FINITELY GENERATED GROUPS AND HARMONIC FUNCTIONS OF SLOW GROWTH

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Abstract. In this paper, we are mainly concerned with $(\mathbb{G}, \mu)$-harmonic functions that grow at most polynomially, where $\mathbb{G}$ is a finitely generated group with a probability measure μ. In the initial part of the paper, we focus on Lipschitz harmonic functions and how they descend onto finite index subgroups. We discuss the relations between Lipschitz harmonic functions and harmonic functions of linear growth and conclude that for groups of polynomial growth, they coincide. In the latter part of the paper, we specialise to positive harmonic functions and give a characterisation for strong Liouville property in terms of the Green's function. We show that the existence of a non-constant positive harmonic function of polynomial growth guarantees that the group cannot have polynomial growth.

1. Introduction

Based on Colding and Minicozzi's solution to a conjecture of Yau [CM97], Kleiner [Kle10] has proved that for any finitely generated group $\mathbb{G}$ of polynomial growth, the space of Lipschitz harmonic functions on $\mathbb{G}$ is a finite-dimensional vector space. Using this result Kleiner has obtained a non-trivial finite-dimensional representation of $\mathbb{G}$, which in turn leads to a new proof of Gromov's theorem [Gro81]: any finitely generated group of polynomial growth is virtually nilpotent.

In this article, we study the spaces of polynomially growing harmonic functions which are $\mathbb{G}$-invariant vector spaces over $\mathbb{C}$ and denoted as $\mathrm{HF}_k(\mathbb{G}, \mu)$ (see subsection 2.5). We say that the measured group $(\mathbb{G}, \mu)$ is Liouville (equivalently the Poisson boundary is trivial) if all the bounded μ-harmonic functions are constant, i.e., the space of bounded harmonic functions, denoted as $\mathrm{BHF}(\mathbb{G}, \mu)$, is isomorphic to $\mathbb{C}$. In [Fur73], the author shows that non-amenable groups are not Liouville for every non-degenerate measure (Definition 2.2). Conversely, when $\mathbb{G}$ is amenable, then there exists some non-degenerate μ such that $(\mathbb{G}, \mu)$ is Liouville, as shown by Kaimanovich and Vershik [KV83] and Rosenblatt [Ros81]. It is well-known that any non-degenerate random walk on a finitely generated group of polynomial growth has a trivial Poisson boundary, while on the other hand, random walks on groups of intermediate growth with non-trivial Poisson boundary are studied in [Ers04, Theorem 2]. A group $\mathbb{G}$ is said to be *Choquet-Deny* if $(\mathbb{G}, \mu)$ is Liouville for every non-degenerate μ. In [FHTVF19], the authors show that a finitely generated group $\mathbb{G}$ is Choquet-Deny if and only if it has polynomial growth. For simple random walks on a finitely generated group, Gournay [Gou15] investigates the space of bounded harmonic functions and studies the triviality of reduced l^p-cohomology for groups of super-polynomial growth.

In this direction, we also refer to the work of Lyons and Sullivan [LS84] on the Liouville and the strong Liouville properties (i.e., positive harmonic functions are constant) of the covering space. It should be emphasised that it is not known if these properties depend only on the topology of the base manifold, or if the Riemannian metric plays a role. Let $p : M \to N$ be a normal Riemannian covering of a closed manifold, with deck transformation group Γ. The authors [LS84, Theorem 3] show that if Γ is non-amenable, then there

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exist non-constant, bounded harmonic functions on M. About the strong Liouville property, the authors [LS84, Theorem 1] show that if Γ is virtually nilpotent, then any positive harmonic function on M is constant. In [LS84, page 305], the authors conjecture that Γ is of exponential growth if and only if M admits non-constant, positive harmonic functions. This is proved in [BBE94], [BE95], under the assumption that Γ is linear, that is, a closed subgroup of $GL_n(\mathbb{R})$, for some $n \in \mathbb{N}$.

In [Ale02], [MPTY17], the authors show that a harmonic function of polynomial growth on a virtually nilpotent group is essentially a polynomial. There are a few natural ways to define a polynomial, each of which is useful in certain contexts. For a general finitely generated group, a particularly natural approach is to define a polynomial as a function that vanishes after taking a bounded number of derivatives (see Definition 2.7 below).

Another possible way to define a polynomial on a group, as used in [Ale02], is in terms of a certain coordinate system (see Definition 2.9 below), defined using the lower central series of the group. As it turns out, all of these definitions are equivalent due to [MPTY17, Proposition 4.9].

Lately, there has been growing interest in the study of spaces of harmonic functions of polynomial growth on groups and other homogeneous spaces (for instance, one can look at [BDCKY15] and references therein). In [MPTY17], the authors show that for a virtually nilpotent group $\mathbb{G}$ and a finitely supported SAS measure μ on $\mathbb{G}$ (see Definition 2.6), $\dim \mathrm{HF}_k(\mathbb{G}, \mu)$ is bounded, where the bounds depend on the rank of the nilpotent subgroup of finite index. The following is also true:

Proposition 1.1. *[MY16, Proposition 3.4] Let $\mathbb{G}$ be a finitely generated group, μ be a* SAS *measure on $\mathbb{G}$ and $\mathbb{H} \leq \mathbb{G}$ a subgroup of finite index. For any $k \geq 0$, the restriction of any $f \in \mathrm{HF}_k(\mathbb{G}, \mu)$ to $\mathbb{H}$ is $\mu_{\mathbb{H}}$-harmonic and in $\mathrm{HF}_k(\mathbb{H}, \mu_{\mathbb{H}})$ (for an explanation of $\mu_{\mathbb{H}}$, see below). Conversely, any $\tilde{f} \in \mathrm{HF}_k(\mathbb{H}, \mu_{\mathbb{H}})$ is the restriction of a unique $f \in \mathrm{HF}_k(\mathbb{G}, \mu)$. Thus, the restriction map is a linear bijection from $\mathrm{HF}_k(\mathbb{G}, \mu)$ to $\mathrm{HF}_k(\mathbb{H}, \mu_{\mathbb{H}})$.*

We show below the analogous statement for Lipschitz harmonic functions for a given group $\mathbb{G}$ and a SAS measure μ, denoted as $\mathrm{LHF}(\mathbb{G}, \mu)$. For a subgroup $\mathbb{H}$ of $\mathbb{G}$, define the hitting time $\tau_{\mathbb{H}} := \inf\{t \geq 1 : X_t \in \mathbb{H}\}$, where X_t is the random walk on $\mathbb{G}$ generated by μ. We say that $\mathbb{H}$ is a recurrent subgroup of $\mathbb{G}$ if $\tau_{\mathbb{H}} < \infty$ almost surely. It is well known that a subgroup of finite index is always recurrent. Furthermore, the expectation of $\tau_{\mathbb{H}}$ is equal to $[\mathbb{G} : \mathbb{H}]$ (see e.g. [HLT14]). For a recurrent subgroup $\mathbb{H}$ the hitting measure is the probability measure on $\mathbb{H}$ defined by

$$\mu_{\mathbb{H}}(x) := \mathbb{P}_e\left[X_{\tau_{\mathbb{H}}} = x\right],$$

where the subscript "e" is used to indicate that the random walk starts at the origin, that is, $X_0 = e$. It is known that if μ is an adapted smooth measure on $\mathbb{G}$, then the hitting measure $\mu_{\mathbb{H}}$ is also a smooth measure (see [MY16, Lemma 3.2] for a proof). We show the following result:

Theorem 1.2. *Let $\mathbb{G}$ be a finitely generated group with a finite index subgroup $\mathbb{H}$. Given a* SAS *measure μ on $\mathbb{G}$, the restriction map is a linear bijection between the sets $\mathrm{LHF}(\mathbb{G}, \mu)$ and $\mathrm{LHF}(\mathbb{H}, \mu_{\mathbb{H}})$.*

Next, we start investigating assorted properties of slowly growing harmonic functions. Our first result is that on virtually nilpotent groups, the classes LHF and HF_1 coincide.

Theorem 1.3. *Let $\mathbb{G}$ be a finitely generated infinite group of polynomial growth. Then $\mathrm{LHF}(\mathbb{G}, \mu) \cong \mathrm{HF}_1(\mathbb{G}, \mu)$, for any* SAS *measure μ.*

As a brief note to the reader: a direct proof of Theorem 1.3 uses the concept of a coordinate system on a group. An interesting question for future investigation could be the following:

Question 1.4. *If* $\mathrm{LHF}(\mathbb{G},\mu)$ *is finite dimensional, does there exist a finite index subgroup of* $\mathbb{G}$ *on which the restriction of each member of* $\mathrm{LHF}(\mathbb{G},\mu)$ *is a linear polynomial?*

Such a result would be a variant of [MPTY17, Theorem 1.3]. In many practical situations, it becomes important to realise that for some $k \geq 0$, HF_k is finite dimensional. Case in point, a central ingredient in [Kle10] is the fact that the space of Lipschitz harmonic functions is finite dimensional. On the other hand, there has been a lot of work on positive harmonic functions, the Martin boundary and the strong Liouville property. It is a natural question to ask what is the growth rate of positive harmonic functions, when they do exist? For example, an interesting question for future investigation could be to see what is the possible growth rate of the positive harmonic function whose existence is stated in [Pol21] for groups of exponential growth. Here, we show that if some HF_k is finite-dimensional, then it cannot contain a positive harmonic function.

