THE NUMBER OF CLIQUES IN HYPERGRAPHS WITH FORBIDDEN SUBGRAPHS

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Abstract. We study the maximum number of r-vertex cliques in $(r-1)$-uniform hypergraphs not containing complete r-partite hypergraphs $K_r^{(r-1)}(a_1,\dots,a_r)$. By using the hypergraph removal lemma, we show that this maximum is $o(n^{r-1/(a_1\cdots a_{r-1})})$. This immediately implies the corresponding results of Mubayi and Mukherjee and of Balogh, Jiang, and Luo for graphs. We also provide a lower bound by using hypergraph Turán numbers.

1. Introduction

Given integers $r\ge 2$ and $n>0$ and two r-uniform hypergraphs T and F, let $\mathrm{ex}(n,T,F)$ denote the maximum number of copies of T in any F-free (i.e., not containing F as a subgraph) r-uniform hypergraph on n vertices. The case when T is an edge (i.e., $T=K_2$) is the Turán problem $\mathrm{ex}(n,F)$. The parameter $\mathrm{ex}(n,T,F)$ has been studied for different choices of graphs T and F by many authors (for example, see [1, 2, 4, 7, 9, 10, 14, 17]).

Given an r-uniform hypergraph F with vertex set $V(F)=\{v_1,\dots,v_\ell\}$, let $F(a_1,\dots,a_\ell)$ denote a blowup of F, i.e., the hypergraph obtained from F by replacing each vertex v_i by a set V_i of size a_i, and every edge $\{v_{i_1},\dots,v_{i_r}\}$ by a complete r-partite graph on the vertex sets $V_{i_1},\dots,V_{i_r}$. If $a_1=\cdots=a_\ell=a$, then we denote $F(a_1,\dots,a_\ell)$ by $F(a)$. We denote by $K_r^{(r)}(a_1,\dots,a_r)$ the complete r-partite r-uniform hypergraph with $a_1,\dots,a_r$ vertices in its parts. In this note, we consider the parameter $\mathrm{ex}(n,T,F)$, when T is an r-uniform hypergraph, and F is a blowup of T.

This problem is related to the following classical result of Erdős [5] on the Turán number $\mathrm{ex}(n,K_r^{(r)}(a_1,\dots,a_r))$. It states that given integers $r\ge 2$ and $1\le a_1\le\cdots\le a_r$,

$$\mathrm{ex}(n,K_r^{(r)}(a_1,\dots,a_r))=O(n^{r-1/(a_1\cdots a_{r-1})}). \tag{1}$$

Given $2\le i<r$ and an r-uniform hypergraph F, the *s-uniform shadow* $\partial^{(s)}F$ of F is an s-uniform hypergraph on $V(F)$ whose edge set consists all s-subsets $A\subseteq V(F)$ such that $A\subseteq B$ for some edge $B\in F$. We observe the following simple fact (and will prove it in Section 2).

Fact 1. *Given $r\ge 3$ and an r-uniform hypergraph F, we have*

$$\mathrm{ex}(n,K_r,\partial^{(2)}F)\le\cdots\le\mathrm{ex}(n,K_r^{(r-1)},\partial^{(r-1)}F)\le\mathrm{ex}(n,F).$$

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Fact 1 and (1) together imply that, given positive integers $r \geq 3$ and $a_1 \leq \cdots \leq a_r$,

$$\mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a_1, \ldots, a_r)) \leq \mathrm{ex}(n, K_r^{(r)}(a_1, \ldots, a_r)) = O(n^{r-1/(a_1 \cdots a_{r-1})}). \quad (2)$$

In [17], it was shown that the upper bound in (2) can be improved in the case where $r = 3$ and $a_1 = 1$. We extend their result in the following theorem.

Theorem 2. *Given positive integers $r \geq 3$ and $a_1 \leq \cdots \leq a_r$, we have,*

$$\mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a_1, \ldots, a_r)) = o(n^{r-1/(a_1 \cdots a_{r-1})}). \quad (3)$$

Further, given integers $a \geq 1$, $\ell \geq r$, and any $(r-1)$-uniform hypergraph F on ℓ vertices,

$$\mathrm{ex}(n, F, F(a)) = o\left(n^{\ell - \frac{1}{a^{\ell-1}}}\right). \quad (4)$$

By Fact 1, Theorem 2 also implies the following recent result in [2] about the number of copies K_r in graphs without certain complete r-partite subgraph.

Corollary 3. *Given integers $r \geq 3$ and $1 \leq a_1 \leq \cdots \leq a_r$,*

$$\mathrm{ex}(n, K_r, K_r(a_1, \ldots, a_r)) = o(n^{r-1/(a_1 \cdots a_{r-1})}).$$

The following lower bound complements Theorem 2 (and will be proved in Section 3).

Proposition 4. *Given integers $1 \leq a_1 \leq \cdots \leq a_r$,*

$$\mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a_1, \ldots, a_r)) = \Omega\left(n \cdot \mathrm{ex}\left(n,\ K_{r-1}^{(r-1)}(a_1, \ldots, a_{r-1})\right)\right). \quad (5)$$

Observe that if $a_1 = \cdots = a_{r-1} = 1$ and $a_r \geq 2$, then the right hand side of (5) is zero and hence the lower bound is trivial. In this case, a construction in [8] implies the following lower bound. Let $r_r(n)$ denotes the size of the largest subset of $[n]$ that does not contain an arithmetic progression of length r.

Proposition 5. *For every $r \geq 3$,*

$$\mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(1, \ldots, 1, 2)) \geq n^{r-2} r_r(n).$$

For the proof of Proposition 5, see Section 3.

2. Proof of Fact 1 and Theorem 2

In this section, we will prove Fact 1 and Theorem 2.

Proof of Fact 1. It suffices to show that, for every $2 \leq s \leq r-1$,

$$\mathrm{ex}(n, K_r^{(s)}, \partial^{(s)} F) \leq \mathrm{ex}(n, K_r^{(s+1)}, \partial^{(s+1)} F),$$

(trivially $\partial^{(r)} F = F$). Indeed, let G be an $\partial^{(s)} F$-free s-graph on $[n]$ with $\mathrm{ex}(n, K_r^{(s)}, \partial^{(s)} F)$ copies of $K_r^{(s)}$. Let H be the $(s+1)$-graph on $[n]$ whose edges are $(s+1)$-sets that span a copy of $K_{s+1}^{(s)}$ in G. We claim that H is $\partial^{(s+1)} F$-free. Suppose instead, that H contains a copy of $\partial^{(s+1)} F$ on some set $S \subset [n]$ under a bijection $\phi : V(F) \to S$. Consider an s-set $A \in \partial^{(s)} F$. We know $A \subset B$ for some $B \in \partial^{(s+1)} F$. Thus, $\phi(B) \in H$ by the definition of ϕ and consequently, $\phi(A) \in G$ by the definition of H. This implies that S spans a copy of $\partial^{(s)} F$ in G, contradicting that G is $\partial^{(s)} F$-free.

Furthermore, it is easy to see that for any r-subset $S \subset [n]$, S spans a copy of $K_r^{(s)}$ in G if and only if S spans a copy of $K_r^{(s+1)}$ in H. Thus, the number of $K_r^{(s+1)}$ in H equals to $\mathrm{ex}(n, K_r^{(s)}, \partial^{(s)} F)$, the number of $K_r^{(s)}$ in G. Since H is $\partial^{(s+1)} F$-free, we conclude that $\mathrm{ex}(n, K_r^{(s)}, \partial^{(s)} F) \leq \mathrm{ex}(n, K_r^{(s+1)}, \partial^{(s+1)} F)$. □

Before proving Theorem 2, we will fix some notation that we use for the rest of the section. We call r-uniform hypergraphs *r-graphs.* Given an $(r-1)$-graph G and a vertex $v \in V(G)$, let $G(v)$ be the $(r-1)$-graph with vertex set $V(G) \setminus \{v\}$, and

$$\{v_1, \ldots, v_{r-1}\} \in G(v) \text{ if } \{v, v_1, \ldots, v_{r-1}\} \text{ induces } K_r^{(r-1)} \text{ in } G.$$

For a positive integer a, let $G(v_1) \cap \cdots \cap G(v_a)$ be the $(r-1)$-graph with vertex set $V(G) \setminus \{v_1, \ldots, v_a\}$ and edge set consisting of all $\{w_1, \ldots, w_{r-1}\}$ such that $\{v_i, w_1, \ldots, w_{r-1}\}$ induces a $K_r^{(r-1)}$ for every $i \in \{1, \ldots, a\}$.

In the following proofs we will use the hypergraph removal lemma, which we state below.

Lemma 6 (Hypergraph Removal Lemma [18, 11])**.** *For every $r \geq 3$, $\varepsilon > 0$ there exists $\delta > 0$ such that for every r-uniform hypergraph G on n vertices the following holds. If G contains at least εn^{r-1} edge disjoint copies of $K_r^{(r-1)}$, then it must contain at least δn^r copies of $K_r^{(r-1)}$.*

We also need the following simple claim.

Claim 7. *For every positive integer a, $r \geq 3$ and $(r-1)$-graph G on n vertices, the following holds. If $\mathscr{I}$ is a collection of cliques $K_r^{(r-1)}$ of G such that every edge $e \in G$ is contained in less than a cliques of $\mathscr{I}$, then G contains at least $\frac{|\mathscr{I}|}{r(a-1)}$ edge disjoint copies of $K_r^{(r-1)}$.*

Proof. For G and $\mathscr{I}$ satisfying the above assumptions, let $\mathscr{I}_1 \subseteq \mathscr{I}$ be a maximum collection of pairwise edge disjoint cliques $K_r^{(r-1)}$ in $\mathscr{I}$ and let $\mathscr{E}$ be the union of edge sets of the cliques in $\mathscr{I}_1$. Clearly $|\mathscr{E}| = r \cdot |\mathscr{I}_1|$. Since by assumption, each edge $e \in \mathscr{E}$ is contained in at most $(a-1)$ cliques $K_r^{(r-1)}$ in $\mathscr{I}$, there are at most $(a-1)r|\mathscr{I}_1|$ cliques in $\mathscr{I}$ containing some edge of $\mathscr{E}$. Due to the maximality of $\mathscr{I}_1$, it follows that $|\mathscr{I}| \leq (a-1)r|\mathscr{I}_1|$ and thus $|\mathscr{I}_1| \geq \frac{|\mathscr{I}|}{(a-1)r}$. □

Now we prove Theorem 2.

Proof of Theorem 2. Fix $r \geq 3$ and integers $a_1 \leq \cdots \leq a_r$. We first consider the case when $a_{r-1} = 1$. Let $\varepsilon > 0$, and let G be a $K_r^{(r-1)}(1, \ldots, 1, a_r)$-free $(r-1)$-graph on n vertices. Assume by contradiction, that the collection $\mathscr{I}$ of all $K_r^{(r-1)}$ in G has size at least εn^{r-1}. In view of Claim 7, G must contain at least $\varepsilon' n^{r-1}$ edge disjoint copies of $K_r^{(r-1)}$ where $\varepsilon' = ((a_r - 1)r)^{-1}\varepsilon$. By the hypergraph removal lemma, this implies that there exists some $\delta > 0$ (depending on ε') such that G contains δn^r copies of $K_r^{(r-1)}$. However, this contradicts (2).

Next, we consider the case where $a_{r-1} \geq 2$. Let G be a $K_r^{(r-1)}(a_1, \ldots, a_r)$-free $(r-1)$-graph on n vertices. First we will show that for every $\{v_1, \ldots, v_{a_{r-1}}\} \subseteq V(G)$, the $(r-1)$-graph

$G(v_1)\cap\cdots\cap G(v_{a_{r-1}})$ is $K_{r-1}^{(r-1)}(a_1,\dots,a_{r-2},a_r)$-free. Indeed, given $\{v_1,\dots,v_{a_{r-1}}\}\subseteq V(G)$, assume by contradiction, that the $(r-1)$-graph $G(v_1)\cap\cdots\cap G(v_{a_{r-1}})$ contains a copy of $K_{r-1}^{(r-1)}(a_1,\dots,a_{r-2},a_r)$ with the vertex set $V_1\sqcup V_2\cdots\sqcup V_{r-2}\sqcup V_r$, where $|V_i|=a_i$. Let $V_{r-1}:=\{v_1,\dots,v_{a_{r-1}}\}$. Then the r-partite graph on $V_1\sqcup V_2\cdots\sqcup V_r$ in G forms a copy of $K_r^{(r-1)}(a_1,\dots,a_r)$, a contradiction.

Consequently, for every $\{v_1,\dots,v_{a_{r-1}}\}\subseteq V(G)$,

$$|G(v_1)\cap\cdots\cap G(v_{a_{r-1}})|\le \mathrm{ex}(n,K_{r-1}^{(r-1)}(a_1,\dots,a_{r-2},a_r))=O\left(n^{r-1-\frac{1}{a_1a_2\cdots a_{r-2}}}\right). \tag{6}$$

Our goal is using the above fact to obtain a large collection of edge disjoint $K_r^{(r-1)}$ in G. To this end we consider the family $\mathscr{A}$, elements of which are collections of a_{r-1} copies of $K_r^{(r-1)}$ that share an edge of G. More formally,

$$\mathscr{A}:=\{\{T_1,\dots,T_{a_{r-1}}\}:T_i\cong K_r^{(r-1)}\text{ and }T_1,T_2,\dots,T_{a_{r-1}}\text{ share an edge of }G\}.$$

Next we give an upper bound on the size of $\mathscr{A}$. Given *any* element in $\mathscr{A}$, there exists vertices $\{v_1,\dots,v_{a_{r-1}}\}\subseteq V(G)$, and an edge $e\in G$ (in particular, $e\in G(v_1)\cap\cdots\cap G(v_{a_{r-1}})$), such that $e\cup\{v_i\}$ form a $K_r^{(r-1)}$ for every $1\le i\le a_{r-1}$. Consequently, the cardinality of $\mathscr{A}$ can be bounded by the number of pairs $(\{v_1,\dots,v_{a_{r-1}}\},e)$ with $e\in G(v_1)\cap\cdots\cap G(v_{a_{r-1}})$. Thus in view of (6),

$$|\mathscr{A}|\le\binom{n}{a_{r-1}}|G(v_1)\cap\cdots\cap G(v_{a_{r-1}})|\le n^{a_{r-1}}O\left(n^{r-1-\frac{1}{a_1a_2\cdots a_{r-2}}}\right). \tag{7}$$

In order to prove (3) of Theorem 2, assume by contradiction, that G contains $N=\Omega(n^{r-1/(a_1\cdots a_{r-1})})$ copies of $K_r^{(r-1)}$. We will find a collection $\mathscr{I}$ of cliques $K_r^{(r-1)}$ in G satisfying

$$|\mathscr{I}|=\Omega(n^{r-1})\quad\text{ and }\quad\mathscr{I}^{(a_{r-1})}\cap\mathscr{A}=\emptyset, \tag{8}$$

i.e., for every $S\subseteq\mathscr{I}$ with $|S|=a_{r-1}$, S is not an element of $\mathscr{A}$. Note that if $|\mathscr{A}|\le N/2$, then one can obtain $\mathscr{I}$ from the collection of $K_r^{(r-1)}$ in G, by deleting a copy of $K_r^{(r-1)}$ for each element of $\mathscr{A}$. Thus, $\mathscr{I}^{(a_{r-1})}\cap\mathscr{A}=\emptyset$ and $|\mathscr{I}|$ is at least $N/2=\Omega(n^{r-1/(a_1\cdots a_{r-1})})$ which is bigger than $\Omega(n^{r-1})$.

Now we consider the case where $|\mathscr{A}|\ge N/2$. Let $\mathbf{I}$ be a random subset of copies of $K_r^{(r-1)}$ where each copy of $K_r^{(r-1)}$ in G is chosen with probability $p>0$ independently. Let $\mathbf{I}^{(a_{r-1})}$ denote the collection of a_{r-1}-subsets of $\mathbf{I}$. We have that,

$$\mathbb{E}[|\mathbf{I}|-|\mathscr{A}\cap\mathbf{I}^{(a_{r-1})}|]=pN-p^{a_{r-1}}|\mathscr{A}|.$$

Let p be chosen such that, $p^{a_{r-1}}|\mathscr{A}|=pN/2$, which implies

$$p=\left(\frac{N}{2|\mathscr{A}|}\right)^{\frac{1}{a_{r-1}-1}}\le 1,\ (\text{since }N\le 2|\mathscr{A}|)\quad\text{ and }\quad pN=\frac{N^{\frac{a_{r-1}}{a_{r-1}-1}}}{(2|\mathscr{A}|)^{\frac{1}{a_{r-1}-1}}}.$$

Consequently, there exists a choice of $\mathscr{I}'$ such that,

$$|\mathscr{I}'|-|\mathscr{A}\cap\mathscr{I}'^{(a_{r-1})}|\ge\frac{pN}{2}.$$

Let $\mathscr{I} \subseteq \mathscr{I}'$ be the collection of $K_r^{(r-1)}$ of G formed by deleting one $K_r^{(r-1)}$ in $\mathscr{I}'$ from every a_{r-1} subset in $\mathscr{A} \cap \mathscr{I}'^{(a_{r-1})}$. Consequently, $\mathscr{A} \cap \mathscr{I}^{(a_{r-1})}$ is empty. Further,

$$|\mathscr{I}| \geq \frac{pN}{2} = \frac{N^{\frac{a_{r-1}}{a_{r-1}-1}}}{(2|\mathscr{A}|)^{\frac{1}{a_{r-1}-1}}}. \tag{9}$$

Using the value of N (by assumption) and $|\mathscr{A}|$ in (7), the exponent of n in the RHS of (9) is equal to,

$$\begin{aligned}&\left(r - \frac{1}{a_1 a_2 a_3 \cdots a_{r-1}}\right) \frac{a_{r-1}}{a_{r-1}-1} - \left(a_{r-1} + r - 1 - \frac{1}{a_1 \cdots a_{r-2}}\right) \frac{1}{a_{r-1}-1} \\ &= \frac{a_{r-1} r - a_{r-1} - (r-1)}{a_{r-1}-1} = r - 1,\end{aligned}$$

which implies $|\mathscr{I}| = \Omega(n^{r-1})$. Hence $\mathscr{I}$ satisfies (8).

Next we obtain a family of edge disjoint $K_r^{(r-1)}$ in G from $\mathscr{I}$. By construction, $\mathscr{I}$ is a collection of cliques $K_r^{(r-1)}$ in G such that every edge $e \in G$ is contained in less than a_{r-1} cliques of $\mathscr{I}$. In view of Claim 7, this implies that G contains at least $|\mathscr{I}|/r(a_{r-1}-1)$ edge disjoint copies of $K_r^{(r-1)}$. Since $|\mathscr{I}| = \Omega(n^{r-1})$, this implies that G contains $\Omega(n^{r-1})$ copies of edge disjoint $K_r^{(r-1)}$.

To summarise, this implies that given any $\varepsilon > 0$, and $(r-1)$-graph G on n vertices that is $K_r^{(r-1)}(a_1, \ldots, a_r)$-free the following holds. Assuming by contradiction that G contains $N = \varepsilon n^{r-1/(a_1 \cdots a_{r-1})}$ copies of $K_r^{(r-1)}$, there exists some $\varepsilon' > 0$ (depending only on ε, r, a_i) such that G contains $\varepsilon' n^{r-1}$ edge disjoint copies of $K_r^{(r-1)}$. By the hypergraph removal lemma, this implies that there exists some $\delta > 0$ (depending only on ε, r, a_i) such that G contains δn^r copies of $K_r^{(r-1)}$. In view of (2), however, this implies that G contains $K_r^{(r-1)}(a_1, \ldots, a_r)$. Thus (3) holds.

Now we prove the upper bound in (4) on $\mathrm{ex}(n, F, F(a))$ for any given $(r-1)$-graph F. Label the vertices of F $v_1, \ldots, v_\ell$. Let G be an $F(a)$-free $(r-1)$-graph on n vertices, and assume by contradiction, that G contains $N = \Omega(n^{\ell - \frac{1}{a^{\ell-1}}})$ copies of F. Given an ℓ-partition of $V(G) = W_1 \sqcup \cdots \sqcup W_\ell$, we call a set $X \subseteq V(G)$ *crossing* if $|X \cap W_i| \leq 1$ for $1 \leq i \leq \ell$. We call a copy of F in G on a vertex set $\{x_1, \ldots, x_\ell\}$ *aligned* with respect to $W_1 \sqcup W_2 \sqcup \cdots \sqcup W_\ell$ if

(1) $x_i \in W_i$ for $i = 1, 2, \ldots, \ell$, and
(2) $x_i \mapsto v_i$ is an isomorphism.

We will denote such a copy by $\overrightarrow{F}$. A simple averaging argument yields that there exists a partition of $V(G) = W_1 \sqcup \cdots \sqcup W_\ell$ with at least $\ell^{-\ell} N$ copies of $\overrightarrow{F}$.

Let $\mathscr{H}$ be an *auxiliary* ℓ-partite $(\ell-1)$-graph with vertex set $W_1 \sqcup \cdots \sqcup W_\ell$. Let the edges of $\mathscr{H}$ be those crossing $(\ell-1)$-tuples that extend to a copy of $\overrightarrow{F}$. Formally,

$$\mathscr{H} = \bigsqcup_{i=1}^{\ell} \left\{ (x_j)_{j\in[\ell]\setminus\{i\}} : \text{ there exists } x_i \in W_i \text{ such that } (x_1,\dots,x_\ell) \text{ is a copy of } \overrightarrow{F} \right\}.$$

Note that each *aligned copy* $\overrightarrow{F}$ in G forms a $K_\ell^{(\ell-1)}$ in $\mathscr{H}$. Consequently, the number of copies of $K_\ell^{(\ell-1)}$ in $\mathscr{H}$ is at least $(\ell^{-\ell})N = \Omega(n^{\ell-\frac{1}{a^{\ell-1}}})$.

By the first part of Theorem 2, this implies that $\mathscr{H}$ contains a copy of $K_\ell^{(\ell-1)}(a)$ with vertex sets $U_i \subseteq W_i$ for $1 \le i \le \ell$. Let $(x_1,\dots,x_\ell) \in U_1 \times \cdots U_\ell$. Since $\ell \ge r$, for every edge $\{v_{i_1},\dots,v_{i_{r-1}}\}$ of F, there exists an $\ell-1$ subset $S \subseteq [\ell]$ such that $\{i_1,\dots,i_{r-1}\} \subseteq S$. By definition of $\mathscr{H}$, the tuple $(x_s : s \in S)$ must extend to some copy of $\overrightarrow{F}$, which implies $\{x_{i_1},\dots,x_{i_{r-1}}\}$ must be an edge in G.

Consequently, for every $(x_1,\dots,x_\ell) \in U_1 \times \cdots U_\ell$, we have that the subgraph of G induced by $\{x_1,\dots,x_\ell\}$ contains an aligned copy $\overrightarrow{F}$. This implies that G contains a copy of $F(a)$, contradicting the assumption that G is $F(a)$-free. □

3. Lower Bound Constructions

In this section, we will prove Propositions 4 and 5.

Proof of Proposition 4. We construct an $(r-1)$-graph H whose vertex set is partitioned into $A \sqcup B$ such that

- $|A| = n/r$ and $|B| = (r-1)n/r$;
- $H[B]$ is $K_{r-1}^{(r-1)}(a_1,\dots,a_{r-1})$-free and has $\mathrm{ex}(\frac{r-1}{r}n, K_{r-1}^{(r-1)}(a_1,\dots,a_{r-1}))$ edges;
- every vertex of A and every $(r-2)$ - subset of B form an edge and there are no other edges intersecting A. In other words, the link of every vertex in A is the complete $(r-2)$-graph on the vertex set B.

The number of $K_r^{(r-1)}$ is at least $\frac{n}{r}\mathrm{ex}(\frac{r-1}{r}n, K_{r-1}^{(r-1)}(a_1,\dots,a_{r-1}))$ because every vertex of A together with any edge of B form a copy of $K_r^{(r-1)}$. It remains to show that H contains no $K_r(a_1,\dots,a_r)$. Assume by contradiction, it does. Since there is no edge containing two vertices from A, and $a_1 \le a_2 \le \cdots \le a_r$, the subgraph induced by H on B needs to contain a $K_{r-1}^{(r-1)}(a_1,\dots,a_{r-1})$, thus contradicting the construction of H. □

The proof of Proposition 5 is based on a construction given in [8].

Proof of Proposition 5. In the proof of [8, Proposition 2.1], it was shown that for every $r \ge 3$, there exists an r-partite r-graph H with parts $V_1,\dots,V_r$ satisfying the following properties.

1. For every $\{x_1,\dots,x_{r-1}\} \subseteq V(H)$, there exists at most one edge in H containing $\{x_1,\dots,x_{r-1}\}$.
2. For every collection of subsets $\{\{x_i,y_i\} \subseteq V_i : 1 \le i \le r\}$, there exist $1 \le i \le r$ such that $\{x_1,\dots,x_r\} \setminus \{x_i\} \cup \{y_i\}$ is not an edge of H.
3. H has $(r-1)rn$ vertices and $n^{r-2}r_r(n)$ edges.

Let G be the $(r-1)$-uniform shadow of H, i.e., $G = \partial^{(r-1)}H$. We claim that G is $K_r^{(r-1)}(1,\ldots,1,2)$-free and contains $n^{r-2}r_r(n)$ copies of $K_r^{(r-1)}$. Since G is the shadow of H, the number of copies of $K_r^{(r-1)}$ in G is at least the number of edges in H.

While the edges of H correspond to a collection of edge disjoint cliques ("real cliques") in G, we will now show that G contains no other cliques $K_r^{(r-1)}$. Assume by contradiction that $\{x_1,\ldots,x_r\}$ induces such a "fake clique" $K_r^{(r-1)}$, i.e., $\{x_1,\ldots,x_r\} \notin H$ but induces a $K_r^{(r-1)}$ in G. Since every edge of this clique belongs to some "real clique", for every $1 \le i \le r$, there must exist $y_i \ne x_i$ in V_i such that $\{x_1,\ldots,x_r\} \setminus \{x_i\} \cup \{y_i\} \in H$, contradicting (2). Consequently, by (1), no two $K_r^{(r-1)}$ in G share an edge and hence G is $K_r^{(r-1)}(1,\ldots,1,2)$-free. □

4. Concluding remarks

As mentioned earlier, in the case where $a_1 = \cdots = a_{r-1} = 1$ and $a_r \ge 2$ the lower bound in (5) is trivial. We ask if there are other sequences of integers $a_1 \le \cdots \le a_r$ for which (5) can be improved.

Question 8. *Given integer $r \ge 3$, for what sequence of integers $1 \le a_1 \le \cdots \le a_r$,*

$$\mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a_1,\ldots,a_r)) \ge n^{1+\varepsilon} \cdot \mathrm{ex}\left(n,\ K_{r-1}^{(r-1)}(a_1,\ldots,a_{r-1})\right)$$

for some $\varepsilon = \varepsilon(n) > 0$?

The order of magnitude for $\mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a_1,\ldots,a_r))$ is not known in any non-trivial case. The case when $r \ge 3$ and $a_1 = \cdots = a_r = 2$ is related to a problem of Erdős, see, e.g., [3, 12, 13]. Theorem 2 and Proposition 4, together with the lower bound in [3] imply that

$$\Omega\left(n^{r-\left\lceil \frac{2^{r-1}-1}{r-1} \right\rceil}\right) \le \mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(2,\ldots,2)) \le o\left(n^{r-\frac{1}{2^{r-1}}}\right).$$

It was conjectured in [16], that $\mathrm{ex}(n, K_{r-1}^{(r-1)}(a_1,\ldots,a_{r-1})) = \Omega(n^{r-1-1/(a_1\cdots a_{r-2})})$. This was confirmed for some cases in [15, 16]. If this conjecture is true, then Theorem 2 and Proposition 4 would imply that,

$$\Omega(n^{r-1/(a_1\cdots a_{r-2})}) \le \mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a_1,\ldots,a_r)) \le o(n^{r-1/(a_1\cdots a_{r-1})}).$$

When $a_1 = \cdots = a_r = a \ge 2$, one can obtain that $\mathrm{ex}(n, K_{r-1}^{(r-1)}(a)) = \Omega(n^{r-1-(r-1)/(a^{r-1}-1)})$ by using the *probabilistic deletion method* [6]. Together with Proposition 4 and Theorem 2, this gives

$$\Omega\left(n^{r-\frac{(r-1)(a-1)}{a^{r-1}-1}}\right) \le \mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a)) \le o\left(n^{r-\frac{1}{a^{r-1}}}\right).$$

When $a_1 = 1$, instead of Proposition 4, one can employ the *deletion method* directly to an random $(r-1)$-uniform hypergraph on n vertices by removing copies of $K_r^{(r-1)}(1,a,\ldots,a)$. