t) |t\rangle_p \quad (9)$$

The eigenbases $\{|s\rangle_q\}$ and $\{|t\rangle_p\}$ are mutually associated via Fourier transform. In fact, the inner product of the quadrature eigenstates satisfies

$${}_q\langle s|t\rangle_p = \frac{1}{\sqrt{2\pi}} \exp(ist) \quad (10)$$

and thus, using (7), we obtain

$$\tilde{\psi}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} ds \exp(-ist) \psi(s) \quad (11)$$

$$\psi(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dt \exp(ist) \tilde{\psi}(t) \quad (12)$$

which exactly corresponds to Fourier transform of complex functions.

A CV quantum state has another convenient representation other than wave function and density operator: the Wigner function. This is a quasi-probability function defined on the phase space that depicts quantum state in quadrature picture [35]. Importantly, a quantum state is called a Gaussian state when its Wigner function is a (single- or multi-variable) Gaussian function. For example, the vacuum state, coherent states and squeezed states are all Gaussian states.

B. Quantum gates in continuous-variable system

In a CV quantum system, any unitary transformation can be uniquely specified by the transformation of quadrature operators it induces in the Heisenberg picture. That is, unitary operation on an n -mode CV systems uniquely corresponds to a $2n$ -variable function f that transforms the quadrature operators as

$$\hat{\mathbf{x}}^{\text{out}} = f(\hat{\mathbf{x}}^{\text{in}}) \quad (13)$$

where $\hat{\mathbf{x}} = (\hat{q}_1, \hat{p}_1, \dots, \hat{q}_n, \hat{p}_n)^\top$. Note that f must preserve the canonical commutation relations, namely $[\hat{q}_j, \hat{p}_k] = i\delta_{jk}$. When f is a linear affine map, then the corresponding quantum operation is called a Gaussian operation since it converts any Gaussian state only into another Gaussian state.

As is shown in Table I, Gaussian operations correspond to time evolution processes under (at most) quadratic Hamiltonians. In contrast, non-Gaussian operations require Hamiltonians of cubic or higher order in the quadrature operators. A non-Gaussian operation is also called

a non-linear operation because it adds non-linear terms to the quadrature operators in the Heisenberg picture.

The relation between Gaussian and non-Gaussian operations is analogous to that of Clifford and non-Clifford operations in qubit systems. A CV quantum process limited to starting with Gaussian states and using only Gaussian operations and measurements can be efficiently simulated by matrix multiplication [36]. Meanwhile, it is also known that one can approximate an arbitrary CV quantum operation if at least one single-mode non-Gaussian operation is available, in addition to a universal multi-mode Gaussian gate set [17]. In other words, the availability of a single non-Gaussian operation is necessary and sufficient for a multi-mode CV universal gate set when arbitrary Gaussian operations are available.

C. Categorical quantum mechanics

Categorical quantum mechanics is a relatively new field that investigates the mathematical backgrounds of quantum theory in terms of monoidal category theory [8]. In more detail, it focuses on the mathematical structures and relations of quantum processes by regrading quantum processes as linear maps between Hilbert spaces. One of the most significant feature of categorical quantum theory is that it comes already equipped with many structures that occupy important places in quantum theory, such as tensor product, dual space, transpose, adjoint, and superposition, to name a few. Another interesting aspect of this field is that a monoidal category is associated with a graphical language called a string diagram [37], and thus it enables one to visualize quantum processes as simple diagrams. These diagrams provide useful tools for finding intuitive reasoning for various quantum processes, including quantum teleportation.

Complementarity and bialgebra [8] are the key mathematical features of quantum processes that must be present in any faithful representation using a graphical language. These two properties, collectively referred as strong complementarity, play important roles in describing how different bases mutually relate to each other. This is made explicit in the ZX calculus for finite-dimensional systems. In the following definitions, let $\mathcal{H}$ be a finite-dimensional Hilbert space with $\dim \mathcal{H} = n$.

Definition 1 (Swap operator). We define the *swap operator* as a linear map $\hat{\sigma} : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ with $\hat{\sigma}(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle$.

Definition 2 (Complementary bases). Let $\{|u_i\rangle\}_{i=1}^n$ and $\{|v_j\rangle\}_{j=1}^n$ each be an orthogonal basis in $\mathcal{H}$. This pair of bases is called *complementary* (or equivalently, *unbiased*) when there exists a positive real number $c > 0$ such that

$$|\langle u_i | v_j \rangle|^2 = c \quad (14)$$

for all i, j . In other words, this condition requires that the inner products of the two bases have constant absolute value—i.e., it is independent of the basis elements chosen.

Table I. List of representative CV quantum gates and their properties. Each index on a quadrature operator in an entangling gate specifies the mode it belongs to. The unitary action of each gate is given by $\hat{U} = e^{-i\hat{H}}$, and the quadrature transformation is defined using the Heisenberg picture. Note that all parameters except α need to be real since Hamiltonians are Hermitian.

Gate	Notation	Number of modes	Hamiltonian	Quadrature transformation	Gaussianity
Displacement gate	$\hat{D}(\alpha)$	1	$i(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$	$\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} \mapsto \begin{pmatrix} \hat{q} + \sqrt{2}\text{Re}(\alpha) \\ \hat{p} + \sqrt{2}\text{Im}(\alpha) \end{pmatrix}$	Gaussian
Phase rotation gate	$\hat{R}(\theta)$	1	$\theta\hat{a}^\dagger\hat{a}$	$\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta\hat{q} + \sin\theta\hat{p} \\ -\sin\theta\hat{q} + \cos\theta\hat{p} \end{pmatrix}$	Gaussian
1-mode squeezing gate	$\hat{S}(r)$	1	$-\frac{r}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})$	$\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} \mapsto \begin{pmatrix} e^{-r}\hat{q} \\ e^r\hat{p} \end{pmatrix}$	Gaussian
Controlled-sum gate	$\widehat{\text{CS}}_{1,2}(g)$	2	$g\hat{q}_1\hat{p}_2$	$\begin{pmatrix} \hat{q}_1 \\ \hat{p}_1 \\ \hat{q}_2 \\ \hat{p}_2 \end{pmatrix} \mapsto \begin{pmatrix} \hat{q}_1 \\ \hat{p}_1 - g\hat{p}_2 \\ g\hat{q}_1 + \hat{q}_2 \\ \hat{p}_2 \end{pmatrix}$	Gaussian
Controlled-Z gate	$\widehat{\text{CZ}}(g)$	2	$-g\hat{q}_1\hat{q}_2$	$\begin{pmatrix} \hat{q}_1 \\ \hat{p}_1 \\ \hat{q}_2 \\ \hat{p}_2 \end{pmatrix} \mapsto \begin{pmatrix} \hat{q}_1 \\ \hat{p}_1 + g\hat{q}_2 \\ \hat{q}_2 \\ g\hat{q}_1 + \hat{p}_2 \end{pmatrix}$	Gaussian
Beamsplitter gate	$\widehat{\text{BS}}(\theta)$	2	$\theta(\hat{q}_1\hat{p}_2 - \hat{p}_1\hat{q}_2)$	$\begin{pmatrix} \hat{q}_1 \\ \hat{p}_1 \\ \hat{q}_2 \\ \hat{p}_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta\hat{q}_1 - \sin\theta\hat{q}_2 \\ \cos\theta\hat{p}_1 - \sin\theta\hat{p}_2 \\ \sin\theta\hat{q}_1 + \cos\theta\hat{q}_2 \\ \sin\theta\hat{p}_1 + \cos\theta\hat{p}_2 \end{pmatrix}$	Gaussian
Cubic phase gate	$\widehat{\text{CPG}}(\gamma)$	1	$-\gamma\hat{q}^3$	$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q \\ p + 3\gamma q^2 \end{pmatrix}$	Non-Gaussian

To define a bialgebra, we first need to introduce the notion of a *monoid* for a Hilbert space.

Definition 3 (Monoid and commutative monoid). Let $\hat{\mu} : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ be a linear map and $|\eta\rangle \in \mathcal{H}$ be a (not necessarily normalized) state. The pair $(\hat{\mu}, |\eta\rangle)$ is called a *monoid* if the following two conditions are satisfied:

1. $\hat{\mu} \circ (\text{id}_{\mathcal{H}} \otimes \hat{\mu}) = \hat{\mu} \circ (\hat{\mu} \otimes \text{id}_{\mathcal{H}})$ (associativity)
2. For arbitrary $|\psi\rangle \in \mathcal{H}$, $\hat{\mu}(|\psi\rangle \otimes |\eta\rangle) = \hat{\mu}(|\eta\rangle \otimes |\psi\rangle) = |\psi\rangle$ (unitality)

where $\text{id}_{\mathcal{H}}$ denotes the identity map on $\mathcal{H}$. If $\hat{\mu} = \hat{\mu} \circ \hat{\sigma}$, then the monoid is called *commutative*.

The first condition implies that $\hat{\mu}$ induces an associative binary operation on $\mathcal{H}$, while the second requires $|\eta\rangle$ to be the identity element for that operation. Therefore, the definition above matches that for a monoid on $\mathcal{H}$. An important example of a monoid is one that is induced by an orthogonal basis:

Definition 4 (Monoid induced by a basis). Let $\{|e_i\rangle\}_{i=1}^n$ be an orthogonal basis in $\mathcal{H}$. Then, the pair $(\hat{\mu}, |\eta\rangle)$, with

$$\hat{\mu} = \sum_{i=1}^n \frac{1}{\langle e_i | e_i \rangle} |e_i\rangle \langle e_i| \langle e_i| \quad (15)$$

$$|\eta\rangle = \sum_{i=1}^n \frac{1}{\langle e_i | e_i \rangle} |e_i\rangle \quad (16)$$

forms a *monoid induced by the basis* $\{|e_i\rangle\}_{i=1}^n$.

We are now ready to define a bialgebra.

Definition 5 (Bialgebra). Let $(\hat{\mu}_1, |\eta_1\rangle)$ and $(\hat{\mu}_2, |\eta_2\rangle)$ each be monoids induced by two different bases on the same Hilbert space $\mathcal{H}$. The pair of monoids is called a *bialgebra* if the following four conditions are satisfied:

$$\hat{\mu}_1^\dagger \circ \hat{\mu}_2 = (\hat{\mu}_2 \otimes \hat{\mu}_2) \circ (\text{id} \otimes \hat{\sigma} \otimes \text{id}) \circ (\hat{\mu}_1^\dagger \otimes \hat{\mu}_1^\dagger) \quad (17)$$

$$\hat{\mu}_1^\dagger |\eta_2\rangle = |\eta_2\rangle \otimes |\eta_2\rangle \quad (18)$$

$$\hat{\mu}_2^\dagger |\eta_1\rangle = |\eta_1\rangle \otimes |\eta_1\rangle \quad (19)$$

$$\langle \eta_1 | \eta_2 \rangle = 1 \quad (20)$$

We will often refer to the bases that induced the monoids within a bialgebra as *bases that form a bialgebra*.

Orthogonal bases that form a bialgebra can be characterized by the next theorem.

Theorem 1. [38] Let $\{|u_i\rangle\}_{i=1}^n$ and $\{|v_i\rangle\}_{i=1}^n$ each be orthogonal bases, and $(\hat{\alpha}, |\phi\rangle)$ and $(\hat{\beta}, |\chi\rangle)$ be monoids induced by the bases, respectively. Then the pair of monoids forms a bialgebra if and only if there exists a commutative group $(G, \cdot)$ with $|G| = n$, and a bijective map $\pi : G \rightarrow \{1, \dots, n\}$, with $|u_g\rangle = |u_{\pi(g)}\rangle$ (similarly for $|v_g\rangle$), such that

$$\hat{\alpha} = \sum_{g,h \in G} \frac{1}{\| |v_g\rangle \|^2 \| |v_h\rangle \|^2} |v_{g \cdot h}\rangle \langle v_g| \langle v_h| \quad (21)$$

$$\hat{\beta} = \sum_{g,h \in G} \frac{1}{\| |u_g\rangle \|^2 \| |u_h\rangle \|^2} |u_{g \cdot h}\rangle \langle u_g| \langle u_h| \quad (22)$$

and

$$|\phi\rangle = |v_e\rangle \quad (23)$$

$$\begin{aligned} |\chi\rangle &= |u_e\rangle \\ \langle u_e | v_e \rangle &= 1 \end{aligned} \quad (24) \quad (25)$$

where $e \in G$ denotes the identity element of $(G, \cdot)$.

Definition 6 (Strongly complementary bases). A pair of bases is called *strongly complimentary* if it is both complimentary and forms a bialgebra.

Typically, orthogonal bases satisfying

$$|u_j\rangle = \sum_{k=1}^n \exp\left(\frac{2\pi i j k}{n}\right) |v_k\rangle \quad (26)$$

$$|v_k\rangle = \sum_{j=1}^n \exp\left(-\frac{2\pi i j k}{n}\right) |u_j\rangle \quad (27)$$

correspond to the special case where $(G, \cdot)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, up to trivial scalar factor. In this case, $\hat{\alpha}$ and $\hat{\beta}$ induced, respectively, by $\{|u_i\rangle\}_{i=1}^n$ and $\{|v_i\rangle\}_{i=1}^n$, satisfy following relations

$$\hat{\alpha} \propto \sum_{l,m=1}^n |v_{l \oplus m}\rangle \langle v_l| \langle v_m| \quad (28)$$

$$\hat{\beta} \propto \sum_{l,m=1}^n |u_{l \oplus m}\rangle \langle u_l| \langle u_m| \quad (29)$$

where $\oplus$ denotes addition modulo n . These bases define the generalized Pauli Z -basis and generalized Pauli X -basis in finite-dimensional Hilbert spaces, whose properties are fully exploited in the ZX calculus, as its name represents.

D. ZX calculus for qubits

The ZX calculus for qubits mainly consists of two elements: (1) ZX diagrams and (2) rewrite rules. A ZX diagram comprises generators that represent certain projective linear operators individually, and connecting wires within these generators corresponds to composition of linear maps. (Note that here we restrict to quantum processes involving pure states only.) A generator comprises two types of “spiders” with an arbitrary number of inputs and outputs,

$$\begin{array}{c} \vdots \\ \alpha \\ \vdots \end{array} := |0 \dots 0\rangle \langle 0 \dots 0| + e^{i\alpha} |1 \dots 1\rangle \langle 1 \dots 1| \quad (30)$$

$$\begin{array}{c} \vdots \\ \alpha \\ \vdots \end{array} := |+\dots+\rangle \langle +\dots+| + e^{i\alpha} |-\dots-\rangle \langle -\dots-| \quad (31)$$

and a “box” denoted by a blank box,

$$\text{---}\square\text{---} := |+\rangle \langle 0| + |-\rangle \langle 1| \quad (32)$$

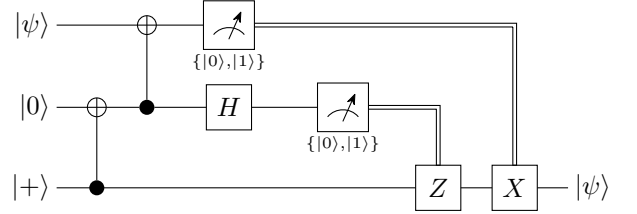
representing the Hadamard gate. Though a ZX diagram has no notion of direction, it should be read from left to right according to standard conventions to explicitly specify input-output relations.

Rewrite rules are formulated as equations between ZX diagrams, identifying how a diagram can be transformed into another one while preserving the quantum process it represents. Below are examples of rewrite rules:

$$\begin{array}{c} \vdots \\ \alpha \\ \vdots \\ \vdots \\ \beta \\ \vdots \end{array} = \begin{array}{c} \vdots \\ \alpha + \beta \\ \vdots \end{array}, \quad \begin{array}{c} \vdots \\ \alpha \\ \vdots \\ \vdots \\ \beta \\ \vdots \end{array} = \begin{array}{c} \vdots \\ \alpha + \beta \\ \vdots \end{array} \quad (33)$$

$$\begin{array}{c} \text{---}\square\text{---} \\ \vdots \end{array} = \begin{array}{c} \vdots \end{array} \quad (34)$$

Using rewrite rules, one may comprehend quantum dynamics by a simple graphical calculus. For instance, the quantum circuit shown below is a schematic illustration of qubit quantum teleportation using conventional quantum circuits.



It is easy to verify that this process is equivalent to the identity operation using the ZX calculus [10]:

$$\begin{array}{c} \text{---}\alpha\pi \\ \vdots \end{array} = \begin{array}{c} \text{---}\alpha\pi \\ \vdots \end{array} \quad (35)$$

$$\begin{array}{c} \text{---}\alpha\pi \\ \vdots \end{array} = \begin{array}{c} \text{---}\alpha\pi \\ \vdots \end{array} \quad (36)$$

$$\begin{array}{c} \text{---}\alpha\pi \\ \vdots \end{array} = \begin{array}{c} \text{---}\alpha\pi \\ \vdots \end{array} \quad (37)$$

In the procedure of the ZX calculus, each diagram transformation is a simple application of a rewrite rule onto a fraction of the diagram, and thus one can straightforwardly confirm that the quantum process remains unchanged. This property is called *soundness*—i.e., equivalence of diagrams implies equivalence of quantum processes. *Completeness* of the ZX calculus is the converse—i.e., whether two diagrams representing the same process are always graphically convertible. The rewrite rules of the ZX calculus (for qubits) are designed to represent equations of the qubit system in the standard interpretation, which means the qubit ZX calculus is sound. Although the original proposal of the ZX calculus is not complete [39], there are several variations of the model equipped with additional diagrams and rewrite rules to assure completeness [40, 41], making it an alternative *formalism* (to Hilbert space) for representing quantum processes. Even without the assurance of completeness, one can still intuitively manipulate diagrams and straightforwardly simplify them by applying rewrite rules one by one to prove two diagrams are equivalent. For more detail, see [8, 42, 43].

The infinite-dimensional nature of CV systems creates challenges for proving soundness and completeness in a mathematically rigorous way. In our proposal for a CV ZX calculus below, we carefully design rewrite rules to achieve soundness within the standard framework of CV information processing and prove completeness of our model when confined to one-mode Gaussian states and operations. We discuss inherent limitations of the CV graphical model in the following sections, including suggested avenues of investigation for overcoming them.

The ZX calculus has a helpful property of being topological, by which we mean that diagrams can be freely deformed as long as the connectivity between nodes is unchanged. This property lets us interpret and reshape diagrams in any way to obtain a different perspective as long as the input-output relationships are maintained. The topological feature is a special feature of the two-dimensional nature of the qubit space, which leads to the Pauli operators (and entangling gates such as CNOT) being self-inverse. This degeneracy is a peculiarity of two-dimensional Hilbert space and generally would not hold for a higher-dimensional ZX calculus. It also does not hold in our CV version. We will discuss this issue in the following sections.

Thanks to the flexibility of diagrams, the ZX calculus has been actively applied to multiple fields of quantum computation. The most typical and successful application is quantum circuit optimization, which seeks a simplified quantum circuit for a given one so that the number of quantum gates and circuit depth are reduced. It has achieved a certain level of success so far [11, 12], and there even exists a convenient Python library to demonstrate this optimization [44]. Another interesting example is diagrammatic reasoning of measurement-based quantum computing (MBQC) using a diagrammatic representation of graph states and projective measurements. This was the first use case of the ZX calculus [45]. Recent research extends to the design and verification of quantum error correcting codes [15], graphical reasoning of lattice surgery [46], analysis of certain computational complexity classes [47], an extended framework with diagrammatic differentiation for quantum machine learning [16], among others.

Our goal in the next section is to propose the basic building blocks of a CV generalization of the ZX calculus. In later sections, we will prove a set of rewrite rules for the diagrams, discuss what properties of our construction can be proven with the tools at hand, and offer suggestions for eventually proving full soundness and completeness of a CV ZX calculus based on the starting point we propose.

III. GRAPHICAL REPRESENTATION OF CONTINUOUS-VARIABLE PROCESSES

Comparing (10) and (14), one might notice that position and momentum eigenbases in CV space satisfy complementarity. This is also true for bialgebra laws by tak-

ing an integral instead of finite sum. In fact, the following relations hold:

$$\int_{\mathbb{R}} ds |s\rangle_{q_1 \ q_2, q_3} \langle s, s| = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dt du |t+u\rangle_{p_1 \ p_2, p_3} \langle t, u| \quad (38)$$

$$\int_{\mathbb{R}} dt |t\rangle_{p_1 \ p_2, p_3} \langle t, t| = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} ds du |s+u\rangle_{q_1 \ q_2, q_3} \langle s, u| \quad (39)$$

$$|0\rangle_q = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dt |t\rangle_p \quad (40)$$

$$|0\rangle_p = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} ds |s\rangle_q \quad (41)$$

This coincides with (21)–(24). For this reason, one may regard the position and momentum eigenbases as strongly complementary by the construction in Theorem 1 with the commutative group being $(\mathbb{R}, +)$. This perspective suggests that a CV graphical computational model theory may be constructed as well by simply replacing Pauli-Z and X eigenbases, respectively, with the position and momentum bases.

In this section, we first introduce two phase spiders as generators of our CV ZX calculus and their standard interpretations with the definition of diagram contraction. Consequently, we show several example of representation of quantum states and basic decompositions of each quantum gate in Table I as an array of phase spiders.

A. Proper diagrams, operators, and their equivalence relations

We begin with some important definitions.

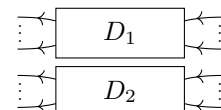
Definition 7 (Diagram, inputs, outputs). A *diagram* D is an open, directed graph with labeled nodes. An *input* of D is an open edge pointing inward. An *output* of D is an open edge pointing outward.

We choose to draw diagrams from right to left so that input-output relations are matched with bra-ket notation (when the diagram does not contain loops). Note this convention is only for readability and not substantial since our model employs directed graphs and is, in fact, topological. Wires in our diagrams are marked with arrows unless their direction can be freely reversed (we show examples later).

Definition 8 (Generator). A diagram is called a *generator* for the CV ZX calculus if it appears in Table II.

Diagrams can be combined in two important ways.

Definition 9 (Parallelization of diagrams). For two diagrams D_1 and D_2 , let $D_1 \otimes D_2$ denote the *parallelization* of D_1 and D_2 . This is a new diagram where D_1 and D_2 are placed in parallel vertically as shown:


(42)

Definition 10 (Composition of diagrams). Given two diagrams D_1, D_2 with the number of outputs in D_1 equal to the number of inputs in D_2 , let $D_2 \circ D_1$ denote the *composition* of D_2 and D_1 , which is a diagram with all outputs of D_1 connected to inputs of D_2 as shown:



This lets us recursively define the important concept of a proper diagram, which will be the main object of the CV ZX calculus.

Definition 11 (Proper diagram). A diagram D is called a *proper diagram* if any of the following hold:

1. D is a generator,
2. D is a parallelization of any two proper diagrams, or
3. D is a composition of any two proper diagrams,

The purpose of proper diagrams is to represent operators, so we need the following definition.

Definition 12 (Operator represented by a proper diagram). Given a proper diagram D , the *operator represented by D* is denoted $\llbracket D \rrbracket$ and defined as follows:

1. When D is a generator, $\llbracket D \rrbracket$ is defined in Table II.
2. $\llbracket D_1 \otimes D_2 \rrbracket$ is the tensor product of the operators representing the two parallelized diagrams, i.e., $\llbracket D_1 \rrbracket \otimes \llbracket D_2 \rrbracket$.
3. $\llbracket D_2 \circ D_1 \rrbracket$ is the composition (operator multiplication) of the operators representing the two composed diagrams, i.e., $\llbracket D_2 \rrbracket \circ \llbracket D_1 \rrbracket$ (or $\llbracket D_2 \rrbracket \llbracket D_1 \rrbracket$).

In order to relate diagrams to each other, we need the following notions of equivalence for operators and diagrams.

Definition 13 (Equivalent operators). Two operators $\hat{A}$ and $\hat{B}$ are called *equivalent*, denoted $\hat{A} \sim \hat{B}$, if there exists nonzero $c \in \mathbb{C} \setminus \{0\}$ such that $\hat{A} = c\hat{B}$.

Definition 14 (Equivalent diagrams). Given two proper diagrams D_1 and D_2 and a set of rewrite rules $\mathcal{S}$ for proper diagrams, D_1 and D_2 are called *equivalent (modulo $\mathcal{S}$)*, denoted $D_1 =_{\mathcal{S}} D_2$, if D_1 and D_2 are mutually rewritable using elements of $\mathcal{S}$. (Note: we will typically fix $\mathcal{S}$ to be the whole set of basic rewrite rules given in Sec. IV A and drop the subscript $\mathcal{S}$ for simplicity.)

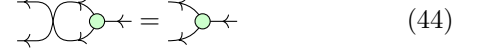
Finally, we can describe the important concept of soundness, which we define with respect to sets of proper diagrams.

Definition 15 (Soundness). *Soundness* of the calculus with respect to a set of diagrams $\mathcal{D}$ and a set of rewrite rules $\mathcal{S}$ is represented formally by $(D_1 =_{\mathcal{S}} D_2) \Rightarrow (\llbracket D_1 \rrbracket \sim \llbracket D_2 \rrbracket)$ for all $D_1, D_2 \in \mathcal{D}$.

If the set of all possible proper diagrams is sound with respect to a set of rewrite rules, then the entire calculus is said to be sound with respect to those rewrite rules.

B. Phase spider and diagram contraction

Functions written inside of a q -spider or a p -spider are what are called *phase functions* that add quadrature-dependent phase factors. In this paper we impose a restriction for phase functions to be real polynomial functions. We denote phase functions of zero by blank spiders. Note that spiders are symmetric under mode permutation, and thus combining swap diagrams with a single spider can be absorbed in the following way:



Using spiders, various quantum states can be drawn as simple diagrams. For example, quadrature eigenstates can be represented by linear phase functions as is shown here:

$$\llbracket \leftarrow \text{green spider} \rrbracket = \int_{\mathbb{R}} ds |s\rangle_q \sim |0\rangle_p \quad (45)$$

$$\llbracket \leftarrow \text{red spider} \rrbracket = \int_{\mathbb{R}} dt |t\rangle_p \sim |0\rangle_q \quad (46)$$

As for multi-mode states, an EPR state and a GHZ state can be represented as follows:

$$\llbracket \text{green spider with 2 outputs} \rrbracket \sim \int_{\mathbb{R}} ds |s, s\rangle_{q_1, q_2} = \int_{\mathbb{R}} dt |t, -t\rangle_{p_1, p_2} \quad (47)$$

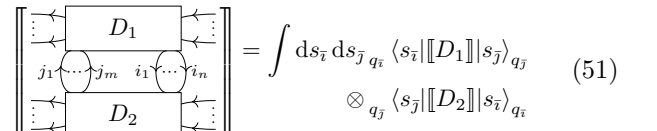
$$\llbracket \text{red spider with 2 outputs} \rrbracket \sim \int_{\mathbb{R}} dt |t, t\rangle_{p_1, p_2} = \int_{\mathbb{R}} ds |s, -s\rangle_{q_1, q_2} \quad (48)$$

$$\llbracket \text{green spider with 3 outputs} \rrbracket \sim \int_{\mathbb{R}} ds |s, s, s\rangle_{q_1, q_2, q_3} \quad (49)$$

$$\llbracket \text{red spider with 3 outputs} \rrbracket \sim \int_{\mathbb{R}} dt |t, t, t\rangle_{p_1, p_2, p_3} \quad (50)$$

Reversing the direction of each arrow gives interconversion between bra and ket, which can be understood as a generalized Choi-Jamiołkowski isomorphism. For example, reversing one of the two output arrows of (47) and (48) yields the identity operation, and reversing both of them yields the projection process onto an EPR state that models a post-selected bell measurement. Also, the two kinds of EPR states above correspond to q - and p -correlated states respectively, both of which are anti-correlated in the other basis.

Connecting the input(s) of one diagram to the output(s) of another represents tensor contraction along those indices, a simple example of which is operator multiplication. However, as our calculus model handles directed diagrams and free connections, it is necessary to establish a generalized rule to admit loops to appear in the diagram. For this purpose, we define the following contraction rule:



$$\llbracket \text{Diagram with D1 and D2 connected} \rrbracket = \int ds_{\bar{i}} ds_{\bar{j}} \langle s_{\bar{i}} | \llbracket D_1 \rrbracket | s_{\bar{j}} \rangle_{q_j} \otimes_{q_j} \langle s_{\bar{j}} | \llbracket D_2 \rrbracket | s_{\bar{i}} \rangle_{q_i} \quad (51)$$

Table II. List of generators for our graphical model and operators they represent. Note that a ket or bra subscript such as q^m is a shorthand for the q basis in m modes.

Name	Diagram D	Arity (outputs $\leftarrow$ inputs)	$\llbracket D \rrbracket$
q -spider		$m \leftarrow n$	$\int_{\mathbb{R}} ds e^{if(s)} s \dots s\rangle_{q^m} \langle s \dots s $
p -spider		$m \leftarrow n$	$\int_{\mathbb{R}} dt e^{-if(t)} t \dots t\rangle_{p^m} \langle t \dots t $
Swap		$2 \leftarrow 2$	$\iint_{\mathbb{R}^2} ds ds' s', s\rangle_{q_1, q_2} \langle s, s' $
Fourier		$1 \leftarrow 1$	$\int_{\mathbb{R}} du u\rangle_p \langle u $
Inverse Fourier		$1 \leftarrow 1$	$\int_{\mathbb{R}} du -u\rangle_p \langle u $
Squared Fourier		$1 \leftarrow 1$	$\int_{\mathbb{R}} ds -s\rangle_q \langle s $

where $\bar{i} = i_1, \dots, i_n$ and $\bar{j} = j_1, \dots, j_m$ are compound subscripts representing multiple copies of the symbol being subscripted, one for each item in the respective list. Notice that the contractions along $\bar{i}$ can be done in any basis across those modes (and similarly for $\bar{j}$). This has an interpretation as a type of partial trace on $\llbracket D_1 \rrbracket \otimes \llbracket D_2 \rrbracket$ over the contracted modes, introducing generalized input-output relationship between the two operators. In fact, when D_2 represents the identity operation, the diagram exactly corresponds to partial trace on the operator $\llbracket D_1 \rrbracket$, which can be confirmed from the following equation:

$$\left[\begin{array}{c} \text{Diagram of } D \text{ with } A \text{ on both sides} \end{array} \right] \quad (52)$$

$$= \iint_{\mathbb{R}^2} ds_1 ds_2 \langle s_1 | \llbracket D \rrbracket | s_2 \rangle_{q_2} \langle s_2 | s_1 \rangle_{q_1} \quad (53)$$

$$= \iint_{\mathbb{R}^2} ds_1 ds_2 \delta(s_2 - s_1) \langle s_1 | \llbracket D \rrbracket | s_2 \rangle_{q_2} \quad (54)$$

$$= \int_{\mathbb{R}} ds \langle s | \llbracket D \rrbracket | s \rangle_q \quad (55)$$

$$= \text{Tr}_A (\llbracket D \rrbracket) \quad (56)$$

where A denotes the qumode to be traced out. Composition of three or more diagrams can be defined by recursively applying the above interpretation.

In some cases, the contraction rule may not function properly because of infinite non-converging integrals. For example, the rule suggests that $\llbracket \text{green circle} \leftarrow \text{green circle} \rrbracket = {}_p \langle 0 | 0 \rangle_p = \int_{\mathbb{R}} 1/\sqrt{2\pi} dt = \delta(0)$, which is technically infinite. Nevertheless, we will treat it as a global scalar that can be ignored (for instance, when discussing operator equivalence for corresponding diagrams). Issues like this are ubiquitous when working with CV systems, as mentioned

in subsection II A, and solving them is beyond the scope of this work. Our purpose is to introduce a toolbox that can be the starting point for further exploration.

C. Graphical representation of CV quantum gates

By combining spiders defined above, one can obtain graphical representation of a wide variety of quantum gates. In this subsection, we show diagrams of the CV quantum gates listed in Table I. We verify the consistency of the diagrammatic representations in Appendix A 1.

1. Displacement gate

The displacement gate has a linear Hamiltonian and thus can be decomposed into two displacements, one each with respect to q and p . With this observation, we obtain the following representation of the displacement gate:

$$\leftarrow D(\alpha) \leftarrow := \leftarrow \text{red oval } \sqrt{2} \text{Re}(\alpha)x \leftarrow \text{green oval } \sqrt{2} \text{Im}(\alpha)x \leftarrow \quad (57)$$

Since any two displacement gates commute with each other up to an irrelevant global phase factor, the order of the q -spider and p -spider in the diagram above can be reversed. In Sec. IV C, we provide diagrams that are equivalent to the definitions of the quantum gates given in this chapter with their derivations based upon rewrite rules we define in the following sections.

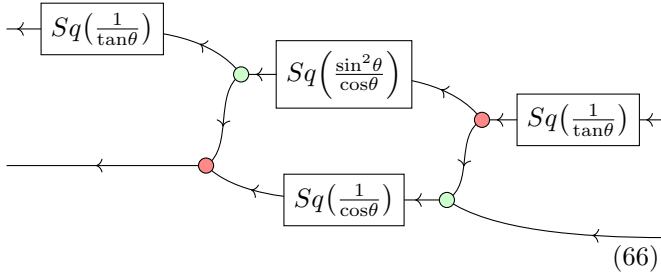
2. Phase rotation gate

A phase rotation uses a quadratic Hamiltonian and can be decomposed into three shear gates. For this reason,

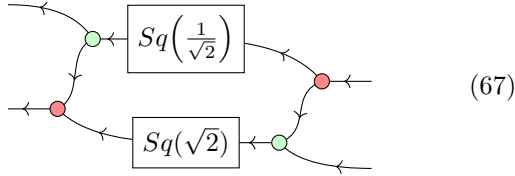
Hence, a graphical representation of any Gaussian gate can be obtained by combining the diagrams defined above [17]. One such decomposition for the beamsplitter operation $\exp(-i\theta(\hat{q}_1\hat{p} - \hat{p}_1\hat{q}_2))$ is given by

$$\begin{aligned} BS(\theta) &\equiv \exp(\theta(\hat{q}_1\hat{p}_2 - \hat{p}_1\hat{q}_2)) \\ &= \widehat{Sq}_{q_1}\left(\frac{1}{\tan\theta}\right) \cdot CS_{1,2} \cdot \widehat{Sq}_{q_1}\left(\frac{\sin^2\theta}{\cos\theta}\right) \\ &\quad \cdot \widehat{Sq}_{q_2}\left(\frac{1}{\cos\theta}\right) \cdot CS_{2,1}^\dagger \cdot \widehat{Sq}_{q_1}\left(\frac{1}{\tan\theta}\right) \end{aligned} \quad (65)$$

up to irrelevant global phase factor, as is confirmed in A 1f. Thus, the beamsplitter gate can be represented by the following diagram.



When $\theta = \frac{\pi}{4}$, the beamsplitter is often called a *balanced beamsplitter* or a *half-splitter* because its reflectance and transmittance are equal. In that case, its diagram can be represented more compactly as follows:



7. Cubic phase gate

By definition, the cubic phase gate (CPG) is represented by a single q -spider with a cubic phase function:

$$\leftarrow CPG(\gamma) \leftarrow := \leftarrow \text{green circle with } \gamma x^3 \leftarrow \quad (68)$$

In our graphical model, the non-Gaussian feature of quantum process lies not so much in the complexity of the appearance of a diagram but rather in how they are transformed by rewrite rules. In the following arguments, we will also discuss the distinct property of Gaussian and non-Gaussian properties of diagrams.

IV. PROPERTIES OF CV ZX CALCULUS

In this section, first we list the rewrite rules that constitute the axioms of our CV graphical calculus. Each rule depicts the corresponding equations of linear maps

over CV modes. Consequently, we derive basic properties of our graphical model and various equations utilized in CV quantum information processing protocols based upon the rules.

A. Basic rewrite rules

Before describing individual rewrite rules, we shall define a conjugate diagram.

Definition 16 (Conjugate diagram). Let D be a proper diagram. The *conjugate diagram* of D , denoted by $D^\dagger$, is the diagram obtained by inverting directions of all arrows and multiplying -1 to all phase functions in D (this operation interchanges Fourier diagram and inverse Fourier diagram.)

Note that conjugation is compatible with parallelizing and combining diagrams—i.e., $(D_1 \otimes D_2)^\dagger = D_1^\dagger \otimes D_2^\dagger$ and $(D_2 \circ D_1)^\dagger = D_1^\dagger \circ D_2^\dagger$. This, together with the fact that $\llbracket G \rrbracket^\dagger = \llbracket G^\dagger \rrbracket$ for each generator G in Table II, it is easy to see that $\llbracket D \rrbracket^\dagger = \llbracket D^\dagger \rrbracket$ holds for any proper diagram D .

In our calculus model, we employ a meta-rule that a rewrite rule is conserved under conjugation. More specifically, for each rewrite rule claiming $D_1 = D_2$, we accept $D_1^\dagger = D_2^\dagger$ as rewrite rule, as well. Thus, whenever two diagrams are shown to be equal via sequential application of rewrite rules, its conjugative counterpart also holds in the same manner. Also, as we are only interested in the quantum dynamics, and global scalars do not affect processes for that purpose, so we will ignore closed diagrams that appear in the derivations. The proof of soundness for each rewrite rule is given in Appendix A 2.

1. Identity rule

The identity rule (*id*) states

$$\leftarrow \text{green circle } a \leftarrow = \leftarrow \text{red circle } b \leftarrow \stackrel{(id)}{=} \leftarrow \quad (69)$$

where a and b denotes arbitrary constant-valued functions. Note that when the constants are zero, these diagrams are denoted by blank spiders.

2. Fusion rule

The fusion rule (*f*) states

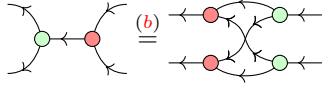
$$\begin{array}{c} n \text{ green } f(x) \text{ spider} \\ \vdots \\ n' \text{ green } g(x) \text{ spider} \end{array} \begin{array}{c} m \\ \vdots \\ m' \end{array} \stackrel{(f)}{=} n+n' \text{ green } (f+g)(x) \text{ spider} \begin{array}{c} m+m' \end{array} \quad (70)$$

$$\begin{array}{c} n \text{ red } f(x) \text{ spider} \\ \vdots \\ n' \text{ red } g(x) \text{ spider} \end{array} \begin{array}{c} m \\ \vdots \\ m' \end{array} \stackrel{(f)}{=} n+n' \text{ red } (f+g)(x) \text{ spider} \begin{array}{c} m+m' \end{array} \quad (71)$$

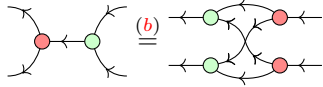
where the undirected lines between the spiders represent an arbitrary number of wires (except zero), with arbitrary directions.

3. Bialgebra rule

The bialgebra rule (*b*) has two forms:


(72)

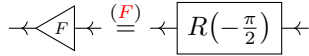
Conjugating the above rule gives the following equation:


(73)

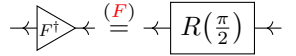
Much of the power of the graphical calculus is contained in this rule. (Recall the importance of the bialgebra property in quantum mechanics discussed in Sec. II C.)

4. Fourier rule

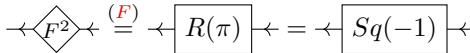
The Fourier rule (*F*) is


(74)

Conjugation gives

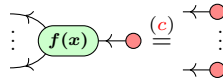

(75)

and composing this with the original gives

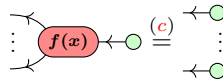

(76)

5. Copy rule

The copy rule (*c*) has two forms:


(77)

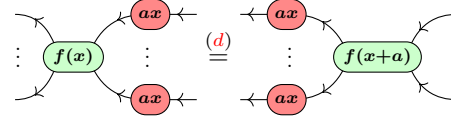
and


(78)

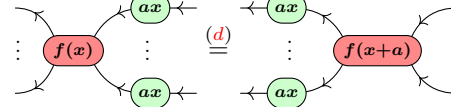
where dotted lines denote arbitrary number of wires and diagrams (and so on in the following equations).

6. Displacement rule

The displacement rule (*d*) also has two forms:


(79)

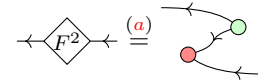
and


(80)

In the case that the diagram has no inputs (or no outputs), the linear spider disappears after acting on the central spider function.

7. Antipode rule

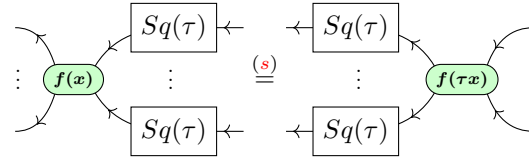
The antipode rule (*a*) states


(81)

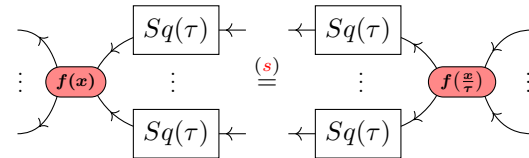
The left-side diagram of the equation is called *antipode* for its algebraic action to the phase space. We will investigate its useful properties in the following sections.

8. Squeezing rule

The squeezing rule (*s*) also has two forms:


(82)

and


(83)

In the case that the diagram has no inputs (or no outputs), the squeezing gate disappears after acting on the spider function.

9. Quadratic rule

The quadratic rule (*q*) is


(84)

$$\stackrel{(q)}{=} \leftarrow \left(\frac{bc}{4abc+a+c} x^2 \right) \leftarrow \left((4abc+a+c)x^2 \right) \leftarrow \left(\frac{ab}{4abc+a+c} x^2 \right) \leftarrow \quad (85)$$

which can also be written

$$\leftarrow \left(ax^2 \right) \leftarrow \left(bx^2 \right) \leftarrow \left(cx^2 \right) \leftarrow \quad (86)$$

$$\stackrel{(q)}{=} \leftarrow \left(\frac{bc}{4abc+a+c} x^2 \right) \leftarrow \left((4abc+a+c)x^2 \right) \leftarrow \left(\frac{ab}{4abc+a+c} x^2 \right) \leftarrow \quad (87)$$

Note that $4abc + a + c \neq 0$ must be satisfied whenever applying the quadratic rule.

10. Inversion rule

The inversion rule (*inv*) can be used to change spider types:

$$\begin{array}{c} \leftarrow F \\ \vdots \\ \leftarrow F \end{array} \leftarrow \left(f(x) \right) \begin{array}{c} \rightarrow F^\dagger \\ \vdots \\ \rightarrow F^\dagger \end{array} \stackrel{(inv)}{=} \begin{array}{c} \rightarrow F \\ \vdots \\ \rightarrow F \end{array} \leftarrow \left(f(x) \right) \begin{array}{c} \leftarrow F^\dagger \\ \vdots \\ \leftarrow F^\dagger \end{array} \quad (88)$$

Sandwiching the above equation between $\hat{F}^\dagger$ and $\hat{F}$, the following equivalent form can also be derived:

$$\begin{array}{c} \rightarrow F^\dagger \\ \vdots \\ \rightarrow F^\dagger \end{array} \leftarrow \left(f(x) \right) \begin{array}{c} \leftarrow F \\ \vdots \\ \leftarrow F \end{array} \stackrel{(inv)}{=} \begin{array}{c} \leftarrow F \\ \vdots \\ \leftarrow F \end{array} \leftarrow \left(f(x) \right) \begin{array}{c} \rightarrow F^\dagger \\ \vdots \\ \rightarrow F^\dagger \end{array} \quad (89)$$

B. Derived rewrite rules

The following rewrite rules can be derived using the basic ones.

1. Commutative and associative property of displacement gate

Direct calculation using the Baker-Campbell-Hausdorff formula shows that the displacement gates are commutative and hence associative—i.e. $\hat{D}(\alpha_1)\hat{D}(\alpha_2) \sim \hat{D}(\alpha_2)\hat{D}(\alpha_1) \sim \hat{D}(\alpha_1 + \alpha_2)$ up to an irrelevant global phase. This fact can be confirmed using our graphical calculus. First, using displacement rule, one obtains the following equation:

$$\leftarrow \left(ax \right) \leftarrow \left(bx \right) \stackrel{(d)}{=} \leftarrow \left(bx \right) \leftarrow \left(a(x+b) \right) \leftarrow \quad (90a)$$

$$\stackrel{(f)}{=} \leftarrow \left(bx \right) \leftarrow \left(ax \right) \leftarrow \left(ab \right) \leftarrow \quad (90b)$$

$$\stackrel{(id)}{=} \leftarrow \left(bx \right) \leftarrow \left(ax \right) \leftarrow \quad (90c)$$

By using this commutation rule, the associative property can be confirmed as is shown below.

$$\leftarrow \left[D(\alpha_1) \right] \leftarrow \left[D(\alpha_2) \right] \leftarrow \quad (91)$$

$$\stackrel{(57)}{=} \leftarrow \left(\sqrt{2} \operatorname{Re}(\alpha_1)x \right) \leftarrow \left(\sqrt{2} \operatorname{Im}(\alpha_1)x \right) \leftarrow \left(\sqrt{2} \operatorname{Re}(\alpha_2)x \right) \leftarrow \left(\sqrt{2} \operatorname{Im}(\alpha_2)x \right) \leftarrow \quad (92)$$

$$\stackrel{(90)}{=} \leftarrow \left(\sqrt{2} \operatorname{Re}(\alpha_1)x \right) \leftarrow \left(\sqrt{2} \operatorname{Re}(\alpha_2)x \right) \leftarrow \left(\sqrt{2} \operatorname{Im}(\alpha_1)x \right) \leftarrow \left(\sqrt{2} \operatorname{Im}(\alpha_2)x \right) \leftarrow \quad (93)$$

$$\stackrel{(f)}{=} \leftarrow \left(\sqrt{2} \operatorname{Re}(\alpha_1 + \alpha_2)x \right) \leftarrow \left(\sqrt{2} \operatorname{Im}(\alpha_1 + \alpha_2)x \right) \leftarrow \quad (94)$$

$$\stackrel{(57)}{=} \leftarrow \left[D(\alpha_1 + \alpha_2) \right] \leftarrow \quad (95)$$

2. Self-conjugateness of antipode

As is mentioned in subsection IV A 7, the antipode diagram represents the square of Fourier transform, which is self-conjugate—i.e. $(\hat{F}^2)^\dagger = \hat{F}^2$. The same property holds for the diagram as well:

$$\begin{array}{c} \leftarrow \text{green} \\ \leftarrow \text{red} \end{array} = \left(\begin{array}{c} \leftarrow \text{green} \\ \leftarrow \text{red} \end{array} \right)^\dagger = \begin{array}{c} \leftarrow \text{red} \\ \leftarrow \text{green} \end{array} \quad (96)$$

This equation can be derived in the following way. First, we have

$$\leftarrow \stackrel{(B22)}{=} \leftarrow \left[Sq(-1) \right] \leftarrow \left[Sq(-1) \right] \leftarrow \quad (97)$$

$$\stackrel{(F)}{=} \leftarrow \left(F^2 \right) \leftarrow \left(F^2 \right) \leftarrow \quad (98)$$

$$\stackrel{(a)}{=} \begin{array}{c} \leftarrow \text{green} \\ \leftarrow \text{red} \end{array} \quad (99)$$

$$\stackrel{(a)}{=} \begin{array}{c} \leftarrow \text{green} \\ \leftarrow \text{red} \end{array} \quad (100)$$

in which we used multiplicativity of squeezing gate to be shown in Appendix B 1. Consequently,

$$\begin{array}{c} \leftarrow \text{green} \\ \leftarrow \text{red} \end{array} \stackrel{(100)}{=} \begin{array}{c} \leftarrow \text{red} \\ \leftarrow \text{green} \end{array} \quad (101)$$

$$\stackrel{(f, id)}{=} \begin{array}{c} \leftarrow \text{red} \\ \leftarrow \text{green} \end{array} \quad (102)$$

$$\stackrel{(f, id)}{=} \begin{array}{c} \leftarrow \text{red} \\ \leftarrow \text{green} \end{array} \quad (103)$$

holds.

3. Antipode reversal

The antipode reversal rule (*rev*) is written in two forms:

$$\begin{array}{c} \text{f}(x) \\ \downarrow F^2 \\ \text{g}(x) \end{array} \stackrel{(rev)}{=} \begin{array}{c} \text{f}(x) \\ \downarrow \\ \text{g}(x) \end{array} \quad (104)$$

$$\begin{array}{c} \text{f}(x) \\ \downarrow F^2 \\ \text{g}(x) \end{array} \stackrel{(rev)}{=} \begin{array}{c} \text{f}(x) \\ \downarrow \\ \text{g}(x) \end{array} \quad (105)$$

The antipode reversal rule claims that the squared Fourier diagram between two spiders can be deleted by reversing its input and output. The first equation can be derived as follows:

$$\begin{array}{c} \text{f}(x) \\ \downarrow F^2 \\ \text{g}(x) \end{array} \stackrel{(a)}{=} \begin{array}{c} \text{f}(x) \\ \downarrow \\ \text{g}(x) \end{array} \stackrel{(f)}{=} \begin{array}{c} \text{f}(x) \\ \downarrow \\ \text{g}(x) \end{array} \quad (106)$$

The second equation can also be derived in the same way by using Eq. (96).

4. Quadrature copy rule

By combining the copy rule (*c*) with the displacement rule (*d*), one obtains a generalized copy rule represented in two forms:

$$\begin{array}{c} \text{f}(x) \\ \downarrow \\ \text{ax} \end{array} \stackrel{(qc)}{=} \begin{array}{c} \text{ax} \\ \vdots \\ \text{ax} \end{array} \quad (107)$$

and

$$\begin{array}{c} \text{f}(x) \\ \downarrow \\ \text{ax} \end{array} \stackrel{(qc)}{=} \begin{array}{c} \text{ax} \\ \vdots \\ \text{ax} \end{array} \quad (108)$$

The first equation can be derived in the following way:

$$\begin{array}{c} \text{f}(x) \\ \downarrow \\ \text{ax} \end{array} \stackrel{(f)}{=} \begin{array}{c} \text{f}(x) \\ \downarrow \\ \text{ax} \end{array} \quad (109)$$

$$\begin{array}{c} \text{ax} \\ \vdots \\ \text{f}(x+a) \end{array} \stackrel{(d)}{=} \begin{array}{c} \text{ax} \\ \vdots \\ \text{f}(x+a) \end{array} \quad (110)$$

$$\begin{array}{c} \text{ax} \\ \vdots \\ \text{ax} \end{array} \stackrel{(c)}{=} \begin{array}{c} \text{ax} \\ \vdots \\ \text{ax} \end{array} \quad (111)$$

$$\begin{array}{c} \text{ax} \\ \vdots \\ \text{ax} \end{array} \stackrel{(f)}{=} \begin{array}{c} \text{ax} \\ \vdots \\ \text{ax} \end{array} \quad (112)$$

and the same applies to the second one. The orientation of wires can also be reversed with an additional factor of -1 . For example,

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{f}(x) \end{array} \stackrel{(f)}{=} \begin{array}{c} \text{f}(x) \\ \downarrow \\ \text{ax} \end{array} \quad (113)$$

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(qc)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \quad (114)$$

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(96)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \boxed{Sq(-1)} \quad (115)$$

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(s)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ -\text{ax} \end{array} \quad (116)$$

holds.

5. Hopf rule

The Hopf rule (*h*) is

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(h)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \quad (117)$$

which can be derived from basic rewrite rules as follows:

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(h)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \quad (118)$$

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(rev)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \quad (119)$$

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(a)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \quad (120)$$

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(f)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \quad (121)$$

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(b)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \quad (122)$$

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(f)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \quad (123)$$

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(c)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \quad (124)$$

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(f)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \quad (125)$$

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(a)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \quad (126)$$

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(s)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \quad (127)$$

$$\begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \stackrel{(s)}{=} \begin{array}{c} \text{ax} \\ \downarrow \\ \text{ax} \end{array} \quad (128)$$

Using the Hopf rule, one can show that conjugation of the CSUM diagram yields its own inverse. In fact,

$$(129)$$

$$(130)$$

$$(131)$$

$$(132)$$

and

$$(133)$$

holds.

6. Rotation rule

The rotation rule (*rot*) is

$$(134)$$

which holds for any value of θ that is not an odd multiple of $\frac{\pi}{2}$. (In this exceptional case, the Fourier rule can be applied instead.) The rotation rule can be shown in the following way:

$$(135)$$

$$(136)$$

$$(137)$$

$$(138)$$

$$(139)$$

$$(140)$$

$$(141)$$

in which we used the following property

$$(142a)$$

$$(142b)$$

to be proven in Lemma 1. Since conjugating the rotation gate yields its own inverse, one also obtains

$$(143)$$

and thus

$$(144)$$

as well.

The rotation rule comes with another useful form of equality as is shown below:

$$(145)$$

$$(146)$$

$$(147)$$

$$(148)$$

$$(149)$$

$$(150)$$

$$(151)$$

and

$$(152)$$

$$(153)$$

$$(154)$$

$$(155)$$

$$(156)$$

$$(157)$$

thus one obtains

$$(158)$$

for any value of θ that is not a multiple of $\frac{\pi}{2}$.

7. Symmetry of Controlled-Z gate

As is shown in Table. I, controlled-Z gate has a symmetric Hamiltonian over two modes and thus it is invariant under swapping operation of the inputs. This fact can also be easily verified with graphical calculus:

$$(159)$$

always exists. Using k' in the the result above, we have

$$\leftarrow \boxed{Sq(\tau)} \leftarrow := \leftarrow \left(\frac{\tau(1-\tau)}{4k'} x^2 \right) \leftarrow \left(-\frac{k'}{\tau} x^2 \right) \leftarrow \left(\frac{\tau-1}{4k'} x^2 \right) \leftarrow (k' x^2) \leftarrow \quad (178)$$

$$\stackrel{(f)}{=} \leftarrow \left(\frac{\tau(1-\tau)}{4k'} x^2 \right) \leftarrow \left(-\frac{k'}{\tau} x^2 \right) \leftarrow \left(\frac{\tau-1}{4k'} x^2 \right) \leftarrow (k' x^2) \leftarrow (-k x^2) \leftarrow (k x^2) \leftarrow \quad (179)$$

$$\stackrel{(q)}{=} \leftarrow \left(\frac{\tau(1-\tau)}{4k'} x^2 \right) \leftarrow \left(-\frac{k'}{\tau} x^2 \right) \leftarrow \left(\frac{4k'^2 k}{(4k'k-1)\tau+1} x^2 \right) \leftarrow \left(\frac{(1-4k'k)\tau-1}{4k'} x^2 \right) \leftarrow \left(\frac{k'(1-\tau)}{(4k'k-1)\tau+1} x^2 \right) \leftarrow (k x^2) \leftarrow \quad (180)$$

$$\stackrel{(f)}{=} \leftarrow \left(\frac{\tau(1-\tau)}{4k'} x^2 \right) \leftarrow \left(\frac{k'(\tau-1)}{\tau(4k'k\tau-\tau+1)} x^2 \right) \leftarrow \left(\frac{(1-4k'k)\tau-1}{4k'} x^2 \right) \leftarrow \left(\frac{k'(1-\tau)}{(4k'k-1)\tau+1} x^2 \right) \leftarrow (k x^2) \leftarrow \quad (181)$$

$$\stackrel{(q)}{=} \leftarrow \left(\frac{\tau-1}{4k\tau^2} x^2 \right) \leftarrow (-k\tau x^2) \leftarrow \left(\frac{(\tau-1)^2}{4k\tau(4k'k\tau-\tau+1)} x^2 \right) \leftarrow \left(\frac{k'(1-\tau)}{(4k'k-1)\tau+1} x^2 \right) \leftarrow (k x^2) \leftarrow \quad (182)$$

$$\stackrel{(f)}{=} \leftarrow \left(\frac{\tau-1}{4k\tau^2} x^2 \right) \leftarrow (-k\tau x^2) \leftarrow \left(\frac{1-\tau}{4k\tau} x^2 \right) \leftarrow (k x^2) \leftarrow \quad (183)$$

where application of quadratic rule is legal since $k \neq 0$, $k' \neq 0$, and $(1-4k'k)\tau \neq 1$. $\square$

We can also go in the other direction, as shown in the following lemma.

Lemma 2. Let a, b, c, d be nonzero real values such that $4abc + a + c = 4bcd + b + d = 0$. Then, $4ab + 1 \neq 0$, and both of the following hold:

$$\leftarrow (ax^2) \leftarrow (bx^2) \leftarrow (cx^2) \leftarrow (dx^2) \leftarrow = \leftarrow \boxed{Sq(4ab+1)} \leftarrow \quad (184)$$

$$\leftarrow (ax^2) \leftarrow (bx^2) \leftarrow (cx^2) \leftarrow (dx^2) \leftarrow = \leftarrow \boxed{Sq\left(\frac{1}{4ab+1}\right)} \leftarrow \quad (185)$$

Proof. By solving $4abc + a + c = 4bcd + b + d = 0$ for a, b , we obtain

$$a = -c(4cd + 1), \quad (186a)$$

$$b = -d(4cd + 1)^{-1}. \quad (186b)$$

Let us define $\tau = 4cd + 1$. Then, these coefficients can be written as follows:

$$a = -c\tau = \frac{\tau^{-1} - 1}{4d\tau^{-2}} = \frac{\tau(1 - \tau^{-1})}{4d} \quad (187a)$$

$$b = -\frac{d}{\tau} = d\tau^{-1} \quad (187b)$$

$$c = \frac{\tau - 1}{4d} = \frac{1 - \tau^{-1}}{4d\tau^{-1}} = \frac{\tau - 1}{4d} \quad (187c)$$

By comparing Eqs. (187) with Eqs. (172) and (173), we obtain the result. $\square$

Finally, we have the main theorem for representing Gaussian unitary gates.

Theorem 2. Any diagram written as a (finite) chain of quadratic spiders such as

$$\leftarrow (a_1 x^2) \leftarrow (a_2 x^2) \leftarrow (a_3 x^2) \leftarrow (a_4 x^2) \leftarrow \cdots \leftarrow (a_N x^2) \leftarrow \quad (188)$$

can be rewritten into at most four quadratic spiders, corresponding to one of the diagrams listed in Table III.

Proof. Without loss of generality, we can assume all coefficients are nonzero. Consider a generalized form of the

diagram composed of a squeezing part and a chain part as shown here:

$$\leftarrow \boxed{Sq(\tau)} \leftarrow (a_1 x^2) \leftarrow (a_2 x^2) \leftarrow (a_3 x^2) \leftarrow (a_4 x^2) \leftarrow \cdots \leftarrow (a_N x^2) \leftarrow \quad (189)$$

By applying the quadratic rule repeatedly to the chain part of (189), the colours of the spiders are reversed (three at a time), and thus they can be merged with neighboring spiders. Based on this approach, the number of spiders of the chain spiders in (189) can be reduced to at most three by repeating the following procedures.

1. If the first three spiders of the chain part are convertible—i.e., $4a_1 a_2 a_3 + a_1 + a_3 \neq 0$, then the quadratic rule can be applied to these three, giving

$$\leftarrow \boxed{Sq(\tau)} \leftarrow (a_1 x^2) \leftarrow (a_2 x^2) \leftarrow (a_3 x^2) \leftarrow (a_4 x^2) \leftarrow \cdots \leftarrow (a_N x^2) \leftarrow \quad (190)$$

$$\stackrel{(q)}{=} \leftarrow \boxed{Sq(\tau)} \leftarrow (a'_1 x^2) \leftarrow (a'_2 x^2) \leftarrow (a'_3 x^2) \leftarrow (a_4 x^2) \leftarrow \cdots \leftarrow (a_N x^2) \leftarrow \quad (191)$$

$$\stackrel{(f)}{=} \leftarrow \boxed{Sq(\tau)} \leftarrow (a'_1 x^2) \leftarrow (a'_2 x^2) \leftarrow (a'_3 + a_4) x^2 \leftarrow \cdots \leftarrow (a_N x^2) \leftarrow \quad (192)$$

where a'_1, a'_2, a'_3 are determined by the quadratic rule. Note the number of spiders in the chain part is reduced by one in the procedure above.

2. (a) Otherwise, check whether $4a_2 a_3 a_4 + a_2 + a_4 \neq 0$ holds or not. If it holds, then the quadratic rule can be applied to the second, third and fourth spiders, and thus the number of spiders can be reduced in the same way as 1.

- (b) If both 1 and 2a fails, then $4a_1 a_2 a_3 + a_1 + a_3 = 4a_2 a_3 a_4 + a_2 + a_4 = 0$ holds and thus, due to Lemma. 2, there exists a nonzero real value τ' such that

$$\leftarrow (a_1 x^2) \leftarrow (a_2 x^2) \leftarrow (a_3 x^2) \leftarrow (a_4 x^2) \leftarrow = \leftarrow \boxed{Sq(\tau')} \leftarrow \quad (193)$$

is satisfied. In this case, the diagram can be rewritten as

$$\leftarrow \boxed{Sq(\tau)} \leftarrow \text{red oval } a_1 x^2 \leftarrow \text{green oval } a_2 x^2 \leftarrow \dots \leftarrow \text{red oval } a_N x^2 \leftarrow \quad (194)$$

$$\stackrel{(q)}{=} \leftarrow \boxed{Sq(\tau)} \leftarrow \boxed{Sq(\tau')} \leftarrow \text{red oval } a_5 x^2 \leftarrow \dots \leftarrow \text{red oval } a_N x^2 \leftarrow \quad (195)$$

$$\stackrel{(B1)}{=} \leftarrow \boxed{Sq(\tau\tau')} \leftarrow \text{red oval } a_5 x^2 \leftarrow \dots \leftarrow \text{red oval } a_N x^2 \leftarrow \quad (196)$$

and reduction can be performed anyway.

Once the number of spiders in the chain part is reduced

to 3 or less, then the whole diagram looks like either of the following two:

$$\leftarrow \boxed{Sq(\tau)} \leftarrow \text{green oval } c_1 x^2 \leftarrow \text{red oval } c_2 x^2 \leftarrow \text{green oval } c_3 x^2 \leftarrow \quad (197)$$

$$\leftarrow \boxed{Sq(\tau)} \leftarrow \text{red oval } c_1 x^2 \leftarrow \text{green oval } c_2 x^2 \leftarrow \text{red oval } c_3 x^2 \leftarrow \quad (198)$$

where we can assume $c_1 = 0$ holds if and only if $c_1 = c_2 = c_3 = 0$ (and the diagram is equivalent to single squeezing gate). When $c_1 \neq 0$, then Eq. (197) can be written as

$$\leftarrow \boxed{Sq(\tau)} \leftarrow \text{green oval } c_1 x^2 \leftarrow \text{red oval } c_2 x^2 \leftarrow \text{green oval } c_3 x^2 \leftarrow \quad (199)$$

$$\stackrel{(172)}{=} \leftarrow \text{red oval } \frac{\tau(\tau-1)}{4c_1} x^2 \leftarrow \text{green oval } \frac{c_1}{\tau} x^2 \leftarrow \text{red oval } \frac{1-\tau}{4c_1} x^2 \leftarrow \text{green oval } -c_1 x^2 \leftarrow \text{green oval } c_1 x^2 \leftarrow \text{red oval } c_2 x^2 \leftarrow \text{green oval } c_3 x^2 \leftarrow \quad (200)$$

$$\stackrel{(f)}{=} \leftarrow \text{red oval } \frac{\tau(\tau-1)}{4c_1} x^2 \leftarrow \text{green oval } \frac{c_1}{\tau} x^2 \leftarrow \text{red oval } \left(\frac{1-\tau}{4c_1} + c_2 \right) x^2 \leftarrow \text{green oval } c_3 x^2 \leftarrow \quad (201)$$

and so is Eq. (198). It is easy to see that the last diagram is either (1) equivalent to a squeezing gate or (2) convertible by the quadratic rule, thereby reducing the number of spiders to at most 3, which must coincide to one of the diagrams specified in Table III (as the list is complete for 1-mode Gaussian operations). $\square$

This theorem assures any chain of quadratic spiders can be uniquely rewritten into standard form given in Table. III. As linear phase functions can be driven out to the edge of the diagram using displacement rule (d), this result immediately implies completeness of our rewrite rules over 1-mode Gaussian diagrams written as sequence of at most quadratic spiders. Also, 1-mode spiders are interpreted as unitary operations that are well-defined on the CV Hilbert space. Thus, soundness is assured by the argument above, and we conclude that the ZX graphical calculus provides a complete characterization of 1-mode Gaussian gates.

By combining this theorem with Eq. (158), one can observe that completeness of the model is not only limited to unitary operations but extends to infinitely-squeezed 1-mode Gaussian states and projections as well. Extension of the result to multi-mode diagrams awaits further exploration. Recent work has shown that Gaussian quantum mechanics can be treated within a graphical calculus by an axiomization based on the Heisenberg picture [48, 49]. In these works, the authors avoid problems with infinite squeezing by confining the target of their investigations to Gaussian processes and utilizing a graphical calculus of affine transformations for these cases. Although that work is applicable only to Gaussian processes and has yet to deal with general CV processes, for the purpose of establishing a CV ZX calculus, it is a promising avenue of future research to integrate these

authors' methodologies with our CV ZX calculus model.

V. EXAMPLES OF GRAPHICAL CALCULUS

In this section, we provide with several examples to see how our diagrammatic calculus serves as a comprehensive tool for graphical reasoning about CV quantum information processing.

Table III. Complete classification of 1-mode Gaussian unitary gates. Each coefficient is nonzero.

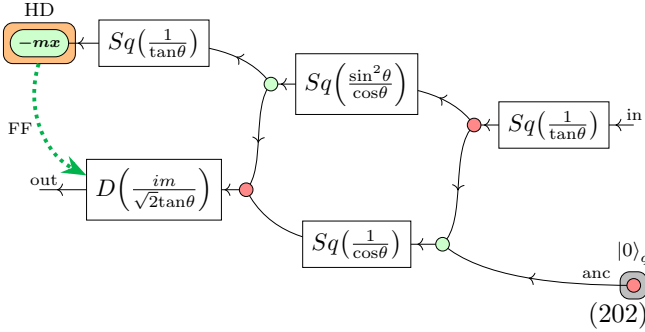
Diagram	Representation matrix
$\leftarrow$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\leftarrow \text{green oval } cx^2 \leftarrow$	$\begin{pmatrix} 1 & 0 \\ 2c & 1 \end{pmatrix}$
$\leftarrow \text{red oval } cx^2 \leftarrow$	$\begin{pmatrix} 1 & 2c \\ 0 & 1 \end{pmatrix}$
$\leftarrow \text{green oval } c_1 x^2 \leftarrow \text{red oval } c_2 x^2 \leftarrow$	$\begin{pmatrix} 1 & 2c_2 \\ 2c_1 & 1 + 4c_1 c_2 \end{pmatrix}$
$\leftarrow \text{red oval } c_1 x^2 \leftarrow \text{green oval } c_2 x^2 \leftarrow$	$\begin{pmatrix} 1 + 4c_1 c_2 & 2c_2 \\ 2c_1 & 1 \end{pmatrix}$
$\leftarrow \text{green oval } c_1 x^2 \leftarrow \text{red oval } -\frac{c_1 + c_2}{4c_1 c_2} x^2 \leftarrow \text{green oval } c_2 x^2 \leftarrow$	$\begin{pmatrix} -\frac{c_2}{c_1} & -\frac{c_1 + c_2}{c_2} \\ 0 & -\frac{c_1 c_2}{c_2} \end{pmatrix}$
$\leftarrow \text{red oval } c_1 x^2 \leftarrow \text{green oval } -\frac{c_1 + c_2}{4c_1 c_2} x^2 \leftarrow \text{red oval } c_2 x^2 \leftarrow$	$\begin{pmatrix} -\frac{c_1}{c_2} & 0 \\ -\frac{c_1 + c_2}{c_1 c_2} & -\frac{c_2}{c_1} \end{pmatrix}$
$\leftarrow \text{green oval } c_1 x^2 \leftarrow \text{red oval } c_2 x^2 \leftarrow \text{green oval } c_3 x^2 \leftarrow$	$\begin{pmatrix} 1 + 4c_2 c_3 & 2c_2 \\ 8c_1 c_2 c_3 + 2c_1 + 2c_3 & 1 + 4c_1 c_2 \end{pmatrix}$
$\leftarrow \boxed{Sq(\tau)} \leftarrow$	$\begin{pmatrix} \tau & 0 \\ 0 & \frac{1}{\tau} \end{pmatrix}$

A. Measurement-induced universal squeezer

The measurement-induced universal squeezer is a device that implements, via measurement, a squeezing gate with the aid of a squeezed state used as an ancilla and an unbalanced beamsplitter [50, 51]. Its (idealised) experimental architecture is illustrated in Fig. 1. Deriving the function of this device is a good example of our graphical formalism.

In this setup, the transmittance T of the beamsplitter determines the squeezing parameter r acting on the target mode, where the squeezing gain e^{-r} is equal to $\sqrt{T} = \sin \theta$. This procedure can be treated within the framework of our graphical calculus.

Recall the representation of $|0\rangle_q$ [Eq. (46)], a beam-splitter [Eq. (66)], a displacement gate [Eq. (57)], and a measurement in the p basis with outcome m (which is just projection onto the bra ${}_p\langle m|$) [first line of Table II, arity $0 \leftarrow 1$, with $f(x) = -mx$]. Combining these, we can write the diagram corresponding to Fig. 1:



Now we can use rewrite rules to simplify this down to its essential effect. Starting from the diagram above, we

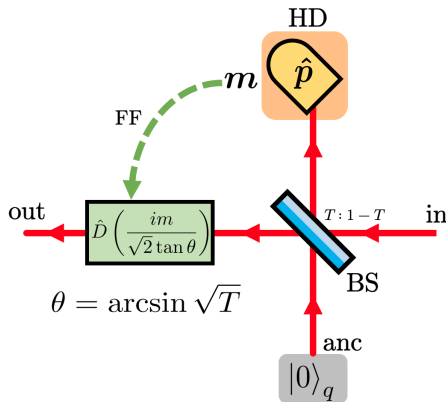
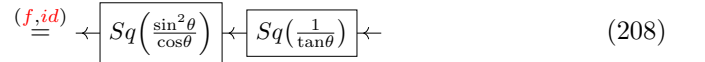
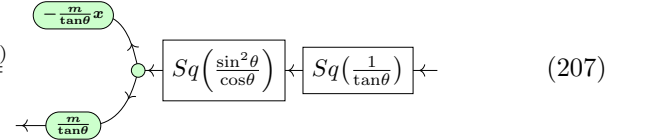
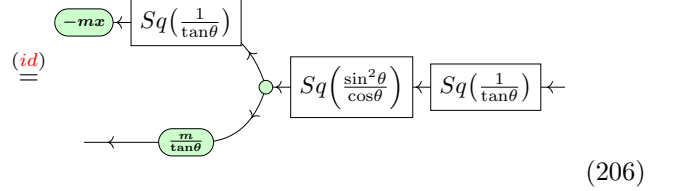
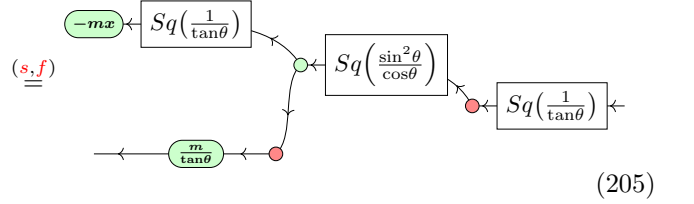
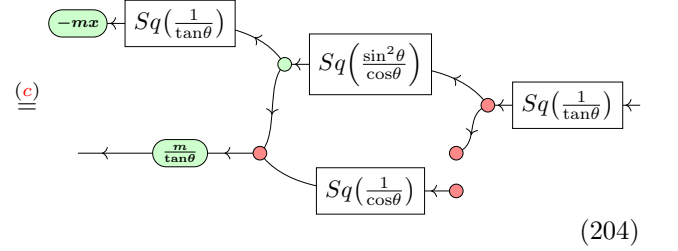
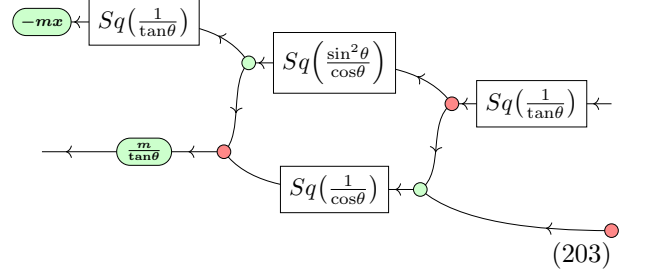


Figure 1. Schematic illustration of measurement-induced universal squeezer [50]. The measurement outcome of homodyne detection is linearly feedforwarded to subsequent displacement operation. The acronyms are as follows: HD = homodyne detection, BS = beamsplitter, FF = feedforward, anc = ancillary input.

have the following chain of equalities:



B. Quantum teleportation

Among various CV processes, quantum teleportation [52] is regarded as one of the most fundamental both conceptually and practically and is shown Fig. 2. In quantum teleportation, performing a joint measurement on an input state and half of an entangled state makes the the input state appear in the remaining system—up to an outcome-dependent displacement that can be corrected by feeding forward the outcomes to the output system and using that data to set the amount of a final displacement.

In our graphical calculus, the quantum teleportation

experiment in Figure 2 can be represented as follows:

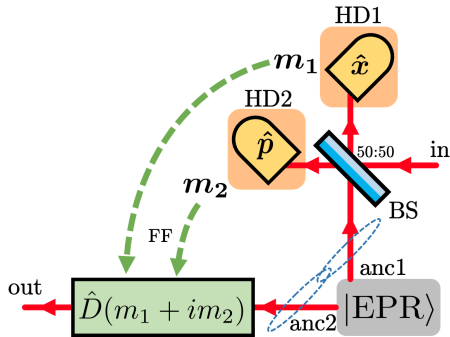
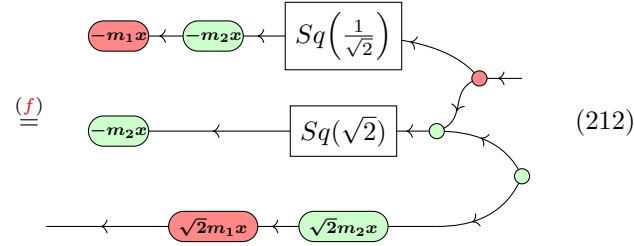
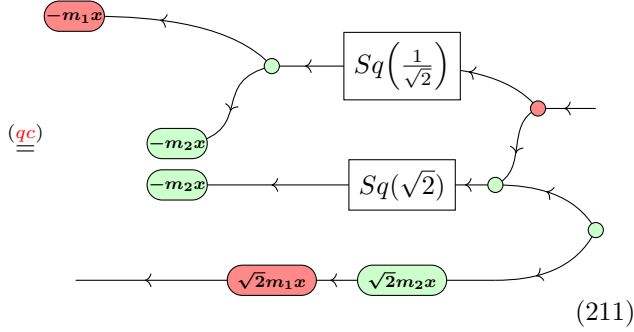
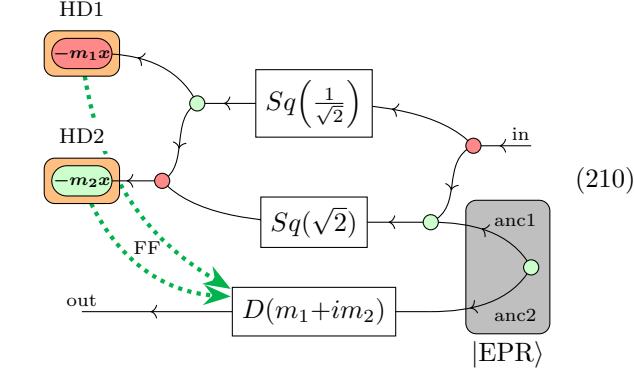
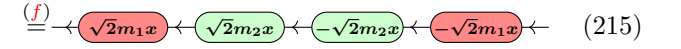
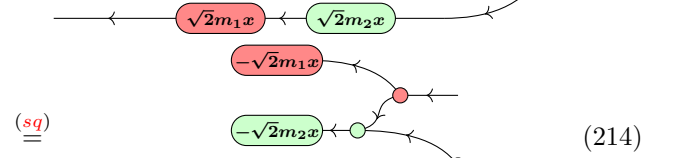
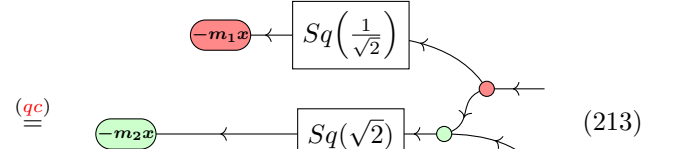


Figure 2. Schematic illustration of quantum teleportation protocol [52]. The measurement outcomes of the two homodyne detections are jointly feedforwarded. The acronyms are as follows: HD = homodyne detection, BS = beamsplitter, FF = feedforward, anc = ancillary input.



where m_1 and m_2 denotes each homodyne measurement outcome. As the last line of the calculus implies, the whole process of quantum teleportation is equivalent to the identity operation.

C. Cubic state injection

Cubic state injection is one possible way to achieve universal quantum computation in an MBQC scheme, as it introduces non-Gaussianity of an ancillary state to the system only with Gaussian operations by implementing the cubic phase gate onto an arbitrary input state. In an MBQC protocol, it is necessary to conduct feedforward operations so that errors induced by previous measurement outcomes are properly corrected. In general, when implementing n^{th} order operations, one needs $(n - 1)^{\text{th}}$ order feedforward to eliminate the effect of projection onto the displaced quadrature eigenstates [53].

The simplest implementation of cubic state injection is based on gate teleportation [55]. As illustrated on the left side of Fig. 3, this consists of a CSUM operation, an ancillary cubic phase state

$$|\psi_{\text{CPG}}^\gamma\rangle := e^{i\gamma\hat{q}^3} |0\rangle_p = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} ds e^{i\gamma s^3} |s\rangle_q, \quad (217)$$

a homodyne measurement, and a tunable Gaussian gate $\hat{G}_\gamma(m)$ defined as

$$\hat{G}_\gamma(m) = \exp(-3i\gamma(m^2x + mx^2)) \quad (218)$$

where m denotes the homodyne measurement outcome.

In our graphical calculus model, the total process of cubic state injection can be graphically calculated as fol-

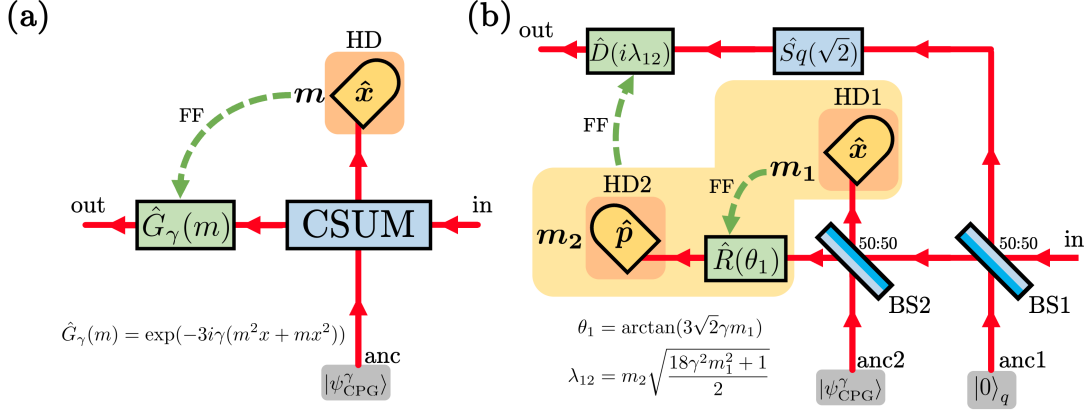


Figure 3. Schematic illustration of (a) simplified cubic state injection and (b) its practical implementation with passive optical components and nonlinear feedforward [54]. Green-colored gates represent feedforwarded operations with tunable parameters and blue-colored gates represent constant operations. (a) The operation $\hat{G}_\gamma$ is determined by the outcome of homodyne detection. (b) The first homodyne measurement outcome determines the measurement angle of the second homodyne detection. Then the two measurement outcomes are jointly feedforwarded to the displacement operation in the last step. The acronyms are as follows: HD = homodyne detection, BS = beamsplitter, FF = feedforward, anc = ancillary input.

lows:

$$\begin{aligned} & \text{out} \leftarrow -3\gamma(mx^2 + m^2x) \leftarrow \text{in} \\ & \text{FF} \leftarrow \text{HD} \leftarrow -mx \leftarrow \text{anc} \leftarrow \gamma x^3 \leftarrow |\psi_{\text{CPG}}^\gamma\rangle \end{aligned} \quad (219)$$

$$\stackrel{(f)}{=} \leftarrow -3\gamma(mx^2 + m^2x) \leftarrow -mx \leftarrow \gamma x^3 \quad (220)$$

$$\stackrel{(d)}{=} \leftarrow -3\gamma(mx^2 + m^2x) \leftarrow \gamma(x+m)^3 \quad (221)$$

$$\stackrel{(f)}{=} \leftarrow \gamma x^3 + \gamma m^3 \quad (222)$$

$$\stackrel{(id)}{=} \leftarrow \gamma x^3 \quad (223)$$

Thus, by operating the Gaussian feedforward operation of $\hat{G}_\gamma(m)$ after the measurement, one can apply the cubic phase gate of arbitrary γ by simply preparing the ancilla $|\psi_{\text{CPG}}^\gamma\rangle$ [Eq. 217] (or a suitable approximation thereof).

Although this protocol is straightforward, it has two drawbacks in practice. First, implementing the CSUM gate is itself a demanding task in optical systems [56] due to experimental constraints. Second, this scheme requires a conditional shear operation for feedforward, which is also challenging to realize, especially when aimed at ultra-fast computation. In order to overcome these difficulties and make the implementation practical, it is necessary to construct a system with feasible operations by an appropriate decomposition.

Such optical implementation of the cubic phase gate has been proposed in Ref. [54] and recently demonstrated in experiment [57]. In this scheme, as is shown on the right side of Fig. 3, the CSUM gate is replaced with two beamsplitters and one additional homodyne measurement, and the feedforward is operated by adaptive homodyne measurement and displacement. The rotation angle before the second homodyne measurement θ_1 is determined by the first measurement outcome m_1 as

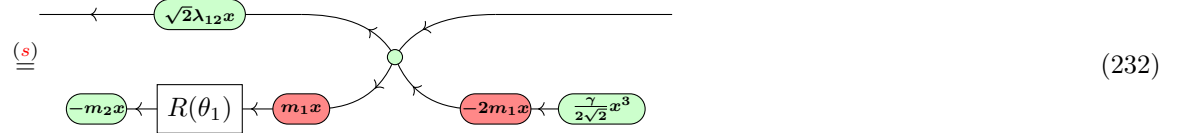
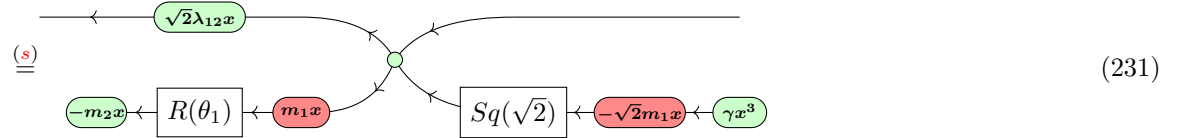
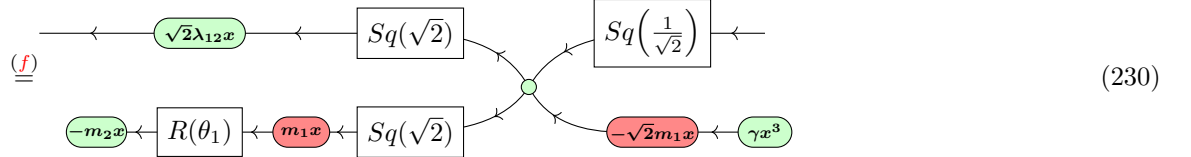
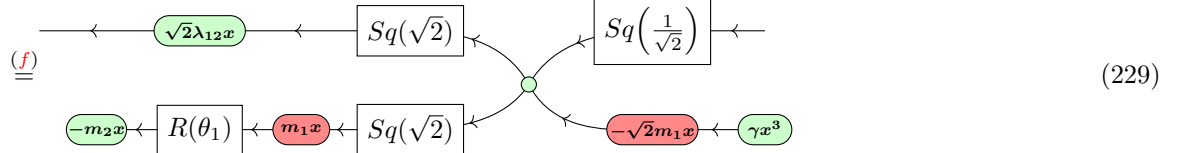
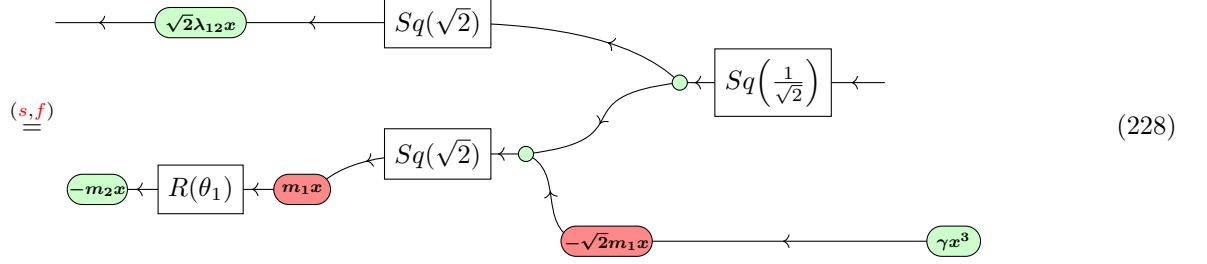
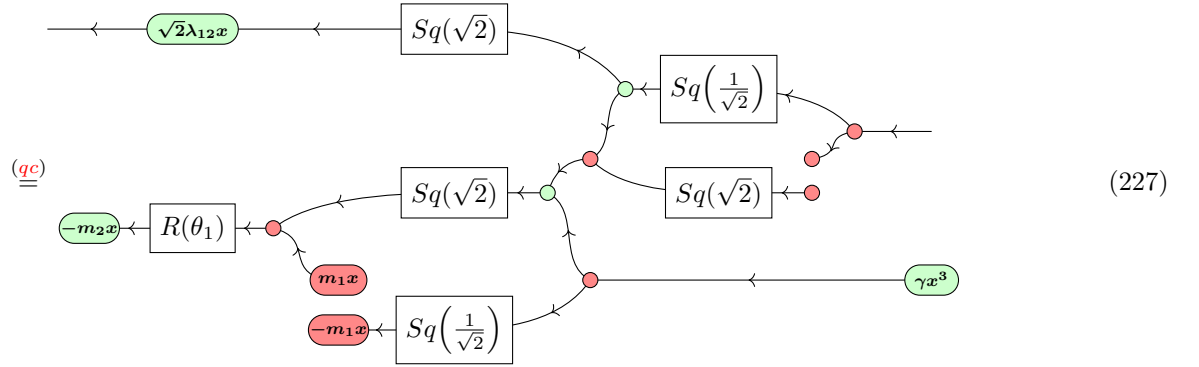
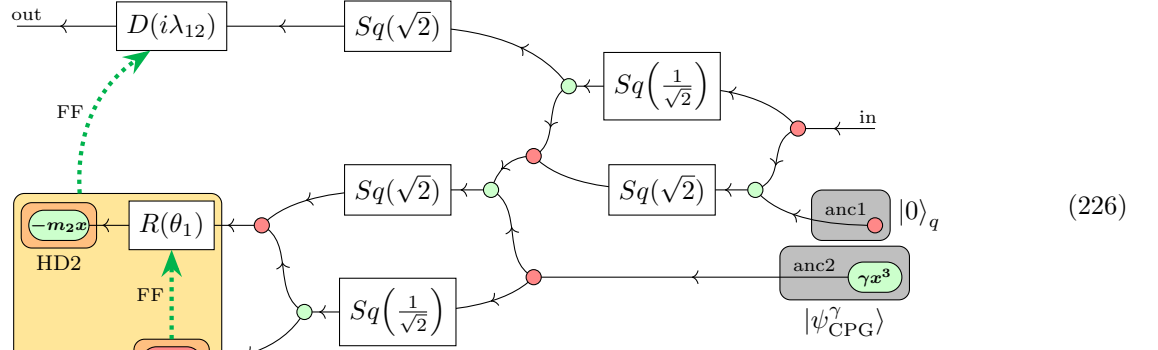
$$\theta_1 = \arctan(3\sqrt{2}\gamma m_1) \quad (224)$$

to cancel out the effect of displacement due to the previous homodyne measurement. Effectively, the rotation operation is equivalent to altering the angle of the measured basis $\hat{q}_\theta$ by $-\theta_1$. After obtaining the second measurement outcome m_2 , the output mode is squeezed by a constant factor of $\sqrt{2}$, and then a feedforwarded displacement follows. The displacement amount in the last step λ_{12} is given by the following equation

$$\lambda_{12} = m_2 \sqrt{\frac{18\gamma^2 m_1^2 + 1}{2}} \quad (225)$$

so that the unwanted displacement introduced by the two homodyne measurement is eliminated. As the constant squeezing operation can be accomplished by the measurement-induced scheme, the entire cubic phase gate protocol is realizable only with passive optical components and ancillary states, which is a great advantage for experimental implementation.

Again, one can follow its dynamics using our graphical calculus. The protocol can be understood as follows:



$$\begin{aligned} & \xleftarrow{(s)} \xleftarrow{\sqrt{2}\lambda_{12}x} \xleftarrow{m_1x} \xleftarrow{-2m_1x} \xleftarrow{\frac{\gamma}{2\sqrt{2}}x^3} \xleftarrow{R(\theta_1)} \xleftarrow{-m_2x} \xleftarrow{\sqrt{2}\lambda_{12}x} \end{aligned} \quad (233)$$

$$\begin{aligned} & \xleftarrow{(239)} \xleftarrow{\sqrt{2}\lambda_{12}x} \xleftarrow{m_1x} \xleftarrow{-2m_1x} \xleftarrow{\frac{\gamma}{2\sqrt{2}}x^3} \xleftarrow{R(\theta_1)} \xleftarrow{-m_2x} \xleftarrow{\sqrt{2}\lambda_{12}x} \end{aligned} \quad (234)$$

$$\begin{aligned} & \xleftarrow{(d)} \xleftarrow{\sqrt{2}\lambda_{12}x} \xleftarrow{m_1x} \xleftarrow{-2m_1x} \xleftarrow{\frac{\gamma}{2\sqrt{2}}x^3} \xleftarrow{R(\theta_1)} \xleftarrow{-m_2x} \xleftarrow{\sqrt{2}\lambda_{12}x} \end{aligned} \quad (235)$$

$$\begin{aligned} & \xleftarrow{(f)} \xleftarrow{\sqrt{2}\lambda_{12}x} \xleftarrow{m_1x} \xleftarrow{-2m_1x} \xleftarrow{\frac{\gamma}{2\sqrt{2}}x^3} \xleftarrow{R(\theta_1)} \xleftarrow{-m_2x} \xleftarrow{\sqrt{2}\lambda_{12}x} \end{aligned} \quad (236)$$

$$\xleftarrow{\frac{\gamma}{2\sqrt{2}}x^3} \quad (237)$$

The last equation directly follows from the definitions (224,225). As can be seen above, the whole process is equivalent to a cubic phase gate with $\gamma' = \frac{\gamma}{\sqrt{2}}$. Here

we used the following commutation property to show the equivalence between (233) and (234):

$$\xleftarrow{R(\theta)} \xleftarrow{ax} \quad (238a)$$

$$\xleftarrow{(58)} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \xleftarrow{-\frac{\sin\theta}{2}x^2} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \xleftarrow{ax} \quad (238b)$$

$$\xleftarrow{(d)} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \xleftarrow{-\frac{\sin\theta}{2}x^2} \xleftarrow{ax} \xleftarrow{\frac{\tan(\theta/2)}{2}(x+a)^2} \quad (238c)$$

$$\xleftarrow{(f,id)} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \xleftarrow{-\frac{\sin\theta}{2}x^2} \xleftarrow{ax} \xleftarrow{atan(\theta/2)x} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \quad (238d)$$

$$\xleftarrow{(f)} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \xleftarrow{ax} \xleftarrow{-\frac{\sin\theta}{2}x^2} \xleftarrow{atan(\theta/2)x} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \quad (238e)$$

$$\xleftarrow{(d)} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \xleftarrow{ax} \xleftarrow{atan(\theta/2)x} \xleftarrow{-\frac{\sin\theta}{2}(x+atan(\theta/2))^2} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \quad (238f)$$

$$\xleftarrow{(90)} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \xleftarrow{atan(\theta/2)x} \xleftarrow{ax} \xleftarrow{-\frac{\sin\theta}{2}(x+atan(\theta/2))^2} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \quad (238g)$$

$$\xleftarrow{(f,id)} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \xleftarrow{atan(\theta/2)x} \xleftarrow{ax} \xleftarrow{-asin\theta\tan(\theta/2)x} \xleftarrow{-\frac{\sin\theta}{2}x^2} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \quad (238h)$$

$$\xleftarrow{(f)} \xleftarrow{atan(\theta/2)x} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \xleftarrow{(acos\theta)x} \xleftarrow{-\frac{\sin\theta}{2}x^2} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \quad (238i)$$

$$\xleftarrow{(d)} \xleftarrow{atan(\theta/2)x} \xleftarrow{(acos\theta)x} \xleftarrow{\frac{\tan(\theta/2)}{2}(x+acos\theta)^2} \xleftarrow{-\frac{\sin\theta}{2}x^2} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \quad (238j)$$

$$\xleftarrow{(f)} \xleftarrow{atan(\theta/2)x} \xleftarrow{(acos\theta)x} \xleftarrow{acos\theta\tan(\theta/2)x} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \xleftarrow{-\frac{\sin\theta}{2}x^2} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \quad (238k)$$

$$\xleftarrow{(90)} \xleftarrow{atan(\theta/2)x} \xleftarrow{acos\theta\tan(\theta/2)x} \xleftarrow{(acos\theta)x} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \xleftarrow{-\frac{\sin\theta}{2}x^2} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \quad (238l)$$

$$\xleftarrow{(f)} \xleftarrow{(asin\theta)x} \xleftarrow{(acos\theta)x} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \xleftarrow{-\frac{\sin\theta}{2}x^2} \xleftarrow{\frac{\tan(\theta/2)}{2}x^2} \quad (238m)$$

$$\xleftarrow{(58)} \xleftarrow{(asin\theta)x} \xleftarrow{(acos\theta)x} \xleftarrow{R(\theta)} \quad (238n)$$

for arbitrary θ, a , (note that the equation holds when θ is multiple of π as well) and thus

$$\xleftarrow{-m_2x} \xleftarrow{R(\theta_1)} \quad (239a)$$

$$\xleftarrow{(c)} \xleftarrow{-(m_2\tan\theta)x} \xleftarrow{-m_2x} \xleftarrow{R(\theta_1)} \quad (239b)$$

$$\xleftarrow{(238)} \xleftarrow{R(\theta_1)} \xleftarrow{-\frac{m_2}{\cos\theta_1}x} \quad (239c)$$

$$\xleftarrow{(144)} \xleftarrow{-\frac{\tan\theta_1}{2}x^2} \xleftarrow{-\frac{m_2}{\cos\theta_1}x} \quad (239d)$$

$$\stackrel{(f)}{=} \left(-\frac{\tan\theta_1}{2}x^2 - \frac{m_2}{\cos\theta_1}x \right) \leftarrow \quad (239e)$$

holds.

VI. DISCUSSIONS

A. Validity of canonical eigenstates and infinite scalars

In this paper, we introduced a graphical calculus based on directed graphs and explored its behavior under a certain set of rewrite rules. As we mentioned earlier, diagrams can be interpreted as CV operators by the mapping $\llbracket \cdot \rrbracket: D \mapsto \llbracket D \rrbracket$, and each rewrite rule equates two diagrams whose interpretation coincides. However, interpretations of q - and p -spiders given in Table I are not well-defined (when $n \neq 1$ or $m \neq 1$). This is due to the fact that canonical operators $\hat{q}, \hat{p}$ are unbounded operators [58] and thus quadrature eigenstates $\{|s\rangle_q\}_{s \in \mathbb{R}}, \{|t\rangle_p\}_{t \in \mathbb{R}}$ are not formally defined in the bosonic Hilbert space. One of the most evident example of such trickiness emerges in the most simple diagram:

$$\llbracket \bigcirc \rrbracket = \int_{\mathbb{R}} ds \, 1 \quad (240)$$

is not defined in $\mathbb{C}$. This diagram can also be evaluated as

$$\llbracket \bigcirc \leftarrow \bigcirc \rrbracket = {}_p \langle 0|0 \rangle_p = \delta(0), \quad (241)$$

in which the delta function emerges as Fourier transform of the constant function. Because of these improper calculation, our graphical model cannot assure soundness in general cases. Even if two diagrams have proper interpretation themselves, the graphical proof of two diagrams being rewritable may include improper transformation and thus leaves ambiguity at the formal level. The reverse is also true: Among our rewrite rules, the identity rule (**id**), fusion rule (**f**), squeezing rule (**s**), displacement rule (**d**), inversion rule (**inv**), and quadratic rule (**q**) do not contain undefinable diagrams when all the inputs and outputs are restricted to a single mode. Since our argument on completeness for 1-mode Gaussian gates, Sec. IV C, relies only on these rules, the proof is strictly valid.

Generally, it is already known that a compact structure does not exist in the category of infinite-dimensional Hilbert spaces [8], which presents a barrier for naive generalization of the framework of the ZX calculus for a standard CV system. This issue raises another problem deeply associated with the standard architecture of CV quantum computing. As we did in this paper, the canonical eigenstates are widely used when exploring the CV systems. Just as we emphasized in the introduction, these eigenstates themselves require a sensitive handling, since they are out of the Hilbert space $\mathcal{H}$ and need careful justification for their expected behaviors in the CV

system. Nevertheless, quadrature eigenstates are often regarded as infinitely-squeezed vacuum, and so are other ideal processes such as homodyne projection and preparation of perfectly correlated EPR states. Although these intuitive interpretations may not suffice to ensure formal mathematical rigor, this is an issue with all of CV quantum information, and these problems are inherited by our formalism. Nevertheless, the quadrature expansion is quite useful to capture the essence of quantum processes in phase space, and canonical eigenstates serve as a handy tool for analysis of ideal processes with infinitely-squeezed states.

For example, a physical EPR state has uncertainty in the quadrature distribution of the two modes due to its finite correlation. By contrast, an ideal EPR state has perfect correlation, thus the homodyne measurement outcome of the first mode exactly coincides to the displaced amount of the residual quantum state in the second mode. By using this property, one can easily show how the quantum teleportation protocol retrieves the original input state without any disturbance, which is essentially what we did through the graphical calculus.

With this perspective, the graphical calculus model proposed in this paper does not achieve the level of rigour used in formal mathematics, but it does so to a standard appropriate for theoretical physics. To wit, it relies on the standard framework of CV quantum computing, and we elucidated how the generalization of the ZX calculus should behave based on it. In order to overcome these ambiguities, an additional framework or structure is needed so that one can handle the infinitely squeezed states in a rigorous way while preserving its expected properties.

One possible approach to deal with these problems is to introduce infinite objects, such as unbounded scalars, quantum states with infinite norms, and others in a rigorous way. Recent research has proposed the application of non-standard analysis [59], a mathematical field of logic that formalizes unbounded scalars and infinitesimals, for illustration of infinite-dimensional quantum mechanics [60, 61] and its categorical representation [62, 63]. Using this approach, one can consider linear algebras on infinite-dimensional Hilbert spaces where infinities and infinitesimals can be treated as normal numbers and where most features of standard Hilbert spaces are transferred without losing mathematical rigor. Further refinements of this approach may help to formalize our graphical model with infinite scalars so that its soundness is strictly guaranteed. The current presentation stands as a useful starting point for a CV ZX calculus, with its more formal analysis—using more sophisticated tools—left to future work.

B. Representation of physical states and error correcting codes

As mentioned in Sec. VI A, our graphical calculus is based on canonical eigenstates which cannot be normalized. This causes another problem in the other direction: Though the model can deal with certain unphysical states, such as infinitely squeezed states and the ideal cubic phase states $|\psi_{\text{CPG}\gamma}\rangle$ [Eq. (217)], as we illustrated in this paper, it cannot give a diagrammatic representation for any physical (finite-norm) state, including Fock eigenstates, displaced vacuum, and finitely-squeezed states.

One possible extension of the model to deal with this problem is to admit complex phase functions for each spider. Since the vacuum state, for instance, can be expanded with regard to the position basis as

$$|0\rangle = \int ds \frac{1}{\pi^{1/4}} \exp\left(-\frac{s^2}{2}\right) |s\rangle_q, \quad (242)$$

it can also be regarded as a momentum eigenstate $|0\rangle_p \propto \int ds |s\rangle_q$ perturbed under imaginary time evolution. With this perspective, one can interpret the following diagram

$$\leftarrow \text{green diamond} \leftarrow = \leftarrow \text{green diamond} \leftarrow \text{circle} \quad (243)$$

as representing the vacuum state. Most of the rewrite rules proposed in this paper can be generalized to complex spiders. The properties of non-unitary Gaussian operators can also be studied using extended rewrite rules including the quadratic rule (q) and the displacement rule (q), which may associate our calculus model with graphical symplectic algebra for general Gaussian processes [48, 49]. Another potential proposal is to introduce diagrams representing non-unitary operations such as damping operator $\mathcal{N}(\gamma) = \exp(-\gamma \hat{a}^\dagger \hat{a})$. Using this operator, the correspondence between infinite- and finitely-squeezed states is given by the equation below [64]:

$$\mathcal{N}(\gamma) |0\rangle_q \propto \widehat{Sq} \left(-\frac{\log \tanh \gamma}{2} \right) |0\rangle \quad (244)$$

and thus the damping operation transforms an unphysical state into physical state (up to some infinite scalar factor). Since the damping operation could be interpreted as a type of decoherence, this process may also be formalized in an extensive framework of ZX calculus generalized for representation of mixed states, decoherence, and classical information [65, 66].

A different extension of the graphical calculus may be found in the direction of quantum error correction codes. As we mentioned in the introduction, CV information processing is highly compatible with error correction, and it is natural to expect the CV calculus model to be unified with DV computing. One possible approach for such application is to introduce new diagrams representing encoded qubit states. For example, the code space of the

Gottesman-Kitaev-Preskill qubit [18] is stabilized by operators

$$\{\exp(-2i\sqrt{\pi}\hat{p}), \exp(2i\sqrt{\pi}\hat{x})\}. \quad (245)$$

Thus, one could define the notion of a GKP diagram, denoted as a yellow diamond in the following equation, such that

$$\leftarrow \text{yellow diamond} \leftarrow = \leftarrow \text{green diamond} \leftarrow \text{yellow diamond} \leftarrow = \leftarrow \text{red diamond} \leftarrow \text{yellow diamond} \quad (246)$$

holds. Moreover, the logical state $|0_{\text{GKP}}\rangle$ is uniquely characterized by the following relation

$$|0_{\text{GKP}}\rangle = \widehat{Sq}(2) \hat{F} |0_{\text{GKP}}\rangle, \quad (247)$$

thus it can be specified with an additional axiom as follows:

$$\leftarrow \text{yellow diamond} \leftarrow = \leftarrow \boxed{Sq(2)} \leftarrow \triangleleft_F \leftarrow \text{yellow diamond} \quad (248)$$

These equations represent the principal properties of the GKP qubit and thus would be sufficient for graphical reasonings of the fundamental error correction schemes proposed in Ref. [18]. It is an interesting question to consider how these non-unitary diagrams and code states could interact with other elements through rewrite rules to be derived in future work.

C. Incompleteness of the set of rewrite rules

The set of rewrite rules we presented in Sec. IV A are determined not only as an extension of the framework of the ZX calculus but also as a handy rule set to conduct diagrammatic calculus operations for what we consider to be the principal processes in CV quantum information processing. Though we have investigated various properties of the CV system within our rule set, it is unknown what rule set can achieve a wider range of application for more general processes. In fact, in the last section, we discussed the completeness of the graphical calculus for 1-mode Gaussian processes in Sec. IV C, but in the discussion we restricted attention to diagrams composed only of 1-to-1 quadratic spiders, and it cannot be directly extended to a general form of Gaussian diagrams including multi-mode spiders. For instance, direct calculation suggests

$$\left[\left[\leftarrow \text{red circle} \leftarrow \begin{array}{c} \boxed{Sq(\tau)} \\ \boxed{Sq(\tau')} \end{array} \right] \right] = \left[\left[\leftarrow \boxed{Sq(\tau+\tau')} \leftarrow \right] \right] \quad (249)$$

should hold. However, rewritability of the two diagrams under the rule set given in this paper is unclear so far.

Previous research has revealed that the qubit ZX calculus is complete for Clifford processes and that all Clifford diagrams can be efficiently rewritten into a standard form [67, 68], and this fact is used in various applications, including graphical characterization of Clifford

process and circuit extraction. As properties of Gaussian processes in a CV system are analogous to that of Clifford operations for qubits in some ways [34], it is an interesting question whether completeness for Gaussian diagrams can be boiled down to similar graphical manipulations.

VII. CONCLUSION

In this paper, we proposed a graphical calculus to model CV dynamics by convertibility of simple diagrams that represent each quantum process under a set of rewrite rules. Although the model contains subtle issues regarding infinities, the framework we propose is a starting point for further investigations. As with the original ZX calculus, we expect our model to have numerous applications in CV quantum information processing.

One potential application is as the foundation of computer software for automated circuit optimization and interactive theorem proving. Recently, the ZX calculus has been used in computing packages that treat DV computations, such as PyZX [44], Quantomatic [69], PennyLane [70] and others. Similar tools for CV quantum computing are far more sparse. As we have seen in this paper, our graphical calculus is powerful enough to interpret various CV protocols, and it may be utilized as a language for abstraction of CV quantum information processing.

Another interesting possibility might be found in the reasoning of MBQC and quantum error-correcting codes. Since one-way quantum computing and topological codes are highly compatible with diagrammatic representation [13–15], a number of investigations reported advantageous in using the (qubit) ZX calculus as a graphical language. Our CV version may be helpful in the analysis and design of CV measurement-based quantum computing architectures [21, 22] and applications.

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Appendix A: Consistency of graphical calculus and CV operators

1. Quantum gates

In this section, we will confirm that the graphical representations of CV quantum gates given in Sec. III C is consistent with the definitions listed in Table I.

a. Displacement gate

$$\llbracket \leftarrow \text{red box } \sqrt{2\text{Re}(\alpha)x} \leftarrow \text{green box } \sqrt{2\text{Im}(\alpha)x} \leftarrow \rrbracket \quad (\text{A1})$$

$$= \left(\int_{\mathbb{R}} dt e^{-i\sqrt{2}\text{Re}(\alpha)t} |t\rangle_p \langle t| \right) \left(\int_{\mathbb{R}} ds e^{i\sqrt{2}\text{Im}(\alpha)t} |s\rangle_q \langle s| \right) \quad (\text{A2})$$

$$= e^{-i\sqrt{2}\text{Re}(\alpha)\hat{p}} e^{i\sqrt{2}\text{Im}(\alpha)\hat{q}} \quad (\text{A3})$$

$$\sim e^{i(-\sqrt{2}\text{Re}(\alpha)\hat{p} + \sqrt{2}\text{Im}(\alpha)\hat{q})} \quad (\text{A4})$$

$$= e^{i(\alpha^* \hat{a} - \alpha \hat{a}^\dagger)} \quad (\text{A5})$$

Here we used Baker-Campbell-Hausdorff formula under the canonical commutation relation $[\hat{q}, \hat{p}] = i$.

b. Phase rotation gate

Instead of directly calculating operator multiplications in the Schrödinger picture, here we employ the Heisenberg picture and compute the quadrature transformation function defined in Eq. (13). For a quadratic Gaussian operator without displacement, one only needs to compute its representation matrix for a unique identification of the operator (up to global scalar factor). Let $M(\hat{A})$ denote the symplectic matrix representing the Heisenberg action of an operator $\hat{A}$ with respect to the vector of quadratures $(\hat{q}, \hat{p})^T$. Then,

$$M(\llbracket \leftarrow \text{red box } \frac{\tan(\theta/2)x^2}{2} \leftarrow \text{green box } -\frac{\sin\theta x^2}{2} \leftarrow \text{red box } \frac{\tan(\theta/2)x^2}{2} \leftarrow \rrbracket) \quad (\text{A6})$$

$$= \begin{pmatrix} 1 & \tan \frac{\theta}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\sin \theta & 1 \end{pmatrix} \begin{pmatrix} 1 & \tan \frac{\theta}{2} \\ 0 & 1 \end{pmatrix} \quad (\text{A7})$$

$$= \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (\text{A8})$$

This is the rotation matrix for the quadrature operators as is defined in Table I.

c. 1-mode squeezing gate

$$M(\llbracket \leftarrow \text{red box } \frac{\tau(1-\tau)}{4}x^2 \leftarrow \text{green box } -\frac{1}{\tau}x^2 \leftarrow \text{red box } \frac{(\tau-1)}{4}x^2 \leftarrow \text{green box } x^2 \leftarrow \rrbracket) \quad (\text{A9})$$

$$= \begin{pmatrix} 1 & \frac{\tau(1-\tau)}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{2}{\tau} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\tau-1}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad (\text{A10})$$

$$= \begin{pmatrix} \tau & 0 \\ 0 & \frac{1}{\tau} \end{pmatrix} \quad (\text{A11})$$

This matrix transformation achieves a squeezing transformation for the quadrature operators, as defined in Table I with $\tau = e^{-r}$.

squeezing gates yields

$$\left[\begin{array}{c} \leftarrow Sq(g^{-1}) \leftarrow \text{green dot} \leftarrow Sq(g) \leftarrow \\ \leftarrow \text{red dot} \leftarrow \end{array} \right] \quad (\text{A17})$$

$$= \left(\int_{\mathbb{R}} ds |s\rangle_q {}_q\langle gs| \right) e^{-i\hat{q}_1 \hat{p}_2} \left(\int_{\mathbb{R}} ds' |gs'\rangle_q {}_q\langle s'| \right) \quad (\text{A18})$$

$$= e^{-ig\hat{q}_1 \hat{p}_2}. \quad (\text{A19})$$

d. Controlled-sum gate

For the unbiased ($g = 1$) controlled-sum gate,

$$\left[\begin{array}{c} \leftarrow \text{green dot} \leftarrow \\ \leftarrow \text{red dot} \leftarrow \end{array} \right] \quad (\text{A12})$$

$$= \left(\int_{\mathbb{R}} dt |t\rangle_{p_2} {}_{p_{\text{anc}}, p_2}\langle t, t| \right) \left(\int_{\mathbb{R}} ds |s, s\rangle_{q_1, q_{\text{anc}}} {}_{q_1}\langle s| \right) \quad (\text{A13})$$

$$= \int_{\mathbb{R}} ds \int_{\mathbb{R}} dt {}_{p_{\text{anc}}} \langle t|s \rangle_{q_{\text{anc}}} |s, t\rangle_{q_1, p_2} {}_{q_1, p_2}\langle s, t| \quad (\text{A14})$$

$$= \int_{\mathbb{R}} ds \int_{\mathbb{R}} dt e^{-ist} |s, t\rangle_{q_1, p_2} {}_{q_1, p_2}\langle s, t| \quad (\text{A15})$$

$$= e^{-i\hat{q}_1 \hat{p}_2} \quad (\text{A16})$$

realizes the unitary transformation defined in Table I. For biased gates, sandwiching the unbiased CSUM gate with

e. Controlled-Z gate

The unbiased CZ gates can be expanded as

$$\left[\begin{array}{c} \leftarrow \text{green dot} \leftarrow \\ \leftarrow \text{green dot} \leftarrow \end{array} \right] \quad (\text{A20})$$

$$= \left(\int_{\mathbb{R}} ds |s\rangle_{q_1} {}_{q_1, q_{\text{anc}}} \langle s, s| \right) \hat{F}_{\text{anc}} \left(\int_{\mathbb{R}} ds' |s'\rangle_{q_{\text{anc}}} {}_{q_2}\langle s'| \right) \quad (\text{A21})$$

$$= \int_{\mathbb{R}} ds \int_{\mathbb{R}} ds' {}_{q_{\text{anc}}} \langle s|\hat{F}|s'\rangle_{q_{\text{anc}}} |s, s'\rangle_{q_1, q_2} {}_{q_1, q_2}\langle s, s'| \quad (\text{A22})$$

$$= \int_{\mathbb{R}} ds \int_{\mathbb{R}} ds' {}_{q_{\text{anc}}} \langle s|s'\rangle_{p_{\text{anc}}} |s, s'\rangle_{q_1, q_2} {}_{q_1, q_2}\langle s, s'| \quad (\text{A23})$$

$$= \int_{\mathbb{R}} ds \int_{\mathbb{R}} ds' e^{iss'} |s, s'\rangle_{q_1, q_2} {}_{q_1, q_2}\langle s, s'| \quad (\text{A24})$$

$$= e^{i\hat{q}_1 \hat{q}_2} \quad (\text{A25})$$

and sandwiching this by squeezing gates works for biased ones in the same way as CSUM gates.

f. Beamsplitter gate

$$M \left(\begin{array}{c} \leftarrow Sq\left(\frac{1}{\tan\theta}\right) \leftarrow \text{green dot} \leftarrow Sq\left(\frac{\sin^2\theta}{\cos\theta}\right) \leftarrow \text{red dot} \leftarrow Sq\left(\frac{1}{\tan\theta}\right) \leftarrow \\ \leftarrow \text{red dot} \leftarrow Sq\left(\frac{1}{\cos\theta}\right) \leftarrow \text{green dot} \leftarrow \end{array} \right) \quad (\text{A26})$$

$$= \begin{pmatrix} \frac{1}{\tan\theta} & 0 & 0 & 0 \\ 0 & \tan\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sin^2\theta}{\cos\theta} & 0 & 0 & 0 \\ 0 & \frac{\cos\theta}{\sin^2\theta} & 0 & 0 \\ 0 & 0 & \frac{1}{\cos\theta} & 0 \\ 0 & 0 & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\tan\theta} & 0 & 0 & 0 \\ 0 & \tan\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A27})$$

$$= \begin{pmatrix} \cos \theta & 0 & -\sin \theta & 0 \\ 0 & \cos \theta & 0 & -\sin \theta \\ \sin \theta & 0 & \cos \theta & 0 \\ 0 & \sin \theta & 0 & \cos \theta \end{pmatrix} \quad (\text{A28})$$

g. Cubic phase gate

$$\llbracket \leftarrow \textcircled{\gamma x^3} \leftarrow \rrbracket = \int_{\mathbb{R}} ds e^{i\gamma x^3} {}_q|s\rangle \langle s|_q = e^{i\gamma q^3} \quad (\text{A29})$$

2. Soundness of basic rewriting rules

As is discussed in Sec IID, to prove soundness of a graphical calculus one just needs to verify that the individual rewrite rules are sound. In this section, we will show soundness for each basic rewriting rules given in IVA by direct calculation.

a. Identity rule

$$\llbracket \leftarrow \textcircled{a} \leftarrow \rrbracket = \int_{\mathbb{R}} ds e^{ia} |s\rangle_q {}_q\langle s| \sim \text{id}_{\mathcal{H}} \quad (\text{A30})$$

$$\llbracket \leftarrow \textcircled{b} \leftarrow \rrbracket = \int_{\mathbb{R}} ds e^{ib} |t\rangle_p {}_p\langle t| \sim \text{id}_{\mathcal{H}} \quad (\text{A31})$$

where $\text{id}_{\mathcal{H}}$ denotes the identity operation.

b. Fusion rule

$$\left[\begin{array}{c} n \text{ } \textcircled{f(x)} \text{ } m \\ \vdots \\ n' \text{ } \textcircled{g(x)} \text{ } m' \end{array} \right] \quad (\text{A32})$$

$$= \int_{\mathbb{R}} ds \int_{\mathbb{R}} ds' e^{if(s)} e^{ig(s')} ({}_q\langle s|s'\rangle_q)^k ({}_q\langle s'|s\rangle_q)^{k'} \\ \times (|s\rangle_q)^{\otimes n} ({}_q\langle s|)^{\otimes m} \otimes (|s'\rangle_q)^{\otimes n'} ({}_q\langle s'|)^{\otimes m'} \quad (\text{A33})$$

$$= \int_{\mathbb{R}} ds \int_{\mathbb{R}} ds' e^{if(s)} e^{ig(s')} \delta(s - s')^{k+k'} \\ \times (|s\rangle_q)^{\otimes n} ({}_q\langle s|)^{\otimes m} \otimes (|s'\rangle_q)^{\otimes n'} ({}_q\langle s'|)^{\otimes m'} \quad (\text{A34})$$

$$= \int_{\mathbb{R}} ds e^{if(s)} e^{ig(s)} \delta(0)^{k+k'-1} (|s\rangle_q)^{\otimes n+n'} ({}_q\langle s|)^{\otimes m+m'} \quad (\text{A35})$$

$$\sim \left[\begin{array}{c} n+n' \text{ } \textcircled{(f+g)(x)} \text{ } m+m' \end{array} \right] \quad (\text{A36})$$

where k and k' denote the number of upward and downward wires between the two spiders, respectively. The colour-flipped counterpart can be shown to be sound in the same way. Note that here we have neglected the infinite scalar term $\delta(0)^{k+k'-1}$.

c. Bialgebra rule

$$\left[\begin{array}{c} \textcircled{a} \text{ } \textcircled{b} \\ \vdots \\ \textcircled{a} \text{ } \textcircled{b} \end{array} \right] \quad (\text{A37})$$

$$= \int_{\mathbb{R}} ds_1 \int_{\mathbb{R}} ds_2 \int_{\mathbb{R}} dt_1 \int_{\mathbb{R}} dt_2 |t_1, t_2\rangle_{p_1 p_2} {}_{pppp}\langle t_1, t_1, t_2, t_2 | s_1, s_2, s_1, s_2 \rangle_{qqqq} {}_{q_1 q_2} \langle s_1, s_2 | \quad (\text{A38})$$

$$\sim \int_{\mathbb{R}} ds_1 \int_{\mathbb{R}} ds_2 \int_{\mathbb{R}} dt_1 \int_{\mathbb{R}} dt_2 e^{-i(s_1 t_1 + s_1 t_2 + s_2 t_1 + s_2 t_2)} |t_1, t_2\rangle_{p_1 p_2} {}_{q_1 q_2} \langle s_1, s_2 | \quad (\text{A39})$$

$$= \int_{\mathbb{R}} ds_1 \int_{\mathbb{R}} ds_2 \int_{\mathbb{R}} dt_1 \int_{\mathbb{R}} dt_2 e^{-i(s_1 + s_2)(t_1 + t_2)} |t_1, t_2\rangle_{p_1 p_2} {}_{q_1 q_2} \langle s_1, s_2 | \quad (\text{A40})$$

$$= \int_{\mathbb{R}} ds_1 \int_{\mathbb{R}} ds_2 \left(\int_{\mathbb{R}} dt_1 e^{-i(s_1 + s_2)t_1} |t_1\rangle_{p_1} \right) \left(\int_{\mathbb{R}} dt_2 e^{-i(s_1 + s_2)t_2} |t_2\rangle_{p_2} \right) {}_{q_1 q_2} \langle s_1, s_2 | \quad (\text{A41})$$

$$\sim \int_{\mathbb{R}} ds_1 \int_{\mathbb{R}} ds_2 |s_1 + s_2\rangle_{q_1} |s_1 + s_2\rangle_{q_2} {}_{q_1 q_2} \langle s_1, s_2 | \quad (\text{A42})$$

$$= \int_{\mathbb{R}} ds \int_{\mathbb{R}} ds_1 \int_{\mathbb{R}} ds_2 \delta(s - s_1 - s_2) |s, s\rangle_{q_1 q_2} {}_{q_1 q_2} \langle s_1, s_2 | \quad (\text{A43})$$

$$= \left(\int_{\mathbb{R}} ds \, |s, s\rangle_{q_1 q_2} {}_q \langle s| \right) \left(\int_{\mathbb{R}} ds_1 \int_{\mathbb{R}} ds_2 \, |s_1 + s_2\rangle_q {}_{q_1 q_2} \langle s_1, s_2| \right) \quad (\text{A44})$$

$$= \left[\left[\begin{array}{c} \text{green circle} \leftarrow \text{red circle} \\ \text{red circle} \leftarrow \text{green circle} \end{array} \right] \right] \quad (\text{A45})$$

d. *Fourier rule*

By definition, the Fourier transform operator $\hat{F}$ can be expanded as

$$\hat{F} = \int_{\mathbb{R}} du \, |u\rangle_p {}_q \langle u| = \int_{\mathbb{R}} du \, |-u\rangle_q {}_p \langle u|. \quad (\text{A46})$$

This operator has the Heisenberg action on the quadrature operators $\hat{q}$ and $\hat{p}$ is as follows:

$$\hat{q} \mapsto \hat{F}^\dagger \hat{q} \hat{F} = \left(\int_{\mathbb{R}} du \, |u\rangle_p {}_q \langle -u| \right) \hat{q} \left(\int_{\mathbb{R}} du' \, |-u'\rangle_q {}_p \langle u'| \right) \quad (\text{A47})$$

$$= \int_{\mathbb{R}} du \int_{\mathbb{R}} du' \, (-u) \delta(u - u') \, |u\rangle_p {}_p \langle u'| \quad (\text{A48})$$

$$= - \int_{\mathbb{R}} du \, u \, |u\rangle_p {}_p \langle u| = -\hat{p}, \quad (\text{A49})$$

$$\hat{p} \mapsto \hat{F}^\dagger \hat{p} \hat{F} = \left(\int_{\mathbb{R}} du \, |u\rangle_q {}_p \langle u| \right) \hat{p} \left(\int_{\mathbb{R}} du' \, |u'\rangle_p {}_q \langle u'| \right) \quad (\text{A50})$$

$$= \int_{\mathbb{R}} du \int_{\mathbb{R}} du' \, (u) \delta(u - u') \, |u\rangle_q {}_q \langle u'| \quad (\text{A51})$$

$$= \int_{\mathbb{R}} du \, u \, |u\rangle_q {}_q \langle u| = \hat{q}. \quad (\text{A52})$$

Thus, one obtains

$$M \left(\left[\left[\leftarrow F \rightarrow \right] \right] \right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (\text{A53})$$

$$= M \left(\left[\left[\leftarrow R(\theta) \rightarrow \right] \right] \right) \Big|_{\theta = -\frac{\pi}{2}} \quad (\text{A54})$$

and the same works for $\hat{F}^\dagger$ and $\hat{F}^2$.

e. *Copy rule*

$$\left[\left[\begin{array}{c} \vdots \\ \text{green circle} \leftarrow \text{red circle} \\ \vdots \end{array} \right] \right] \sim \int_{\mathbb{R}} ds \, e^{if(s)} \, |s \dots s\rangle_{q_1 \dots q_n} {}_q \langle s|_0 \rangle_q \quad (\text{A55})$$

$$= e^{if(s)} \, |0 \dots 0\rangle_{q_1 \dots q_n} \quad (\text{A56})$$

$$\sim |0\rangle_q^{\otimes n} \quad (\text{A57})$$

$$= \left[\left[\begin{array}{c} \text{red circle} \\ \vdots \\ \text{red circle} \end{array} \right] \right] \quad (\text{A58})$$

The same works for its colour-flipped counterpart.

f. *Displacement rule*

$$\left[\left[\begin{array}{c} \vdots \\ \text{green circle} \leftarrow \text{red circle} \, ax \\ \vdots \\ \text{red circle} \leftarrow \text{green circle} \, ax \\ \vdots \end{array} \right] \right] \quad (\text{A59})$$

$$= \int_{\mathbb{R}} ds \, e^{if(s)} \left(|s\rangle_q \right)^{\otimes n} \left({}_q \langle s| e^{-ia\hat{p}} \right)^{\otimes m} \quad (\text{A60})$$

$$= \int_{\mathbb{R}} ds \, e^{if(s)} \left(|s\rangle_q \right)^{\otimes n} \left({}_q \langle s - a| \right)^{\otimes m} \quad (\text{A61})$$

$$= \int_{\mathbb{R}} ds \, e^{if(s+a)} \left(|s + a\rangle_q \right)^{\otimes n} \left({}_q \langle s| \right)^{\otimes m} \quad (\text{A62})$$

$$= \int_{\mathbb{R}} ds \, e^{if(s+a)} \left(e^{-ia\hat{p}} |s\rangle_q \right)^{\otimes n} \left({}_q \langle s| \right)^{\otimes m} \quad (\text{A63})$$

$$= \left[\left[\begin{array}{c} \text{red circle} \, ax \\ \vdots \\ \text{green circle} \leftarrow \text{red circle} \, ax \\ \vdots \\ \text{red circle} \, ax \end{array} \right] \right] \quad (\text{A64})$$

The colour-flipped counterpart holds as well.

g. *Antipode rule*

$$\left[\left[\begin{array}{c} \text{green circle} \\ \text{red circle} \end{array} \right] \right] = \int_{\mathbb{R}} ds \int_{\mathbb{R}} dt \, |s\rangle_{q_{out} \, p} \langle t|_s \rangle_q {}_{p_{in}} \langle t| \quad (\text{A65})$$

$$= \int_{\mathbb{R}} ds \int_{\mathbb{R}} dt \, |-s\rangle_{p_{out} \, p} \langle t|_s \rangle_p {}_{p_{in}} \langle t| \quad (\text{A66})$$

$$= \int_{\mathbb{R}} ds \, |-s\rangle_{p_{out} \, p_{in}} \langle s| \quad (\text{A67})$$

$$= \left[\left[\leftarrow F^2 \rightarrow \right] \right] \quad (\text{A68})$$

h. Squeezing rule

$$\left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \left[\begin{array}{c} Sq(\tau) \\ \vdots \\ Sq(\tau) \end{array} \right] \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \quad (A69)$$

$$= \int_{\mathbb{R}} ds e^{if(s)} \left(|s\rangle_q \right)^{\otimes n} \left({}_q\langle s| \widehat{Sq}(\tau) \right)^{\otimes m} \quad (A70)$$

$$\sim \int_{\mathbb{R}} ds e^{if(s)} \left(|s\rangle_q \right)^{\otimes n} \left({}_q\left\langle \frac{s}{\tau} \right| \right)^{\otimes m} \quad (A71)$$

$$\sim \int_{\mathbb{R}} ds e^{if(\tau s)} \left(|\tau s\rangle_q \right)^{\otimes n} \left({}_q\langle s| \right)^{\otimes m} \quad (A72)$$

$$\sim \int_{\mathbb{R}} ds e^{if(\tau s)} \left(\widehat{Sq}(\tau) |s\rangle_q \right)^{\otimes n} \left({}_q\langle s| \right)^{\otimes m} \quad (A73)$$

$$= \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \left[\begin{array}{c} Sq(\tau) \\ \vdots \\ Sq(\tau) \end{array} \right] \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \quad (A74)$$

Again, the colour-flipped counterpart holds in the same way.

i. Quadratic rule

$$M \left(\left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \left[\begin{array}{c} ax^2 \\ bx^2 \\ cx^2 \end{array} \right] \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \right) \quad (A75)$$

$$= \begin{pmatrix} 1 & 0 \\ 2a & 1 \end{pmatrix} \begin{pmatrix} 1 & 2b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2c & 1 \end{pmatrix} \quad (A76)$$

$$= \begin{pmatrix} 4bc + 1 & 2b \\ 8abc + 2a + 2c & 4ab + 1 \end{pmatrix} \quad (A77)$$

On the other hand,

$$M \left(\left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \left[\begin{array}{c} \frac{bc}{4abc+a+c} x^2 \\ (4abc+a+c)x^2 \\ \frac{ab}{4abc+a+c} x^2 \end{array} \right] \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \right) \quad (A78)$$

$$= \begin{pmatrix} 1 & \frac{2bc}{4abc+a+c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2(4abc+a+c) & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{2bc}{4abc+a+c} \\ 0 & 1 \end{pmatrix} \quad (A79)$$

$$= \begin{pmatrix} 4bc + 1 & 2b \\ 8abc + 2a + 2c & 4ab + 1 \end{pmatrix} \quad (A80)$$

holds. Thus, these two unitary operators are equivalent up to a scalar factor.

j. Inversion rule

$$\left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \left[\begin{array}{c} F \\ \vdots \\ F \end{array} \right] \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \quad (A81)$$

$$= \int_{\mathbb{R}} du e^{if(u)} \left(\hat{F} |u\rangle_q \right)^{\otimes n} \left({}_q\langle u| \hat{F}^\dagger \right)^{\otimes m} \quad (A82)$$

$$= \int_{\mathbb{R}} du e^{if(u)} \left(|u\rangle_p \right)^{\otimes n} \left({}_p\langle u| \right)^{\otimes m} \quad (A83)$$

$$= \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \left[\begin{array}{c} f(x) \\ \vdots \\ f(x) \end{array} \right] \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \quad (A84)$$

Appendix B: Graphical calculus of 1-mode Gaussian gates

1. Associativity of squeezing gates

Theorem 3. For arbitrary nonzero real τ and κ ,

$$\leftarrow Sq(\tau) \leftarrow Sq(\kappa) \leftarrow = \leftarrow Sq(\tau\kappa) \leftarrow \quad (B1)$$

holds.

Proof. Using Theorem. 1, one obtains

$$\leftarrow Sq(\tau) \leftarrow = \leftarrow \left[\begin{array}{c} \frac{\tau(1-\tau)}{4} x^2 \\ -\frac{1}{\tau} x^2 \\ \frac{\tau-1}{4} x^2 \\ x^2 \end{array} \right] \leftarrow \quad (B2)$$

and

$$\leftarrow Sq(\kappa) \leftarrow = \leftarrow \left[\begin{array}{c} -x^2 \\ \frac{\kappa-1}{4\kappa} x^2 \\ \kappa x^2 \\ -\frac{\kappa-1}{4\kappa^2} x^2 \end{array} \right] \leftarrow \quad (B3)$$

Therefore,

$$\leftarrow \boxed{Sq(\tau)} \leftarrow \boxed{Sq(\kappa)} \leftarrow \quad (B4)$$

$$\stackrel{(f)}{=} \leftarrow \left(\frac{\tau(1-\tau)}{4} x^2 \right) \leftarrow \left(-\frac{1}{\tau} x^2 \right) \leftarrow \left(\frac{\tau-1}{4} x^2 \right) \leftarrow \left(\frac{\kappa-1}{4\kappa} x^2 \right) \leftarrow \left(\kappa x^2 \right) \leftarrow \left(-\frac{\kappa-1}{4\kappa^2} x^2 \right) \leftarrow \quad (B5)$$

$$\stackrel{(f)}{=} \leftarrow \left(\frac{\tau(1-\tau)}{4} x^2 \right) \leftarrow \left(-\frac{1}{\tau} x^2 \right) \leftarrow \left(\frac{\kappa\tau-1}{4\kappa} x^2 \right) \leftarrow \left(\kappa x^2 \right) \leftarrow \left(-\frac{\kappa-1}{4\kappa^2} x^2 \right) \leftarrow \quad (B6)$$

$$\stackrel{(q)}{=} \leftarrow \left(\frac{\tau(1-\tau)}{4} x^2 \right) \leftarrow \left(-\frac{1}{\tau} x^2 \right) \leftarrow \left(\frac{1-\kappa}{\tau-1} x^2 \right) \leftarrow \left(\frac{\tau-1}{4\kappa} x^2 \right) \leftarrow \left(\frac{\kappa(\kappa\tau-1)}{\tau-1} x^2 \right) \leftarrow \quad (B7)$$

$$\stackrel{(f)}{=} \leftarrow \left(\frac{\tau(1-\tau)}{4} x^2 \right) \leftarrow \left(\frac{1-\kappa\tau}{\tau(\tau-1)} x^2 \right) \leftarrow \left(\frac{\tau-1}{4\kappa} x^2 \right) \leftarrow \left(\frac{\kappa(\kappa\tau-1)}{\tau-1} x^2 \right) \leftarrow \quad (B8)$$

which is equivalent to

$$\leftarrow \boxed{Sq(\tau\kappa)} \leftarrow \leftarrow \left(\frac{\tau\kappa(1-\tau\kappa)}{4k} x^2 \right) \leftarrow \left(-\frac{k}{\tau\kappa} x^2 \right) \leftarrow \left(\frac{\tau\kappa-1}{4k} x^2 \right) \leftarrow \left(kx^2 \right) \leftarrow \quad (B9)$$

by substituting $k = \frac{\kappa(\kappa\tau-1)}{\tau-1}$ in Theorem. 1. $\square$

2. Associativity of rotation gates

Theorem 4. For arbitrary real θ and ϕ ,

$$\leftarrow \boxed{R(\theta)} \leftarrow \boxed{R(\phi)} \leftarrow = \leftarrow \boxed{R(\theta+\phi)} \leftarrow \quad (B10)$$

can be shown under the identity rule (*id*), the fusion rule (*f*), and the quadratic rule (*q*).

Proof. If either $\theta = 2n\pi$ or $\phi = 2n\pi$ holds, then the rotation is equivalent to identity operation, and thus (B10) is obvious. Likewise, if $\theta = (2n+1)\pi$ for some $n \in \mathbb{Z}$, then

$$\leftarrow \boxed{R((2n+1)\pi)} \leftarrow \boxed{R(\phi)} \leftarrow \quad (B11)$$

$$\stackrel{(58)}{=} \leftarrow \boxed{Sq(-1)} \leftarrow \left(\frac{\tan(\theta/2)}{2} x^2 \right) \leftarrow \left(-\frac{\sin\theta}{2} x^2 \right) \leftarrow \left(\frac{\tan(\theta/2)}{2} x^2 \right) \leftarrow \quad (B12)$$

$$\stackrel{(173)}{=} \leftarrow \left(\frac{1}{\tan(\phi/2)} x^2 \right) \leftarrow \left(-\frac{\tan(\phi/2)}{2} x^2 \right) \leftarrow \left(\frac{1}{\tan(\phi/2)} x^2 \right) \leftarrow \left(-\frac{\tan(\theta/2)}{2} x^2 \right) \leftarrow \left(\frac{\tan(\theta/2)}{2} x^2 \right) \leftarrow \left(-\frac{\sin\phi}{2} x^2 \right) \leftarrow \left(\frac{\tan(\phi/2)}{2} x^2 \right) \leftarrow \quad (B13)$$

$$\stackrel{(f)}{=} \leftarrow \left(\frac{1}{\tan(\phi/2)} x^2 \right) \leftarrow \left(-\frac{\tan(\phi/2)}{2} x^2 \right) \leftarrow \left(\left(\frac{1}{\tan(\phi/2)} - \frac{\sin\phi}{2} \right) x^2 \right) \leftarrow \left(\frac{\tan(\phi/2)}{2} x^2 \right) \leftarrow \quad (B14)$$

$$\stackrel{(q)}{=} \leftarrow \left(\frac{1}{\tan(\phi/2)} x^2 \right) \leftarrow \left(-\frac{1}{2\tan(\phi/2)} x^2 \right) \leftarrow \left(-\frac{\sin\phi}{2} x^2 \right) \leftarrow \left(\frac{1}{2\tan(\phi/2)} x^2 \right) \leftarrow \quad (B15)$$

$$\stackrel{(f)}{=} \leftarrow \left(\frac{1}{2\tan(\phi/2)} x^2 \right) \leftarrow \left(\frac{\sin\phi}{2} x^2 \right) \leftarrow \left(\frac{1}{2\tan(\phi/2)} x^2 \right) \leftarrow \quad (B16)$$

$$\stackrel{(q)}{=} \leftarrow \left(-\frac{1}{2\tan(\theta/2)} x^2 \right) \leftarrow \left(\frac{\sin\theta}{2} x^2 \right) \leftarrow \left(-\frac{1}{2\tan(\theta/2)} x^2 \right) \leftarrow \quad (B17)$$

$$\stackrel{(58)}{=} \leftarrow \boxed{R(\phi+\pi)} \leftarrow \quad (B18)$$

for all ϕ satisfying $\sin\phi \neq 0$, and

$$\leftarrow \boxed{R((2n+1)\pi)} \leftarrow \boxed{R((2n+1)\pi)} \leftarrow \quad (B19)$$

$$= \leftarrow \boxed{Sq(-1)} \leftarrow \boxed{Sq(-1)} \leftarrow \quad (B20)$$

$$\stackrel{(B1)}{=} \leftarrow \boxed{Sq(1)} \leftarrow \quad (B21)$$

$$= \leftarrow \leftarrow \quad (B22)$$

holds (by Theorem. 3). The same argument applies when $\phi = (2n+1)\pi$. Therefore, all we need to confirm is when $\sin\theta \neq 0$ and $\sin\phi \neq 0$.

- M. J. Hartmann, A. Ho, M. Hoffmann, T. Huang, T. S. Humble, S. V. Isakov, E. Jeffrey, Z. Jiang, D. Kafri, K. Kechedzhi, J. Kelly, P. V. Klimov, S. Knysh, A. Korotkov, F. Kostritsa, D. Landhuis, M. Lindmark, E. Lucero, D. Lyakh, S. Mandrà, J. R. McClean, M. McEwen, A. Megrant, X. Mi, K. Michielsen, M. Mohseni, J. Mutus, O. Naaman, M. Neeley, C. Neill, M. Y. Niu, E. Ostby, A. Petukhov, J. C. Platt, C. Quintana, E. G. Rieffel, P. Roushan, N. C. Rubin, D. Sank, K. J. Satzinger, V. Smelyanskiy, K. J. Sung, M. D. Trevithick, A. Vainsencher, B. Villalonga, T. White, Z. J. Yao, P. Yeh, A. Zalcman, H. Neven, and J. M. Martinis, Quantum supremacy using a programmable superconducting processor, *Nature* **574**, 505 (2019).
- [2] H.-S. Zhong, H. Wang, Y.-H. Deng, M.-C. Chen, L.-C. Peng, Y.-H. Luo, J. Qin, D. Wu, X. Ding, Y. Hu, P. Hu, X.-Y. Yang, W.-J. Zhang, H. Li, Y. Li, X. Jiang, L. Gan, G. Yang, L. You, Z. Wang, L. Li, N.-L. Liu, C.-Y. Lu, and J.-W. Pan, Quantum computational advantage using photons, *Science* **370**, 1460 (2020).
- [3] M. Fernández, *Models of Computation: An Introduction to Computability Theory*, edited by I. Mackie, S. Abramsky, C. Hankin, D. Kozen, A. Pitts, H. R. Nielson, S. Skiena, I. Stewart, and D. Zhang, Undergraduate Topics in Computer Science (Springer London, London, 2009).
- [4] J. Miszczak, *High-Level Structures for Quantum Computing*, edited by M. Lanzagorta and J. Uhlmann, Synthesis Lectures on Quantum Computing, Vol. #6 (Morgan & Claypool Publishers, 2012).
- [5] D. Deutsch, Quantum theory, the Church–Turing principle and the universal quantum computer, *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **400**, 97 (1985).
- [6] P. Selinger and B. Valiron, A lambda calculus for quantum computation with classical control, in *Typed Lambda Calculi and Applications*, edited by P. Urzyczyn (Springer Berlin Heidelberg, Berlin, Heidelberg, 2005) pp. 354–368.
- [7] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000).
- [8] C. Heunen and J. Vicary, *Categories for Quantum Theory: An Introduction* (Oxford University Press, 2019).
- [9] B. Coecke and R. Duncan, A graphical calculus for quantum observables (2007).
- [10] B. Coecke and R. Duncan, Interacting Quantum Observables: Categorical Algebra and Diagrammatics, *New Journal of Physics* **13**, 043016 (2011).
- [11] A. Kissinger and J. van de Wetering, Reducing T-count with the ZX-calculus, *Physical Review A* **102**, 022406 (2020).
- [12] N. de Beaudrap, X. Bian, and Q. Wang, Techniques to Reduce $\pi/4$ -Parity-Phase Circuits, Motivated by the ZX Calculus, *Electronic Proceedings in Theoretical Computer Science* **318**, 131 (2020).
- [13] M. Backens, H. Miller-Bakewell, G. de Felice, L. Lobski, and J. van de Wetering, There and back again: A circuit extraction tale, *Quantum* **5**, 421 (2021).
- [14] A. Kissinger and J. van de Wetering, Universal MBQC with generalised parity-phase interactions and Pauli measurements, *Quantum* **3**, 134 (2019).
- [15] D. Horsman, Quantum pictorialism for topological cluster-state computing, *New Journal of Physics* **13**, 095011 (2011).
- [16] A. Toumi, R. Yeung, and G. De Felice, Diagrammatic Differentiation for Quantum Machine Learning, *Electronic Proceedings in Theoretical Computer Science* **343**, 132 (2021).
- [17] S. Lloyd and S. L. Braunstein, Quantum computation over continuous variables, *Physical Review Letters* **82**, 1784 (1999).
- [18] D. Gottesman, A. Kitaev, and J. Preskill, Encoding a qubit in an oscillator, *Physical Review A* **64**, 012310 (2001).
- [19] M. H. Michael, M. Silveri, R. T. Brierley, V. V. Albert, J. Salmilehto, L. Jiang, and S. M. Girvin, New class of quantum error-correcting codes for a bosonic mode, *Physical Review X* **6**, 031006 (2016).
- [20] A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Unconditional Quantum Teleportation, *Science* **282**, 706 (1998).
- [21] W. Asavanant, Y. Shiozawa, S. Yokoyama, B. Charoensombutamon, H. Emura, R. N. Alexander, S. Takeda, J.-i. Yoshikawa, N. C. Menicucci, H. Yonezawa, and A. Furusawa, Generation of time-domain-multiplexed two-dimensional cluster state, *Science* **366**, 373 (2019).
- [22] M. V. Larsen, X. Guo, C. R. Breum, J. S. Neergaard-Nielsen, and U. L. Andersen, Deterministic generation of a two-dimensional cluster state, *Science* **366**, 369 (2019).
- [23] S. Konno, W. Asavanant, F. Hanamura, H. Nagayoshi, K. Fukui, A. Sakaguchi, R. Ide, F. China, M. Yabuno, S. Miki, H. Terai, K. Takase, M. Endo, P. Marek, R. Filip, P. Van Loock, and A. Furusawa, Logical states for fault-tolerant quantum computation with propagating light, *Science* **383**, 289 (2024).
- [24] C. Flühmann, T. L. Nguyen, M. Marinelli, V. Negnevitsky, K. Mehta, and J. P. Home, Encoding a qubit in a trapped-ion mechanical oscillator, *Nature* **566**, 513 (2019).
- [25] L. Podhora, L. Lachman, T. Pham, A. Lešundák, O. Číp, L. Slodička, and R. Filip, Quantum Non-Gaussianity of Multiphonon States of a Single Atom, *Physical Review Letters* **129**, 013602 (2022).
- [26] B. De Neeve, T.-L. Nguyen, T. Behrle, and J. P. Home, Error correction of a logical grid state qubit by dissipative pumping, *Nature Physics* **18**, 296 (2022).
- [27] C. Wang, Y. Y. Gao, P. Reinhold, R. W. Heeres, N. Ofek, K. Chou, C. Axline, M. Reagor, J. Blumoff, K. M. Sliwa, L. Frunzio, S. M. Girvin, L. Jiang, M. Mirrahimi, M. H. Devoret, and R. J. Schoelkopf, A Schrödinger cat living in two boxes, *Science* **352**, 1087 (2016).
- [28] Y. Y. Gao, B. J. Lester, Y. Zhang, C. Wang, S. Rosenblum, L. Frunzio, L. Jiang, S. M. Girvin, and R. J. Schoelkopf, Programmable Interference between Two Microwave Quantum Memories, *Physical Review X* **8**, 021073 (2018).
- [29] P. Campagne-Ibarcq, A. Eickbusch, S. Touzard, E. Zalts-Geller, N. E. Frattini, V. V. Sivak, P. Reinhold, S. Puri, S. Shankar, R. J. Schoelkopf, L. Frunzio, M. Mirrahimi, and M. H. Devoret, Quantum error correction of a qubit encoded in grid states of an oscillator, *Nature* **584**, 368 (2020).
- [30] V. V. Sivak, A. Eickbusch, B. Royer, S. Singh, I. Tsioutsios, S. Ganjam, A. Miano, B. L. Brock, A. Z. Ding, L. Frunzio, S. M. Girvin, R. J. Schoelkopf, and M. H. Devoret, Real-time quantum error correction beyond break-even, *Nature* **616**, 50 (2023).

- [31] R. P. Feynman, Quantum mechanical computers, *Foundations of Physics* **16**, 507 (1986).
- [32] N. C. Menicucci, S. T. Flammia, and P. van Loock, Graphical calculus for Gaussian pure states, *Physical Review A* **83**, 042335 (2011).
- [33] Q. Wang, Qufinite ZX-calculus: A unified framework of qudit ZX-calculi, arXiv:2104.06429 [quant-ph] 10.48550/arXiv.2104.06429 (2021).
- [34] W. Asavanant and A. Furusawa, *Optical Quantum Computers: A Route to Practical Continuous Variable Quantum Information Processing* (AIP Publishing, 2022).
- [35] E. Wigner, On the Quantum Correction For Thermodynamic Equilibrium, *Physical Review* **40**, 749 (1932).
- [36] S. D. Bartlett, B. C. Sanders, S. L. Braunstein, and K. Nemoto, Efficient Classical Simulation of Continuous Variable Quantum Information Processes, *Physical Review Letters* **88**, 097904 (2002).
- [37] P. Selinger, A Survey of Graphical Languages for Monoidal Categories, in *New Structures for Physics*, edited by B. Coecke (Springer Berlin Heidelberg, Berlin, Heidelberg, 2011) pp. 289–355.
- [38] B. Coecke, R. Duncan, A. Kissinger, and Q. Wang, Strong Complementarity and Non-locality in Categorical Quantum Mechanics, in *2012 27th Annual IEEE Symposium on Logic in Computer Science* (IEEE, Dubrovnik, Croatia, 2012) pp. 245–254.
- [39] C. Schröder de Witt and V. Zamdzhiev, The ZX-calculus is incomplete for quantum mechanics, *Electronic Proceedings in Theoretical Computer Science* **172**, 285 (2014).
- [40] K. F. Ng and Q. Wang, A universal completion of the ZX-calculus (2017).
- [41] E. Jeandel, S. Perdrix, and R. Vilmart, A complete axiomatisation of the ZX-Calculus for clifford+T quantum mechanics, in *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '18* (Association for Computing Machinery, New York, NY, USA, 2018) pp. 559–568.
- [42] J. van de Wetering, *ZX-calculus for the working quantum computer scientist* (2020).
- [43] B. Coecke and A. Kissinger, *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning* (Cambridge University Press, Cambridge, 2017).
- [44] A. Kissinger and J. van de Wetering, PyZX: Large Scale Automated Diagrammatic Reasoning, *Electronic Proceedings in Theoretical Computer Science* **318**, 229 (2020).
- [45] R. Duncan and S. Perdrix, Rewriting measurement-based quantum computations with generalised flow, in *Automata, Languages and Programming*, edited by S. Abramsky, C. Gavaille, C. Kirchner, F. Meyer auf der Heide, and P. G. Spirakis (Springer Berlin Heidelberg, Berlin, Heidelberg, 2010) pp. 285–296.
- [46] N. de Beaudrap and D. Horsman, The ZX calculus is a language for surface code lattice surgery, *Quantum* **4**, 218 (2020).
- [47] N. de Beaudrap, A. Kissinger, and K. Meichanetzidis, Tensor Network Rewriting Strategies for Satisfiability and Counting, *Electronic Proceedings in Theoretical Computer Science* **340**, 46 (2021).
- [48] R. I. Booth, T. Carette, and C. Comfort, *Complete equational theories for classical and quantum Gaussian relations* (2024).
- [49] R. I. Booth, T. Carette, and C. Comfort, *Graphical Symplectic Algebra* (2024).
- [50] R. Filip, P. Marek, and U. L. Andersen, Measurement-induced continuous-variable quantum interactions, *Physical Review A* **71**, 042308 (2005).
- [51] K. Miyata, H. Ogawa, P. Marek, R. Filip, H. Yonezawa, J.-i. Yoshikawa, and A. Furusawa, Experimental realization of a dynamic squeezing gate, *Physical Review A* **90**, 060302 (2014).
- [52] S. L. Braunstein and H. J. Kimble, Teleportation of continuous quantum variables, *Physical Review Letters* **80**, 869 (1998).
- [53] P. Marek, R. Filip, H. Ogawa, A. Sakaguchi, S. Takeda, J.-i. Yoshikawa, and A. Furusawa, General implementation of arbitrary nonlinear quadrature phase gates, *Physical Review A* **97**, 022329 (2018).
- [54] K. Miyata, H. Ogawa, P. Marek, R. Filip, H. Yonezawa, J.-i. Yoshikawa, and A. Furusawa, Implementation of a quantum cubic gate by an adaptive non-Gaussian measurement, *Physical Review A* **93**, 022301 (2016).
- [55] S. D. Bartlett and W. J. Munro, Quantum Teleportation of Optical Quantum Gates, *Physical Review Letters* **90**, 117901 (2003).
- [56] Y. Shiozawa, J.-i. Yoshikawa, S. Yokoyama, T. Kaji, K. Makino, T. Serikawa, R. Nakamura, S. Suzuki, S. Yamazaki, W. Asavanant, S. Takeda, P. van Loock, and A. Furusawa, Quantum Nondemolition Gate Operations and Measurements in Real Time on Fluctuating Signals, *Physical Review A* **98**, 052311 (2018).
- [57] A. Sakaguchi, S. Konno, F. Hanamura, W. Asavanant, K. Takase, H. Ogawa, P. Marek, R. Filip, J.-i. Yoshikawa, E. Huntington, H. Yonezawa, and A. Furusawa, Nonlinear feedforward enabling quantum computation, *Nature Communications* **14**, 3817 (2023).
- [58] A. Wintner, The Unboundedness of Quantum-Mechanical Matrices, *Physical Review* **71**, 738 (1947).
- [59] A. Robinson, *Non-Standard Analysis*, Princeton Landmarks in Mathematics and Physics (Princeton University Press, 1974).
- [60] M. Ozawa, Phase Operator Problem and Macroscopic Extension of Quantum Mechanics, *Annals of Physics* **257**, 65 (1997).
- [61] A. Raab, An approach to nonstandard quantum mechanics, *Journal of Mathematical Physics* **45**, 4791 (2004).
- [62] S. Gogioso and F. Genovese, Infinite-dimensional Categorical Quantum Mechanics, *Electronic Proceedings in Theoretical Computer Science* **236**, 51 (2017).
- [63] S. Gogioso and F. Genovese, Quantum Field Theory in Categorical Quantum Mechanics, *Electronic Proceedings in Theoretical Computer Science* **287**, 163 (2019).
- [64] B. W. Walshe, B. Q. Baragiola, R. N. Alexander, and N. C. Menicucci, Continuous-variable gate teleportation and bosonic-code error correction, *Physical Review A* **102**, 062411 (2020).
- [65] T. Carette, E. Jeandel, and S. Perdrix, Completeness of Graphical Languages for Mixed State Quantum Mechanics, *ACM Trans. Quant. Comput.* **2**, 17 (2021).
- [66] B. Coecke and A. Kissinger, Categorical quantum mechanics II: Classical-quantum interaction, *International Journal of Quantum Information* **14**, 1640020 (2016).
- [67] R. Duncan and S. Perdrix, Graph states and the necessity of euler decomposition, in *Mathematical Theory and Computational Practice*, edited by K. Ambos-Spies, B. Löwe, and W. Merkle (Springer Berlin Heidelberg,

- Berlin, Heidelberg, 2009) pp. 167–177.
- [68] R. Duncan, A. Kissinger, S. Perdrix, and J. van de Wetering, Graph-theoretic Simplification of Quantum Circuits with the ZX-calculus, *Quantum* **4**, 279 (2020).
 - [69] A. Kissinger and V. Zamdzhiev, Quantomatic: A Proof Assistant for Diagrammatic Reasoning, in *Automated Deduction - CADE-25*, edited by A. P. Felty and A. Middeldorp (Springer International Publishing, Cham, 2015) pp. 326–336.
 - [70] V. Bergholm, J. Izaac, M. Schuld, C. Gogolin, S. Ahmed, V. Ajith, M. Sohaib Alam, G. Alonso-Linaje, B. Akash-Narayanan, A. Asadi, J. M. Arrazola, U. Azad, S. Banning, C. Blank, T. R. Bromley, B. A. Cordier, J. Ceroni, A. Delgado, O. Di Matteo, A. Dusko, T. Garg, D. Guala, A. Hayes, R. Hill, A. Ijaz, T. Isacsson, D. Ittah, S. Jangiri, P. Jain, E. Jiang, A. Khandelwal, K. Kottmann, R. A. Lang, C. Lee, T. Loke, A. Lowe, K. McKiernan, J. J. Meyer, J. A. Montañez-Barrera, R. Moyard, Z. Niu, L. J. O’Riordan, S. Oud, A. Panigrahi, C.-Y. Park, D. Polatajko, N. Quesada, C. Roberts, N. Sá, I. Schoch, B. Shi, S. Shu, S. Sim, A. Singh, I. Strandberg, J. Soni, A. Száva, S. Thabet, R. A. Vargas-Hernández, T. Vincent, N. Vitucci, M. Weber, D. Wierichs, R. Wiersema, M. Willmann, V. Wong, S. Zhang, and N. Killoran, PennyLane: Automatic differentiation of hybrid quantum-classical computations, *arXiv e-prints*, [arXiv:1811.04968](https://arxiv.org/abs/1811.04968) (2018).