Theorem 1.5. *Let* k *be a positive integer and* μ *be a* SAS *measure. If there exists a non-constant positive* $h \in \mathrm{HF}_k(\mathbb{G},\mu)$ *then*

$$\dim \mathrm{HF}_k(\mathbb{G},\mu) = \infty.$$

The proof of Theorem 1.5 is a modification of the ideas in [Yad, Theorem 3.7.1] in conjunction with [MY16, Proposition 5.4]. One can also wonder about the converse of Theorem 1.5:

Question 1.6. *If* $\dim \mathrm{HF}_k(\mathbb{G},\mu) = \infty$ *for some non-degenerate measure on a finitely generated group* $\mathbb{G}$*, does this imply that there exists* $h \in \mathrm{HF}_k(\mathbb{G},\mu)$ *which is non-constant and positive?*

A partial answer to Question 1.6 is provided in Observation 5.3.

Remark 1.7. *From [Kle10, Theorem 1.3] it follows that the groups appearing in Theorem 1.5 cannot have polynomial growth.*

Specialising to minimal positive harmonic functions (see [Woe00, Definition 24.3]), we have the following result:

Theorem 1.8. *If* $(\mathbb{G},\mu)$ *is strong Liouville and* μ *generates a transient random walk with super-exponential moments, then*

$$l_{\mathbb{G}} = 0, \tag{1}$$

where $l_{\mathbb{G}}$ *is the* Green speed *of the random walk generated by* μ *on* $\mathbb{G}$.

For the notion of superexponential moments, see Definition 2.3 below. Here, the Green speed is defined by the almost sure limit

$$l_{\mathbb{G}} = \lim_{n\to\infty} \frac{d_G(e, X_n)}{n},$$

where

$$d_G(x,y) := \log G(e,e) - \log G(x,y) \tag{2}$$

is the Green's distance. In the particular case where $\mathbb{G}$ is not a finite extension of $\mathbb{Z}$ or $\mathbb{Z}^2$, using [Woe00, Section I.3.B] and [BHM08, Theorem 1], we have that

$$\rho(\mu) = 0,$$

where $\rho(\mu) := \lim \frac{H(\mu)}{n}$ is the *asymptotic entropy* of the random walk on $\mathbb{G}$ generated by μ. Here $H(\mu) = -\sum_{x\in\mathbb{G}} \mu(x) \log \mu(x)$ denotes the usual (Shannon) entropy of the random walk generated by μ.

In [Pol21, Theorem 1.2], it is claimed that in every group $\mathbb{G}$ of exponential growth and every non-degenerate measure μ (assuming the measure to be transient) on $\mathbb{G}$ there is a non-constant positive μ-harmonic function, i.e., the group fails to be strong Liouville. A similar approach is considered by the previously appearing work [AK17] for finite supported random walks. The latter work considers a functional ε (defined in (18)) to prove the following general result: any directed graph, which consists of a finitely supported Markov chain that is invariant under some transitive group of automorphisms of the ambient graph and for which the directed balls grow exponentially, supports a non-constant positive harmonic function.

We show an equivalent condition of the existence of a non-constant positive harmonic function using Δ. We mainly use the gradient and heat kernel (return probability) estimates from [HSC93]. Finally, in the context of positive harmonic functions, we consider a variant of a functional $\epsilon(S)$ defined in [AK17, pp 2] and implicit in several earlier works (eg. see [SY94, Chapter 2, Sections 2, 3]). For a finite $S \subset \mathbb{G}$, we define the quantity,

$$\Delta(S; a, b) := \max_{x\in\partial S} \frac{|G(a, x) - G(b, x)|}{|G(a, x)|}, \tag{3}$$

where we have dropped the subscript μ with a slight abuse of notation. We also introduce the notation $S \nearrow \mathbb{G}$ to denote an exhausting sequence $S_k \subseteq \mathbb{G}$ such that $S_k \subseteq S_{k+1}$ for all $k \in \mathbb{N}$ and $\cup_{k\geq 1} S_k = \mathbb{G}$. We prove the following result:

Theorem 1.9. *Let $\mathbb{G}$ be a finitely generated infinite group. If $\mathbb{G}$ has polynomial growth and μ is a finitely supported, non-degenerate transient measure on $\mathbb{G}$ then for all $a, b \in \mathbb{G}$, we have $\Delta(S; a, b) \to 0$ as $S \nearrow \mathbb{G}$.*

The functional $\Delta(S; a, b)$ is important because of its relation with the strong Liouville property (see Propositions 6.2, 6.3 and 6.4 below). In particular, the proofs of Propositions 6.2 and 6.4 implicitly contain the following:

Corollary 1.10. *If $\mathbb{G}$ is a finitely generated group of exponential growth and μ is a non-degenerate measure on $\mathbb{G}$ having superexponential moments, then $\mathbb{G}$ supports a nonconstant positive μ-harmonic function.*

This is somewhat weaker than the result in [AK17], but a saving grace could be that our measure is not necessarily finitely supported. Our proof here essentially follows the ideas in [Pol21], and we include some of the details for the sake of completeness.

In [Pol21], [AK17], the authors ask the following question:

Question 1.11. *Does the Grigorchuk group (more generally, groups of intermediate growth) support a non-trivial positive harmonic function?*

Using [FHTVF19, Theorem 1], one can show the existence of a non-degenerate measure on groups of intermediate growth that supports a non-constant positive harmonic function. However, for finitely supported measures the answer to the above question is not known to the present authors.

2. PRELIMINARIES

2.1. **Harmonic functions and SAS measures.** Throughout the paper, $\mathbb{G}$ is a finitely generated group and μ is a probability measure (not necessarily finitely supported) on $\mathbb{G}$. Normally, we need μ to satisfy certain extra conditions, otherwise corresponding harmonic functions might not have nice properties. We start with some definitions.

Definition 2.1 (Symmetric measure). *The probability measure μ is called symmetric if $\mu(g) = \mu(g^{-1})$ for all $g \in \mathbb{G}$.*

Definition 2.2 (Non-degenerate/adapted measure). *The probability measure μ is called non-degenerate or adapted if* $\operatorname{supp}(\mu)$ *generates $\mathbb{G}$ as a semi-group.*

Definition 2.3 (Smooth measure). *A measure μ on a group $\mathbb{G}$ is called smooth if the generating function*

$$\Psi(\zeta) = \sum_{x \in \mathbb{G}} \mu(x) e^{\zeta |x|} < \infty \tag{4}$$

for some positive real number ζ. The measure μ is said to have superexponential moments if (4) *holds for all real $\zeta > 0$.*

Clearly, the above definition stipulates a uniform control on all the moments of X, where X is a random variable taking values in $\mathbb{G}$ with law μ. Since the definition looks somewhat unintuitive, we include a few words on the importance of smooth measures. Firstly, we say that a random variable has *exponential tail* if $\mathbb{P}\left(|X| > t\right) < c_1 e^{-c_2 t}$ for positive constants c_j. One can calculate that this guarantees $\mathbb{E}\left(e^{\alpha |X|}\right) < \infty$ for some $\alpha > 0$. In other words, a measure μ on $\mathbb{G}$ is smooth if the length of a μ-random element of $\mathbb{G}$ has an exponential tail.

Definition 2.4 (Harmonic function). *Let $\mathbb{G}$ be a group and μ be a probability measure on $\mathbb{G}$. A function $f : \mathbb{G} \to \mathbb{C}$ is μ-harmonic at $k \in \mathbb{G}$ if*

$$f(k) = \sum_{g \in G} \mu(g) f(kg), \tag{5}$$

and the above sum converges absolutely. f is said to be μ-harmonic on $\mathbb{G}$ if (5) *holds at all $k \in \mathbb{G}$.*

By slight abuse of notation, we shall call a function f harmonic if the corresponding measure μ is tacitly understood, or not critical to the discussion. By way of further notation, we denote by $\mathrm{BHF}(\mathbb{G}, \mu)$ the set of all bounded μ-harmonic functions on $\mathbb{G}$.

Remark 2.5. *(a) $\mathbb{G}$ acts naturally on the set of harmonic functions by $g.f(k) = f(g^{-1}k)$ for all $g, k \in \mathbb{G}$.*
(b) It can proved by induction that if $f : \mathbb{G} \to \mathbb{C}$ is μ-harmonic then it is μ^{*n}-harmonic. By μ^{*n} we mean the convolution of μ with itself n times.

Definition 2.6. *We shall use the notation* SAS *for any measure which is symmetric, adapted and smooth (recall that [MY16] calls such measures courteous).*

2.2. **Polynomials and co-ordinate polynomials.** In this sub-section we define polynomials on a given group due to [Ale02], [Lei02].

Definition 2.7. *Given $f : \mathbb{G} \to \mathbb{C}$ and an element $u \in \mathbb{G}$ we define the left derivative $\partial_u f$ of f with respect to u by $\partial_u f(x) = f(ux) - f(x)$. Analogously, define the right derivative $\partial^u f$ of f with respect to u by $\partial^u f(x) = f(xu^{-1}) - f(x)$. Let $H \subset \mathbb{G}$ be any subset. A function $f : \mathbb{G} \to \mathbb{C}$ is called a polynomial with respect to H if there exists some integer $k \geq 0$ such that*

$$\partial_{u_1} \cdots \partial_{u_{k+1}} f = 0$$

for all $u_1, \cdots, u_{k+1} \in H$. The degree (with respect to H) of a non-zero polynomial f is the smallest such k. When $H = \mathbb{G}$ we simply say that $f : \mathbb{G} \to \mathbb{C}$ is a polynomial. We denote the space of polynomials of degree at most k by $P^k(G)$. For notational convenience we also define $P^k(G) = \{0\}$ for $k < 0$.

Remark 2.8. (1) *Observe that choosing left or right derivatives in Definition 2.7 does not change the set of polynomials (see [Lei02, Corollary 2.13]).*

(2) *It is known that if S_1 and S_2 are two generating sets of $\mathbb{G}$, then a function $f : \mathbb{G} \to \mathbb{C}$ is a polynomial of degree at most k with respect to S_1 if and only if f is a polynomial of degree at most k respect to S_2 (see [Lei02]). Since $\mathbb{G}$ generates itself, it follows that $f : \mathbb{G} \to \mathbb{C}$ is a polynomial of degree atmost k if and only if*

$$\partial_{u_1} \cdots \partial_{u_{k+1}} f = 0$$

for all $u_1, \cdots, u_{k+1} \in S$, where S is a generating set of $\mathbb{G}$.

Let $\mathbb{G}$ be a finitely generated group. One can show (by [MPTY17, Lemma 4.5]) that $\hat{\mathbb{G}}_i/\hat{\mathbb{G}}_{i+1}$ is torsion free abelian, where $\hat{\mathbb{G}}_i = \{x \in \mathbb{G} \mid x^n \in \mathbb{G}_i \text{ for some } n\}$ and $\mathbb{G}_i = [\mathbb{G}, \mathbb{G}_{i-1}]$ (we assume $\mathbb{G}_1 = \mathbb{G}$). Note that we have the decreasing sequence

$$\mathbb{G} = \mathbb{G}_1 \geq \mathbb{G}_2 \geq \cdots \mathbb{G}_i \geq \mathbb{G}_{i+1} \geq \cdots .$$

For each $i \in \mathbb{N}$, let $e_{n_{i-1}+1}, \cdots, e_{n_i}$ be the elements whose images in $\mathbb{G}/\hat{\mathbb{G}}_{i+1}$ form a basis for $\hat{\mathbb{G}}_i/\hat{\mathbb{G}}_{i+1}$. Then for each $k \in \mathbb{N}$, every element x of $\mathbb{G}$ can be represented, modulo $\hat{\mathbb{G}}_{k+1}$, by a unique expression of the form

$$x\hat{\mathbb{G}}_{k+1} = e_1^{x_1} \dots e_{n_k}^{x_{n_k}} \hat{\mathbb{G}}_{k+1} \tag{6}$$

with $x_i \in \mathbb{Z}$. Moreover, the value of each x_i is independent of the choice of k, and so this defines, for each $x \in \mathbb{G}$, a unique (possibly finite) sequence $x_1, x_2, \cdots$.

We call $e_1, e_2, \cdots$ a **coordinate system** on $\mathbb{G}$. For each $x \in \mathbb{G}$, we define the sequence $x_1, x_2, \cdots$ in (6) to be the **coordinates** of x with respect to the coordinate system $e_1, e_2, \cdots$.

Definition 2.9 (Coordinate polynomial on $\mathbb{G}$). *Let $\mathbb{G}$ be a finitely generated group with coordinate system $e_1, e_2, \cdots$. Then a coordinate monomial on $\mathbb{G}$ with respect to $e_1, e_2, \cdots$ is a function $q : \mathbb{G} \to \mathbb{C}$ of the form*

$$q(x) = \lambda x_1^{a_1} \cdots x_r^{a_r},$$

with $\lambda \in \mathbb{C}$, $r \leq n_k$, each a_i a non-negative integer, and $x_1, x_2, \cdots x_r$ the coordinates of x given in (6). A coordinate polynomial on $\mathbb{G}$ with respect to $e_1, e_2, \cdots$ is a finite sum of coordinate monomials.

For each e_i we define $\sigma(i) := \sup\{k \in \mathbb{N} \mid e_i \in \hat{G}_k\}$. We then define the *degree* of the monomial

$$q(x) = \lambda x_1^{a_1} \cdots x_r^{a_r}$$

by $\deg q = \sigma(1)a_1 + \cdots + \sigma(r)a_r$. If $q_1, \cdots q_t$ are monomials then we define the degree of the polynomial $p = q_1 + \cdots + q_t$ by $\deg p = \max_i \deg q_i$.

The above two definitions of polynomial on a group are equivalent due to [Lei02].

Proposition 2.10 ([Lei02]). *Let $\mathbb{G}$ be a finitely generated group with coordinate system $e_1, e_2, \cdots$ and suppose that $f : \mathbb{G} \to \mathbb{C}$. Then $f \in P^k(\mathbb{G})$ if and only if f is a coordinate polynomial of degree at most k.*

2.3. **Growth rate of functions.** We say that a function $a : [0, \infty) \to \mathbb{R}$ is a growth function if it is monotonically increasing, if $a(0) \geq 1$, and if a is weakly sub-multiplicative in the sense that, for all $r, s \geq 0$,

$$a(r+s) \leq C_a a(r) a(s).$$

Besides the constant function 1, the function $(r+1)^\alpha$ and $e^{\alpha r}$ with $\alpha > 0$ are the most important growth functions and give rise to the concepts of polynomial and exponential growth. Another interesting class are the functions e^{cr^α} with $c > 0$ and $0 < \alpha < 1$, which are between polynomial and exponential growth. We say that a growth function a is of sub-exponential growth if

$$\lim_{r\to\infty} \frac{1}{r} \ln a(r) = 0.$$

The above functions $(r+1)^\alpha$ with $\alpha > 0$ and e^{cr^α} with $c > 0$ and $0 < \alpha < 1$ are examples of sub-exponential growth functions.

2.4. **Lipschitz harmonic functions.** For a function $f : \mathbb{G} \to \mathbb{C}$ and a symmetric generating set S of $\mathbb{G}$, define the gradient $\nabla f : \mathbb{G} \to \mathbb{C}^{|S|}$ by $(\nabla f(x))_s = \partial^s f(x)$. f is said to be Lipschitz if the semi-norm

$$\|\nabla_S f\| := \sup_{s\in S} \sup_{x\in\mathbb{G}} |\partial^s f(x)|$$

is finite. One can check the following fact:

Lemma 2.11. *Let $\mathbb{G}$ be a finitely generated infinite group. If S_1, S_2 are two generating sets of $\mathbb{G}$, there exists $C > 0$ such that*

$$\|\nabla_{S_1} f\|_\infty \leq C.\|\nabla_{S_2} f\|_\infty \tag{7}$$

Hence, the definition of Lipschitz functions does not depend on the choice of the generating set. The subspace of Lipschitz functions which are also μ-harmonic functions is of interest to us, and we denote this space by $\mathrm{LHF}(\mathbb{G}, \mu)$.

2.5. **Harmonic functions of polynomial growth.** Let $t \geq 0$ be a real number. Given a symmetric generating set S and a function $f : \mathbb{G} \to \mathbb{C}$, define the t-th degree semi-norm by

$$\|f\|_{S,t} := \limsup_{r\to\infty} r^{-t}. \max_{|x|\leq r} |f(x)|.$$

It can be checked that $\|f\|_{S,t}$ and $\|f\|_{T,t}$ generate equivalent semi-norms for different finite symmetric generating sets S and T, so will drop the corresponding subscript and simply write $\|f\|_t$ henceforth with minor abuse of notation. Let

$$\mathrm{HF}_t(\mathbb{G}, \mu) = \{f \in \mathbb{G} \to \mathbb{C} \mid f \text{ is } \mu\text{-harmonic}, \|f\|_t < \infty\},$$

Lemma 2.12. *The following are immediate:*

(1) $\mathrm{HF}_t(\mathbb{G}, \mu)$ *is a $\mathbb{G}$-invariant vector space*
(2) $\|x.f\|_t = \|f\|_t$ *for all $x \in \mathbb{G}$, $f \in \mathrm{HF}_t(\mathbb{G}, \mu)$.*

The proof is straightforward, and we omit it. $\mathrm{HF}_t(\mathbb{G}, \mu)$ denotes the space of μ-harmonic functions of polynomial growth of degree at most t.

Remark 2.13. *Let $f \in \mathrm{HF}_t(\mathbb{G}, \mu)$ such that $\|f\|_t = 0$, then it is easy to check that $\|f\|_{t'} = 0$ for every $t' > t$. Also, it is easily checked that*

$$\mathrm{HF}_{t_1} \subseteq \mathrm{HF}_{t_2} \iff t_1 \leq t_2.$$

Lastly, the following chain of inclusions is clear:

$$\mathbb{C} \subseteq \mathrm{BHF}(\mathbb{G}, \mu) \subseteq \mathrm{LHF}(\mathbb{G}, \mu) \subseteq \mathrm{HF}_1(\mathbb{G}, \mu) \subseteq \mathrm{HF}_2(\mathbb{G}, \mu) \subseteq \cdots.$$

For a group of polynomial growth along with a SAS measure, the above spaces can be studied using spaces of polynomials. One can explicitly compute the dimensions of such spaces due to the following result.

Theorem 2.14 (Theorem 1.6, [MPTY17]). *Let $\mathbb{G}$ be a finitely generated group with finite index nilpotent subgroup N, and let μ be a* SAS *measure on $\mathbb{G}$. Then* $\dim \mathrm{HF}_k(\mathbb{G}, \mu) = \dim P^k(N) - \dim P^{k-2}(N)$.

3. RESTRICTIONS TO FINITE INDEX SUBGROUPS

3.1. Proof of Theorem 1.2.

Proof. We first show that a Lipschitz harmonic function on $\mathbb{H}$ can be extended to one on $\mathbb{G}$. We consider without loss of generality that $S_{\mathbb{G}}$ is a symmetric generating set of $\mathbb{G}$ which is an extension of a symmetric generating set $S_{\mathbb{H}}$ of $\mathbb{H}$. For given $\tilde{f} \in \mathrm{LHF}(\mathbb{H}, \mu_{\mathbb{H}})$ and $x \in \mathbb{G}$, we define $f(x) = \mathbb{E}_x(\tilde{f}(X_\tau))$, where the subscript "$x$" denotes that $X_0 = x$. This is a well-known method for extending a harmonic function from a recurrent subgroup (for example, see [MY16, Proposition 3.4]). Let K be the Lipschitz constant for $\tilde{f}$. First, note that constant functions are harmonic and therefore whenever it is defined on $\mathbb{H}$, it is uniquely extended to the harmonic function with the same constant value on $\mathbb{G}$. By rescaling if necessary, we may assume that $\tilde{f}(e) = 0$. Let τ_1, τ_2 denote the (random) return times to $\mathbb{H}$ of the random walks starting from x and xs^{-1} respectively. Then, for a large enough $R > 0$, we have that

$$\begin{aligned}
f(x) - f(xs^{-1}) &= \sum_{h \in \mathbb{H}} \tilde{f}(h)\mathbb{P}_x(X_{\tau_1} = h) - \sum_{g \in \mathbb{H}} \tilde{f}(g)\mathbb{P}_{xs^{-1}}(X_{\tau_2} = g) \\
&= \sum_{h \in \mathbb{H}} \tilde{f}(h)\left(\mathbb{P}_x(X_{\tau_1} = h) - \mathbb{P}_{xs^{-1}}(X_{\tau_2} = h)\right) \\
&\le C_1 + \sum_{h \in B_{\mathbb{G}}(x,R) \cap \mathbb{H}} \tilde{f}(h)\left(\mathbb{P}_x(X_{\tau_1} = h) - \mathbb{P}_{xs^{-1}}(X_{\tau_2} = h)\right) \\
&= C_1 + \sum_{h \in B_{\mathbb{G}}(x,R) \cap \mathbb{H}} (\tilde{f}(h) - \tilde{f}(e))\left(\mathbb{P}_x(X_{\tau_1} = h) - \mathbb{P}_{xs^{-1}}(X_{\tau_2} = h)\right), \text{ since } \tilde{f}(e) = 0 \\
&\le C_1 + K_1 \sum_{h \in B_{\mathbb{G}}(x,R) \cap \mathbb{H}} \mathrm{dist}_{\mathbb{H}}(h, e)\left(\mathbb{P}_x(X_{\tau_1} = h) - \mathbb{P}_{xs^{-1}}(X_{\tau_2} = h)\right), \text{ since } \tilde{f} \in \mathrm{LHF}(\mathbb{H}, \mu_{\mathbb{H}}) \\
&\le C_1 + K_1K_2(R + \mathrm{dist}_{\mathbb{G}}(x, e)) \sum_{h \in B_{\mathbb{G}}(x,R) \cap \mathbb{H}} \left(\mathbb{P}_x(X_{\tau_1} = h) - \mathbb{P}_{xs^{-1}}(X_{\tau_2} = h)\right) \\
&\le C_1 + C_2 + K(R + \mathrm{dist}_{\mathbb{G}}(x, e)) \sum_{h \in \mathbb{H}} \left(\mathbb{P}_x(X_{\tau_1} = h) - \mathbb{P}_{xs^{-1}}(X_{\tau_2} = h)\right) \\
&= C_1 + C_2,
\end{aligned}$$

where K_1 is a constant depending on the Lipschitz constant of f, and $K_2 = K_2\left([\mathbb{G} : \mathbb{H}]\right)$ is a constant depending on the index of the subgroup $\mathbb{H}$. In the third and second last steps, we have used the fact that the random walk generated by μ has exponential tail. The last step follows as

$$\sum_{h \in \mathbb{H}} \left(\mathbb{P}_x(X_\tau = h) - \mathbb{P}_{xs^{-1}}(X_\tau = h)\right) = 0,$$

since $\sum_{h \in \mathbb{H}} \mathbb{P}_x(X_\tau = h)$ is the unique extension of the constant harmonic function 1, which is itself identically 1 for all $x \in \mathbb{G}$.

To go the other way: to show that this is indeed a unique extension, it suffices to show that if $f|_{\mathbb{H}} \equiv 0$ and $f \in \mathrm{HF}_k(\mathbb{G}, \mu)$ then $f \equiv 0$ on all of $\mathbb{G}$. This is well-known, and we skip the details. □

Remark 3.1. *We take the space to remark that if* $\mathrm{LHF}(\mathbb{G}, \mu)$ *is finite dimensional for all* SAS *measures, then* $\dim \mathrm{BHF}(\mathbb{G}, \mu) < \infty$ *for all* μ SAS, *which means that* $(\mathbb{G}, \mu)$ *is Liouville.*

If there exists some SAS *measure* μ *on* $\mathbb{G}$ *such that* $\dim \mathrm{BHF}(\mathbb{G}, \mu) = \infty$, *then by Tits alternative [Tit72], either* $\mathbb{G}$ *is virtually solvable or* $\mathbb{G}$ *has exponential growth. If* $\mathbb{G}$ *is virtually solvable there exists* $\mathbb{H}$, *a finite index solvable sub-group of* $\mathbb{G}$. *From [Mil68],* $\mathbb{H}$ *is either polycyclic or* $\mathbb{H}$ *has exponential growth. If* $\mathbb{H}$ *is polycyclic by [Wol68, Theorem 4.3],* $\mathbb{H}$ *is virtually nilpotent. As* $[\mathbb{G} : \mathbb{H}] < \infty$, *it follows that* $\mathbb{G}$ *is virtually nilpotent and hence has polynomial growth. This is a contradiction as polynomial growth groups are Choquet-Deny [FHTVF19] and hence* $\dim \mathrm{BHF}(\mathbb{G}, \mu) < \infty$ *for all* SAS *measure* μ. *So* $\mathbb{G}$ *must be of exponential growth.*

4. HARMONIC FUNCTIONS OF POLYNOMIAL GROWTH AND LIPSCHITZ HARMONIC FUNCTIONS

Let $\mathbb{G}$ be a finitely generated infinite group and μ be a SAS measure on $\mathbb{G}$. Then the following inclusion is well-known:

$$\mathbb{C} \subseteq \mathrm{BHF}(\mathbb{G}, \mu) \subseteq \mathrm{LHF}(\mathbb{G}, \mu) \subseteq \mathrm{HF}_1(\mathbb{G}, \mu) \subseteq \mathrm{HF}_2(\mathbb{G}, \mu) \ldots.$$

This raises the following questions:

Question 4.1. *Given a group* $\mathbb{G}$ *with a SAS measure* μ,

(1) *when do the spaces* $\mathrm{LHF}(\mathbb{G}, \mu)$ *and* $\mathrm{HF}_1(\mathbb{G}, \mu)$ *coincide?*
(2) *does there exist* $f \in \mathrm{HF}_1(\mathbb{G}, \mu)$ *such that* f *is not Lipschitz, i.e., when is the inclusion* $\mathrm{LHF}(\mathbb{G}, \mu) \subseteq \mathrm{HF}_1(\mathbb{G}, \mu)$ *strict?*

The construction of harmonic functions via polynomials in [MPTY17] might be helpful here. In fact, one might be able to prove that for virtually abelian groups, linear polynomials are Lipschitz.

Here is a general lemma which was suggested to us by Ariel Yadin (personal communication):

Lemma 4.2. *Let* $\mathbb{G}$ *be a finitely generated group and* μ *be a SAS measure. If* $\dim \mathrm{HF}_1(\mathbb{G}, \mu) < \infty$, *then* $\mathrm{HF}_1(\mathbb{G}, \mu) = \mathrm{LHF}(\mathbb{G}, \mu)$.

Proof. By [MPTY17, Theorem 1.3] there is a finite index subgroup of $\mathbb{G}$, say $\mathbb{H}$, on which the restriction of all functions in $\mathrm{HF}_1(\mathbb{G})$ on $\mathbb{H}$ are polynomials of degree 1. Then,

$$\partial^{h_1} \partial^{h_2} f(x) = 0 \text{ for all } h_1, h_2, x \in \mathbb{H}$$

implies

$$f(xh_1^{-1}h_2^{-1}) - f(xh_1^{-1}) - f(xh_2^{-1}) - f(x) = 0 \text{ for all } h_1, h_2, x \in \mathbb{H}.$$

Putting $x = e$ we get

$$f(h_1^{-1}h_2^{-1}) = f(h_1^{-1}) + f(h_2^{-1}) + f(e) \text{ for all } h_1, h_2, x \in \mathbb{H},$$

i.e., all functions in $\mathrm{HF}_1(\mathbb{G}, \mu)$ restrict to $\mathbb{H}$ as "complex-valued homomorphisms up to some constant". Thus, all functions in $\mathrm{HF}_1(\mathbb{G}, \mu)$ restrict to a Lipschitz function on $\mathbb{H}$. Indeed, if $S_{\mathbb{H}}$ is a symmetric generating set of $\mathbb{H}$ and $s \in S_{\mathbb{H}}$ we get,

$$\begin{aligned} |\partial^s f(x)| &= |f(xs^{-1}) - f(x)| = |f(x) + f(s^{-1}) + K - f(x)| \\ &\leq |f(s^{-1})| + K \leq M + K, \text{ where } M = \sup_{s \in S_{\mathbb{H}}} |f(s)| \text{ and } K = |f(e)|. \end{aligned}$$

Therefore,

$$\sup_{s\in S_{\mathbb{H}}}\sup_{x\in\mathbb{H}}|\partial^s f(x)| \leq M+K.$$

Hence $f|_{\mathbb{H}} \in \mathrm{LHF}(\mathbb{H},\mu_{\mathbb{H}})$. By Theorem 1.2, we get that the original function is Lipschitz. □

The above lemma implies that for virtually nilpotent groups $\mathrm{HF}_1(\mathbb{G},\mu) = \mathrm{LHF}(\mathbb{G},\mu)$, if one uses the additional fact $\dim \mathrm{HF}_1 < \infty$ for finitely supported SAS measures on virtually nilpotent groups (Theorem 2.14). Here we give a direct demonstration of the former fact bypassing the application of the latter fact, because we believe that the proof is instructive, at any rate, to the present authors!

Remark 4.3. *Note that if $\mathbb{G}$ is finitely generated with finite index nilpotent subgroup N, then by Theorem 2.14, we have*

$$\mathrm{HF}_1(\mathbb{G},\mu) \cong \mathrm{HF}_1(N,\mu_N) = \ker(\Delta : P^1(N) \to P^{-1}(N)) = P^1(N).$$

Example 1 (For free abelian groups comparison of LHF and HF_1 for a given SAS measure)**.** Let $\mathbb{G} = \mathbb{Z}^d$ and μ be a given SAS measure on $\mathbb{G}$. Then one can show that $\dim \mathrm{HF}_1(\mathbb{G},\mu) = \dim P^1(\mathbb{G})$ (one can precisely calculate these dimensions; see [MPTY17]).

As $\mathbb{G}$ is abelian, $\mathbb{G} = \mathbb{G}_1$ and $\mathbb{G}_2 = \{0\}$. Also $\hat{\mathbb{G}}_1 = \mathbb{G}_1 = \mathbb{G}$ and $\hat{\mathbb{G}}_2 = \mathbb{G}_2 = \{0\}$. Thus one can choose the coordinates $\{e_i\}_{i=1}^d$, which is same as the standard basis for $\mathbb{Z}^d$, where d denotes the rank of $\mathbb{G}$. Hence if $x = \sum_{i=1}^d x_i e_i$, where $x_i \in \mathbb{Z}$ and one can define a degree 1 polynomial $f : \mathbb{G} \to \mathbb{C}$ by

$$f(x) = \sum_{i=1}^{d} c_i x_i,$$

where $c_i \in \mathbb{C}$.

We choose $S = \{e_i, -e_i\}_{i=1}^d$, then S is a symmetric generating set of $\mathbb{Z}^d$. Now for $e_i \in S$ and

$$\partial^{e_i} f(x) = f(x-e_i) - f(x) = -c_i.$$

So we have,

$$\|\nabla_S f\|_\infty = \sup_{s\in S}\sup_{x\in G}|\partial^s f(x)| < \infty.$$

Also by Lemma 2.11, if S_1, S_2 are two generating sets there exists $C > 0$ such that

$$\|\nabla_{S_1} f\|_\infty \leq C.\|\nabla_{S_2} f\|_\infty. \tag{8}$$

Hence, every coordinate polynomial and hence any polynomial (by Proposition 2.10) of degree 1 are Lipschitz so by Remark 4.3, we have $\mathrm{HF}_1(\mathbb{G},\mu) = P^1(\mathbb{G}) = \mathrm{LHF}(\mathbb{G},\mu)$ for any SAS measure μ.

Example 2. Let $\mathbb{G}$ be a finitely generated abelian group and $f \in P^1(\mathbb{G})$. By the structure theorem of abelian groups, $\mathbb{G} \cong \mathbb{Z}^d \oplus T$, where T is a finitely generated torsion group. In this case, we get $\mathbb{G}_1 = \mathbb{G}$, $\mathbb{G}_2 = 0$ and $\hat{\mathbb{G}}_2 = T$. Here $\hat{\mathbb{G}}_1/\hat{\mathbb{G}}_2 \cong \mathbb{Z}^d$, which is generated by the image of the standard basis $\{e_i\}_{i=1}^d$ of $\mathbb{Z}^d \subseteq \mathbb{G}$. Note that if $x \in T$, then

$$x\hat{\mathbb{G}}_2 = \hat{\mathbb{G}}_2 = e_1^0\hat{\mathbb{G}}_2.$$

Therefore, $f(x)$ must be zero for all $x \in T$. Next, consider an arbitrary element $y \in \mathbb{G}$, then $y = y_1 + y_2$, where $y_1 \in \mathbb{Z}^d$ and $y_2 \in T$. Then

$$y_1\hat{\mathbb{G}}_2 = e_1^{x_1}\cdots e_d^{x_d}\hat{\mathbb{G}}_2$$

and

$$y_2\hat{\mathbb{G}}_2 = e_1^0\hat{\mathbb{G}}_2.$$

Thus we get, $y = \sum_{i=1}^{d} x_i e_i + 0$ modulo $\hat{\mathbb{G}}_2$ and $f(y) = \sum_{i=1}^{d} c_i x_i$ where, $\{c_i \mid i = 1, \cdots, d\} \subseteq \mathbb{C}$. Using similar argument as free abelian case we get, $\partial^{e_i} f(y) = -c_i$ and if $t \in T$ then also $\partial^t f(y) = f(y-t) - f(y) = 0$ and $\mathrm{LHF}(\mathbb{G}, \mu) \cong \mathrm{HF}_1(\mathbb{G}, \mu)$, for any SAS measure μ.

Question 4.4. *Is the above essentially the only situation where elements appearing in the coordinate system do not generate the group? In other words, suppose $\mathbb{G}$ is a nilpotent group and S is a symmetric generating set does it follow that the coordinate systems are precisely given by those elements in S which have infinite order?*

Example 3 (For virtually abelian)**.** Let $\mathbb{G}$ be virtually abelian and N' be a finite index abelian subgroup of $\mathbb{G}$. Then there is a free abelian subgroup of finite index say N. By Theorem 2.14, $\dim \mathrm{HF}_1(\mathbb{G}, \mu) = \dim P^1(N)$. In particular, they are same space as (since $[G : N] < \infty$)

$$\mathrm{HF}_1(\mathbb{G}, \mu) \cong \mathrm{HF}_1(N, \mu_N) = \ker(\Delta : P^1(N) \to P^{-1}(N)).$$

By Theorem 1.2 we get, $\mathrm{LHF}(\mathbb{G}, \mu) = \mathrm{LHF}(N, \mu_N)$. Thus, as in the abelian case, we conclude that $\mathrm{LHF}(\mathbb{G}, \mu) \cong \mathrm{LHF}(N, \mu_N) = P^1(N) = \mathrm{HF}_1(N, \mu) \cong \mathrm{HF}_1(\mathbb{G}, \mu)$.

Example 4. Let H be the (3-dimensional) Heisenberg group over $\mathbb{Z}$, defined by

$$\mathbb{H} := H_3(\mathbb{Z}) = \{\langle x, y, z\rangle \mid x, y, z \in \mathbb{Z}\}$$

where

$$\langle x, y, z\rangle = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

with the operation of matrix multiplication. The group $\mathbb{H}$ is virtually nilpotent, finitely generated by $S' = \{e_1 = \langle 1, 0, 0\rangle, e_2 = \langle 0, 1, 0\rangle\}$, for example. Let $S = S' \cup S'^{-1}$, then S is a symmetric generating set of $\mathbb{H}$. One can easily check that the commutator subgroup $\mathbb{H}_2 = [\mathbb{H}, \mathbb{H}]$ of $\mathbb{H} = \{\langle 0, 0, \alpha\rangle \mid \alpha \in \mathbb{Z}\}$. Therefore, the images of e_1, e_2 in $\mathbb{H}/\hat{\mathbb{H}}_2$, form a basis of $\mathbb{H}/\hat{\mathbb{H}}_2$. To show this, take an element of $\mathbb{H}$ given by

$$\langle x, y, z\rangle = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\langle x, y, z\rangle = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

In (9) the first matrix on the right-hand side can be written as

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so (9) becomes

$$\langle x, y, z\rangle = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mod \hat{\mathbb{H}}_2. \tag{10}$$

Thus by commutativity of $\mathbb{H}/\hat{\mathbb{H}}_2$ we get,

$$\langle x, y, z\rangle = e_1^x e_2^y \mod \hat{\mathbb{H}}_2$$

where, $e_1^x = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $e_2^y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$.

Going modulo $\hat{\mathbb{H}}_2$, we get $\mathbb{H}/\hat{\mathbb{H}}_2$ is generated by the images of e_1 and e_2 under the projection $\mathbb{H} \to \mathbb{H}/\hat{\mathbb{H}}_2$. Thus this is a coordinate system that generates the group $\mathbb{H}$. We will check that the above Theorem 1.3 holds for this group.

From (10), we get that any degree 1 coordinate polynomial, f, on $\mathbb{H}$ can be written as, $f(X = \langle x, y, z\rangle) = c_1 x + c_2 y$ for $c_1, c_2 \in \mathbb{C}$. Now

$$\partial^{e_1} f(X) = f(Xe_1^{-1}) - f(X) = -c_1$$

and

$$\partial^{e_2} f(X) = f(Xe_2^{-1}) - f(X) = -c_2.$$

Hence, $\|\nabla_S(f)\|_\infty < \infty$, which implies that $P^1(\mathbb{H}) = \mathrm{HF}_1(\mathbb{H}, \mu) \subseteq \mathrm{LHF}(\mathbb{H}, \mu)$ for every SAS measure μ. So we conclude that $\mathrm{LHF}(\mathbb{H}, \mu) \cong \mathrm{HF}_1(\mathbb{H}, \mu)$ for every SAS measure μ.

Proof of Theorem 1.3. Since $\mathbb{G}$ has polynomial growth, by Gromov's theorem $\mathbb{G}$ is virtually nilpotent. Let us assume that $\mathbb{G}$ is nilpotent. We first show that $P^1(\mathbb{G}) = \mathrm{HF}_1(\mathbb{G}, \mu)$. By [MPTY17, Proposition 2.7], $\|f\|_1 < \infty$ for all $f \in P^1(\mathbb{G})$. For any $f \in P^1(\mathbb{G})$ and $k \in \mathbb{G}$, we have

$$\begin{aligned} f(k) - \sum_{g\in\mathbb{G}} f(kg)\mu(g) &= \sum_{g\in\mathbb{G}} [f(kg) - f(k)]\mu(g) \\ &= \frac{1}{2}\sum_{g\in\mathbb{G}} [f(kg) + f(kg^{-1}) - 2f(k)] \\ &= \frac{1}{2}\sum_{g\in\mathbb{G}} \partial^g\partial^g f(kg^{-1}) = 0 \end{aligned}$$

Hence, $P^1(\mathbb{G}) \subseteq \mathrm{HF}_1(\mathbb{G})$. Since $\mathbb{G}$ has polynomial growth, $\dim \mathrm{HF}_1(\mathbb{G}, \mu) < \infty$ by Kleiner's Theorem. Theorem 2.14 tells us that $P^1(\mathbb{G}) = \mathrm{HF}_1(\mathbb{G}, \mu)$.

Let k be the nilpotency class of $\mathbb{G}$, i.e. we have the following decreasing sequence

$$\mathbb{G} = \mathbb{G}_1 \geq \mathbb{G}_2 \geq \cdots \geq \mathbb{G}_k \geq \mathbb{G}_{k+1} = \{e\}.$$

For each $i \in \mathbb{N}$, let $e_{n_{i-1}+1}, \cdots, e_{n_i}$ be the elements whose images in $\mathbb{G}/\hat{\mathbb{G}}_{ii+1}$ form a basis for $\hat{\mathbb{G}}_i/\hat{\mathbb{G}}_{i+1}, i = 1, 2, \ldots k$. We want to show that for any SAS measure μ on $\mathbb{G}$, $\mathrm{LHF}(\mathbb{G}, \mu) = \mathrm{HF}_1(\mathbb{G}, \mu)$. For this let $x \in \mathbb{G}$, then there exist $\{x_i\}_{i=1}^{n_k} \subseteq \mathbb{Z}$ such that

$$x\hat{\mathbb{G}}_{k+1} = e_1^{x_1} \cdots e_{n_k}^{x_{n_k}} \hat{\mathbb{G}}_{k+1}.$$

Therefore, if $f \in P^1(\mathbb{G})$, there exist $\{c_i\}_{i=1}^{n_k} \subseteq \mathbb{C}$ so that $f(x) = \sum_{i=1}^{n_k} c_i x_i$. Now let $s \in S$. Then we have $s^{-1}\hat{\mathbb{G}}_{k+1} = e_1^{s_1} e_2^{s_2} \ldots e_{n_k}^{s_{n_k}} \hat{\mathbb{G}}_{k+1}$ for a unique sequence $(s_1, s_2, \ldots, s_{n_k})$. Also, $xs^{-1}\hat{\mathbb{G}}_{k+1} = e_1^{x_1+s_1} e_2^{x_2+s_2} \ldots e_{n_k}^{x_{n_k}+s_{n_k}} \hat{\mathbb{G}}_{k+1}$. Hence,

$$\partial^s f(x) = f(xs^{-1}) - f(x) = \sum_{i=1}^{n_k} c_i(x_i + s_i) - \sum_{i=1}^{n_k} c_i x_i = \sum_{i=1}^{n_k} c_i s_i$$

.

Therefore, $\|\nabla_S f\|_\infty = \sup_{s\in S} \sup_{x\in\mathbb{G}} |\partial^s f(x)| \leq \sup_{s\in S} \sum_{i=1}^{n_k} |c_i s_i| < \infty$. Hence, f is Lipchitz and $\mathrm{HF}_1(\mathbb{G}, \mu) = \mathrm{LHF}(\mathbb{G}, \mu)$.

For the general case, let N be a finite index nilpotent subgroup of $\mathbb{G}$. Then, we have

$$\mathrm{HF}_1(\mathbb{G}, \mu) \cong \mathrm{HF}_1(N, \mu_N) = \mathrm{LHF}(N, \mu_N) \cong \mathrm{LHF}(\mathbb{G}, \mu)$$

. The first isomorphism is due to the finite index fact, the second one follows from what we proved above and the last one follows from Theorem 1.2. Hence the result is proved.

□

Question 4.5. *What about the converse? Does a group* $\mathbb{G}$ *with* $\mathrm{LHF}(\mathbb{G},\mu) \cong \mathrm{HF}_1(\mathbb{G},\mu)$ *for every* SAS *measure* μ *always have polynomial growth?*

5. POSITIVE HARMONIC FUNCTIONS IN HF_k

We first get familiar with some notations that will be used in this section. For the rest of this section, let k denote a positive integer. If $S = \{s_1, s_2, ..., s_n\}$ is a finite symmetric generating set for $\mathbb{G}$, and $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi index, let us define

$$D^\alpha f = \partial_{s_1}^{\alpha_1} ... \partial_{s_n}^{\alpha_n} f$$

where $\partial_s^m f = \underbrace{\partial_s \dots \partial_s}_{m\text{-times}} f$. Since $\mathrm{HF}_k(\mathbb{G},\mu)$ is a $\mathbb{G}$-invariant vector space, it is easy to see that $\partial_s f \in \mathrm{HF}_k(\mathbb{G},\mu)$ for all $s \in \mathbb{G}$, $f \in \mathrm{HF}_k(\mathbb{G},\mu)$. It follows by induction that $D^\alpha f \in \mathrm{HF}_k(\mathbb{G},\mu)$ for any multi-index α whenever $f \in \mathrm{HF}_k(\mathbb{G},\mu)$.

We now prove a preliminary lemma before proving Theorem 1.5.

Lemma 5.1. *Let* $\mathbb{G}$ *be a finitely generated group and* μ *be a* SAS *measure on* $\mathbb{G}$*. Assume that* $\dim \mathrm{HF}_k(\mathbb{G},\mu) < \infty$*. Then there exists a finite index normal subgroup* $\mathbb{H}$ *of* $\mathbb{G}$ *with the following assertions:*

(1) *There exists a finite subset* B *of* $\mathbb{H}$ *with the property that* $f|_B = 0$ *implies* $f \equiv 0$ *for all* $f \in \mathrm{HF}_k(\mathbb{H},\mu_\mathbb{H})$*. Furthermore, the seminorm* $\|.\|_B$ *defined by,*

$$\|f\|_B := \max_{x\in B} \sum_{|\alpha|=k-1} |D^\alpha f(x) - D^\alpha f(e)|$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{|S|})$*, is a multi-index has kernel* $\mathrm{HF}_{k-1}(\mathbb{H},\mu_\mathbb{H})$*. Hence,* $\|.\|_B$ *is a norm on* $V = \mathrm{HF}_k(\mathbb{H},\mu_\mathbb{H})/\mathrm{HF}_{k-1}(\mathbb{H},\mu_\mathbb{H})$*.*

(2) *The converse of [MPTY17, Proposition 2.7] holds true, i.e., any function* f *with* $\|f\|_k < \infty$ *satisfies* $\|f\|_{k+1} = 0$*.*

(3) *The semi-norm* $\|.\|_k$ *defined on* $\mathrm{HF}_k(\mathbb{H},\mu_\mathbb{H})$ *has kernel* $\mathrm{HF}_{k-1}(\mathbb{H},\mu_\mathbb{H})$*.*

Proof. Using [MPTY17, Theorem 1.3], there exists a finite index normal subgroup $\mathbb{H}$ of $\mathbb{G}$ such that the restriction to $\mathbb{H}$ of any $f \in \mathrm{HF}_k(\mathbb{G},\mu)$ is a polynomial of degree k. Let $\mu_\mathbb{H}$ be the induced hitting measure on $\mathbb{H}$. As $\dim \mathrm{HF}_k(\mathbb{G},\mu) < \infty$, using [MY16, Propostion 3.4] we get, $\mathrm{HF}_k(\mathbb{H},\mu_\mathbb{H})$ is finite dimensional.

(1) We first show the existence of a finite set $B \subset \mathbb{H}$, with the property that $f|_B = 0$ implies $f \equiv 0$ for all $f \in \mathrm{HF}_k(\mathbb{H},\mu_\mathbb{H})$. We use the fact that $\dim \mathrm{HF}_k(\mathbb{H},\mu_\mathbb{H}) < \infty$. Let $B_1 = B(e,1)$, where the ball is considered with respect to the word length metric on $\mathbb{H}$. If for any function $f \in \mathrm{HF}_k(\mathbb{H},\mu_\mathbb{H})$, $f|_{B_1} = 0$ implies $f \equiv 0$ then we are done. So let $f_1 \in \mathrm{HF}_k(\mathbb{H},\mu_\mathbb{H})$ be such that $f_1 = 0$ on B_1 and $f_1 \neq 0$ at $x_1 \notin B_1$. Next, we consider a larger ball $B_2 \supset B_1$ such that $x_1 \in B_2$. Again, if for any $f \in \mathrm{HF}_k(\mathbb{H},\mu)$, $f|_{B_2} = 0$ implies $f \equiv 0$ then we are done. Otherwise we can continue this process and if it goes on indefinitely, we get an increasing sequence of finite sets $\{B_n\}_{n=1}^\infty$ and functions $\{f_n\}_{n=1}^\infty$ in $\mathrm{HF}_k(\mathbb{H},\mu)$ with the properties that $f_i|_{B_i} = 0$, $f_i(x_i) \neq 0$ and $f_i(x_j) = 0$ for $j < i$. It is easy to verify that $\{f_n\}_{n=1}^\infty$ is a linearly independent set in $\mathrm{HF}_k(\mathbb{H},\mu_\mathbb{H})$. As $\dim \mathrm{HF}_k(\mathbb{H},\mu_\mathbb{H}) < \infty$, we arrive at a contradiction. Thus, we have a finite set $B \subset \mathbb{H}$ such that any $f \in \mathrm{HF}_k(\mathbb{H},\mu_\mathbb{H})$ vanishing on B should be the zero function on $\mathbb{H}$. Now, the kernel of $\|.\|_B$ is the set all complex valued functions on $\mathbb{H}$ that are harmonic and are polynomials of degree at most $k-1$ on $\mathbb{H}$ (see Remark 2.8). Hence the kernel of this semi-norm is $\mathrm{HF}_{k-1}(\mathbb{H},\mu_\mathbb{H})$ by [MPTY17, Theorem 1.3]. Therefore, the claim follows.

(2) We have that

$$
\begin{aligned}
\|f\|_{k+1} &= \limsup_{r\to\infty} r^{-(k+1)} \sup_{|x|\le r} |f(x)| = \lim_{r\to\infty} \sup_{s\ge r} s^{-(k+1)} \sup_{|x|\le s} |f(x)| \\
&= \lim_{r\to\infty} \sup_{s\ge r} s^{-1} s^{-k} \sup_{|x|\le s} |f(x)| \le \lim_{r\to\infty} \left(\sup_{s\ge r} \frac{1}{s}\right) \left(\sup_{s\ge r} s^{-k} \sup_{|x|\le s} |f(x)|\right) \\
&= 0.\|f\|_k = 0.
\end{aligned}
$$

(3) This follows from [MPTY17, Theorem 1.3, Proposition 2.7] and (2) above.

$\square$

Now we start proving Theorem 1.5.

Proof of Theorem 1.5. Let k be the minimum for which there exists a non-constant positive harmonic function $h \in \mathrm{HF}_k(\mathbb{G}, \mu)$ such that $\|h\|_k > 0$. Assume to the contrary that $\dim \mathrm{HF}_k(\mathbb{G}, \mu) < \infty$.

Let $\tilde{h} = h|_{\mathbb{H}}$, whence by [MPTY17, Theorem 1.3] it is a k-degree polynomial on $\mathbb{H}$, where $\mathbb{H}$ is a finite index normal subgroup of $\mathbb{G}$. Let $V = \mathrm{HF}_k(\mathbb{H}, \mu_{\mathbb{H}})/\mathrm{HF}_{k-1}(\mathbb{H}, \mu_{\mathbb{H}})$. Then using Lemma 5.1, $\|.\|_k$ induces a norm on V. From Lemma 5.1, we get $\|.\|_B$ is a norm on V for some finite set $B \subset \mathbb{H}$. As V is finite-dimensional, we observe that these norms are equivalent.

Then, considering a random walk X_t in $\mathbb{H}$, we see that $\tilde{h}(X_t)$ is a positive martingale, implying that it converges almost surely by the martingale convergence theorem. Thus, for any fixed m, we have $|D^\alpha \tilde{h}(X_{t+m}) - D^\alpha \tilde{h}(X_t)| \to 0$ almost surely. The latter claim follows by induction, and we skip the details.

Fix $x \in \mathbb{H}$ and let m be such that $\mathbb{P}_e[X_m = x] = \delta > 0$. To see that such a lower bound exists, take any path $e = x_0, x_1, \dots, x_l = x$. Then, we can see that $\mathbb{P}[X_l = x] \ge \prod_{j=0}^{l-1} p(x_j, x_{j+1})$. Since $p(x, y) = \mu_{\mathbb{H}}(x^{-1}y)$, and $\mu_{\mathbb{H}}$ is adapted, such a bound follows. Using the Markov property of the random walk, we see that $\mathbb{P}[X_{t+m} = X_t x \mid X_t] = \delta$, independently of t. Almost sure convergence implies convergence in probability, so for any $\epsilon > 0$,

$$
\mathbb{P}\left[|D^\alpha \tilde{h}(X_t x) - D^\alpha \tilde{h}(X_t)| > \epsilon\right] \le \delta^{-1} \mathbb{P}[|D^\alpha \tilde{h}(X_{t+m}) - D^\alpha \tilde{h}(X_t)| > \epsilon] \to 0.
$$

So $|D^\alpha \tilde{h}(X_t x) - D^\alpha \tilde{h}(X_t)| \to 0$ in probability, for any $x \in \mathbb{H}$. Since B is a finite set this implies that $\max_{x\in B} \sum_{|\alpha|=k-1} |D^\alpha \tilde{h}(X_t x) - D^\alpha \tilde{h}(X_t)| \to 0$ in probability. Now we use the fact from Lemma 2.12, that $\|x.\tilde{h}\|_k = \|\tilde{h}\|_k$. So for all t we have

$$
\|\tilde{h}\|_k = \|X_t^{-1} \tilde{h}\|_k \le K.\|X_t^{-1}\tilde{h}\|_B = K.\max_{x\in B} \sum_{|\alpha|=k-1} |D^\alpha \tilde{h}(X_t x) - D^\alpha \tilde{h}(X_t)|.
$$

Since the latter quantity converges to 0 in probability, we have $\|\tilde{h}\|_k = 0$ and $\tilde{h} \in \mathrm{HF}_{k-1}(\mathbb{H}, \mu_{\mathbb{H}})$. Via the correspondence, we have that $h \in \mathrm{HF}_{k-1}(\mathbb{G}, \mu)$, which is a contradiction. $\square$

Question 5.2. *What about the converse of Theorem 1.5? In other words, does* HF_k *being infinite dimensional imply that there exists a non-constant positive harmonic function in* HF_k*?*

We end this section with the following

Observation 5.3. *Let* $\mathbb{G}$ *be a finitely generated group and* μ *be a non-degenerate measure on* $\mathbb{G}$*. If* $\mathrm{HF}_k(\mathbb{G}, \mu)$ *is infinite-dimensional, then there exists some non-degenerate measure* ν *which supports a non-constant positive harmonic function.*

Proof. By work in [Kle10, Theorem 1.4], we know that our hypothesis forces $\mathbb{G}$ to have super-polynomial growth. From [FHTVF19, Theorem 1], the claim follows. $\square$

6. Positive harmonic functions, asymptotic entropy and Green's function estimates

Throughout this section, we let μ be a non-degenerate transient measure on a finitely generated group $\mathbb{G}$.

6.1. Green speed and asymptotic entropy.

First, we establish a relation between strong Liouville property and zero asymptotic entropy.

Recall that the *Green's function* $G(x, y)$ on $\mathbb{G} \times \mathbb{G}$ is defined by

$$G(x,y) = \sum_{k=1}^{\infty} \mu^{(k)}(x^{-1}y) = \sum_{k=1}^{\infty} p_k(x,y), \tag{11}$$

where $\mu^{(k)}$ is the k-fold convolution of μ with itself, and let $K_y(x,z) := \frac{G(x,z)}{G(y,z)}$ be a Martin kernel.

Let $x \in \mathbb{G}$ be given by $x = s_1 s_2 \dots s_n$, where $s_j \in S$, and denote $x_i := s_1 \dots s_i$. Then one can compute

$$G(e,x) = \prod_{i=0}^{n-1} K_{s_{i+1}}(e, x_i^{-1}x) G(e,e). \tag{12}$$

In particular, this implies that

$$\log G(x,e) - \log G(e,e) = \sum_{i=0}^{n-1} \log K_{s_{i+1}}(e, x_i^{-1}x).$$

Now, using the definition of Green's distance from (2), we have that

$$\begin{aligned} -\frac{d_G(e,x)}{n} &= \frac{1}{n}\sum_{i=0}^{n-1} \log K_{s_{i+1}}(e, x_i^{-1}x) \\ &= \frac{1}{n}\sum_{i=0}^{n_0-1} \log K_{s_{i+1}}(e, x_i^{-1}x) + \frac{1}{n}\sum_{i=n_0}^{n-1} \log K_{s_{i+1}}(e, x_i^{-1}x), \text{ where } n_0 \ll n. \end{aligned} \tag{13}$$

For large n, the left-hand side in the above equation tends to the asymptotic entropy $\rho(\mu)$. The first summation on the right-hand side goes to 0 anyway. Now, for large n, the term $K_{s_{i+1}}(e, x_i^{-1}x)$ is arbitrarily close to $K_{s_{i+1}}(e, \xi)$, where ξ lies on the Martin boundary of $\mathbb{G}$.

Now, we establish a result linking Green's distance d_G to the strong Liouville property of $\mathbb{G}$. Let S denote a set of generators of $\mathbb{G}$, and let the sequence $x_n := s_1 s_2 \dots s_n$ denote a diverging sequence going out to infinity, where $s_j \in S$. Then, up to a subsequence, $x_n \to \xi$, a point on the Martin boundary, as the latter is metrisable, and hence sequentially compact (for such details and more, we refer to [KL22, Section 3.2] and references therein). Consequently, the Martin kernels $K_e(z, x_j^{-1})$ converge to a positive harmonic function h, using the fact that μ has superexponential moments (see [GGPY21, Lemma 7.1]). If $(\mathbb{G}, \mu)$ is strong Liouville, then $h(e) = 1$ forces h to be identically 1 everywhere. Also, $K_y(x,\xi) = \lim_{z\to\xi} \dfrac{G(x,z)}{G(y,z)}$ satisfies that $K_y(y) = 1$, which forces $K_y(x,\xi)$ to be identically equal to 1.

Proof of Theorem 1.8. Let $x = s_1 \dots s_n$, and denote $x_0 := e$, $x_j := s_1 \dots s_j, 1 \leq j \leq n$. Then, one can calculate that

$$G(x,y) = \prod_{j=1}^{n} K_e\left(s_j, x_{j-1}^{-1} y\right) G(e,y) \tag{14}$$

for all $y \in \mathbb{G}$. In particular, for $y = e$, one gets that

$$G(x,e) = \prod_{j=1}^{n} K_e\left(s_j, x_{j-1}^{-1}\right) G(e,e). \tag{15}$$

This implies that

$$e^{-\frac{d_G(e,x)}{n}} = \left(\prod_{j=n_0+1}^{n} K_e\left(s_j, x_{j-1}^{-1}\right)\right)^{1/n} \left(\prod_{j=1}^{n_0} K_e\left(s_j, x_{j-1}^{-1}\right)\right)^{1/n} \tag{16}$$

Since $K_e(., z) \to 1$ as z goes out to infinity, we see that given any $\delta > 0$, there exists some n_0 large enough depending on δ such that $K_e\left(., x_j^{-1}\right) \geq (1-\delta)$. This in turn implies that

$$e^{-\frac{d_G(e,x)}{n}} \geq (1-\delta)^{\frac{n-n_0}{n}} c^{n_0/n}. \tag{17}$$

Taking $n \to \infty$ in the above estimate in the regime $n_0 = o(n)$, and observing that δ is arbitrary, we get the claim. □

6.2. **Positive harmonic functions and a variant of a functional due to Amir-Kozma.** Let $S \subseteq \mathbb{G}$ and μ be a non-degenerate probability measure on $\mathbb{G}$. We say that $x \in \partial S$, if $x \notin S$ and there exists $y \in S$ such that x and y are adjacent. Observe that the definition of harmonicity at x as in (5) can be rewritten as

$$h(x) = \sum_{y \in \mathbb{G}} \mu(x^{-1}y) h(y).$$

The following discussion is intended to be a variant of the harmonic measure construction in [AK17], [SY94, Chapter 2]. Recall that Amir-Kozma define the following functional:

$$\epsilon(S; a, b) := \max_{x \in \partial S} \frac{|\mu_S(a,x) - \mu_S(b,x)|}{|\mu_S(a,x)|}, \tag{18}$$

where $\mu_S(p, x)$ denotes the probability that the μ-random walk starting at p exits S at $x \in \partial S$. Here, we work with the functional $\Delta(S; a, b)$ defined in (3) because we believe it is analytically more amenable. This raises the following

Question 6.1. *Are the functionals $\epsilon(S)$ and $\Delta(S)$ comparable as $S \nearrow \mathbb{G}$?*

We suspect that this question should have an affirmative answer, but we have not checked the details explicitly.

Recall that [Pol21, Proposition 2.1] essentially proves the following proposition; we remark that [Pol21] does not seem to have the hypothesis of transience of μ or of superexponential moments, without which we are not sure whether the argument works (see the discussion below Lemma 1 of [AK17]).

Proposition 6.2. *Let $(\mathbb{G}, \mu)$ be strong Liouville and μ have superexponential moments. Then, for all $a, b \in \mathbb{G}$ and all $S \nearrow \mathbb{G}$, we have that $\Delta(S; a, b) \to 0$.*

Proof. Suppose there exist $a, b \in \mathbb{G}$ and an exhausting sequence $(S_n)_{n \in \mathbb{N}}$ such that $\Delta(S_n; a, b) \not\to 0$. Choose $x_n \in \partial S_n$ such that

$$\frac{|G(a, x_n) - G(b, x_n)|}{G(a, x_n)} \geq \varepsilon.$$

For $v \in \mathbb{G}$, define $\psi_n(v) := \frac{G(v,x_n)}{G(a,x_n)}(= K_a(v,x_n))$. Since each ψ_n is a Martin kernel, as $x_n \to \infty$ (up to a subsequence), using [GGPY21, Lemma 7.1], we get, $\psi_n(v) \to \psi(v)$, a positive harmonic function. Also, $\psi(a) = \lim_{n\to\infty} \psi_n(a) = 1$, $|\psi_n(b) - 1| = \left|\frac{G(b,x_n)}{G(a,x_n)} - 1\right| \geq \varepsilon$. This leads to a nonconstant positive harmonic function on $\mathbb{G}$. □

We now prove the following converse:

Proposition 6.3. *(G, μ) has the strong Liouville property if for all $a, b \in \mathbb{G}$, $\Delta(S; a, b) \to 0$ as $S \nearrow \mathbb{G}$.*

Proof. Let ξ be a point on the Martin boundary of $\mathbb{G}$ and $(y_n)_{n-1}^{\infty}$ be a representing sequence of ξ. Suppose that $h(x) = \lim_{y\to\xi} K_e(x,y) = \lim_{y\to\xi} \dfrac{G(x,y)}{G(e,y)}$. We first show that h is constant. Let $S = (S_n)_{n\in\mathbb{N}}$ be the exhausting sequence of $\mathbb{G}$ given by $S_n = \{x \in \mathbb{G} \mid |x| < |y_n|\}$. Choose any $a, b \in \mathbb{G}$. By our hypothesis, for every $\varepsilon > 0$ there exists a large enough $m \in \mathbb{N}$ such that

$$\begin{aligned}
\Delta(S_m; e, a) &< \varepsilon, \\
\Delta(S_m; e, b) &< \varepsilon, \text{ and} \\
|h(a) - h(b)| &\leq |K_e(a, y_m) - K_e(b, y_m)| + \varepsilon
\end{aligned}$$

Since $y_m \in \partial S_m$, we have

$$\begin{aligned}
|h(a) - h(b)| &\leq \left|\frac{G(a,y_m)}{G(e,y_m)} - \frac{G(b,y_m)}{G(e,y_m)}\right| + \varepsilon \\
&\leq \left|\frac{G(a,y_m)}{G(e,y_m)} - 1\right| + \left|\frac{G(b,y_m)}{G(e,y_m)} - 1\right| + \varepsilon \\
&\leq \Delta(S_m; e, a) + \Delta(S_m; e, b) + \varepsilon \leq 2\varepsilon.
\end{aligned}$$

Arbitraryness of ε, a and b implies that h is constant. Now, via the Poisson-Martin representation theorem [Woe00, Theorem 24.7], the proof is immediate. □

We note that if $\mathbb{G}$ has polynomial growth of degree $D > 2$, and μ is finitely supported, then for all a, b, we have that $\Delta(S; a, b) \to 0$ as $S \nearrow \mathbb{G}$. This is a straightforward application of Green's function bounds for transient random walks, $|G(x,y)| \sim \frac{1}{d(x,y)^{D-2}}$, which is classical. Below, we prove a stronger result:

Proof of Theorem 1.9. Let $q(a,k), q(b,k) \in \mathbb{N}$ denote the least positive integers j such that $p_j(a, x_k)$ and $p_j(b, x_k)$ are positive respectively. If x_k is far away, it is clear that $q(a,k) \sim q(b,k)$. Due to the support properties of convolutions, namely, $\operatorname{supp}(\mu*\mu) \subset \operatorname{supp}\mu \operatorname{supp}\mu$, we see that (up to constants), $q(a, x_k) \sim q(b, x_k) \sim d(a, x_k) \sim d(b, x_k) =: d$. In the ensuing calculations, we indulge in minor abuse of notation by not writing all the constants explicitly. Let G be of polynomial growth of degree D. Now, using [HSC93, (14), (15)], we calculate that

$$\frac{|G(a,x_k) - G(b,x_k)|}{G(a,x_k)} \lesssim \frac{\sum_{n=d}^{\infty} n^{-\frac{D+1}{2}} e^{-c_1 d^2/n}}{\sum_{n=d}^{\infty} n^{-\frac{D}{2}} e^{-c_2 d^2/n}},$$

where $c_1 < c_2$. We now see that the sum

$$\sum_{n=d}^{\infty} n^{-m} e^{-cd^2/n}$$

can be approximated by the integral

$$\int_d^\infty t^{-m} e^{-cd^2/t}\, dt.$$

Writing $cd^2/t = r$, we see that the last integral can be written as

$$(cd^2)^{1-m} \int_0^{cd} e^{-r} r^{-m-2}\, dr = (cd^2)^{1-m} \gamma(-m-1, cd),$$

where $\gamma(s,x)$ is the usual lower incomplete Gamma function. Finally, we see that

$$\frac{|G(a,x_k) - G(b,x_k)|}{G(a,x_k)} \lesssim \frac{1}{d} \frac{\gamma(-\frac{D+1}{2} - 1, c_1 d)}{\gamma(-\frac{D}{2} - 1, c_2 d)}.$$

Letting $d \nearrow \infty$, we have our claim. □

We remark that it is natural to wonder what happens to $\Delta(S; a, b)$ when $\mathbb{G}$ has exponential growth. We have that,

Proposition 6.4. *If $\mathbb{G}$ has exponential growth, then there exists $a, b \in \mathbb{G}$ such that $\Delta(S; a, b) \not\to 0$.*

Proof. Recall that if $\mathbb{G}$ has exponential growth, any symmetric and non-degenerate measure μ on $\mathbb{G}$ generates a transient random walk. Now, if the conclusion were to be false, then Proposition 6.3 would imply that $(\mathbb{G}, \mu)$ is strong Liouville.

From [BHM08, Proposition 3.1], it is known that if $\mathbb{G}$ has superpolynomial growth, then it has exponential growth with respect to the Green metric. Furthermore, one has that

$$\limsup_{r \to \infty} \frac{1}{r} |B_g(r)| \leq 1. \tag{19}$$

From (17), we see that if $r_n = -(n - n_0)\log(1-\delta) - n_0 \log c$, then $B(n) \setminus B(n_0) \subseteq B_g(r_n)$. Then,

$$\limsup_n \frac{1}{n} |B(n)| = \limsup_n \frac{1}{n} |B(n) \setminus B(n_0)| \leq \limsup_n \frac{1}{n} |B_g(r_n)| \leq -\log(1-\delta),$$

which is a contradiction. □

Propositions 6.2 and 6.4 together contain the ideas for Corollary 1.10. The above is essentially the argument in [Pol21, Theorem 1.2], which we could check only under the additional assumption of the measure μ having superexponential moments. It would be interesting to investigate a possible converse of the last fact.

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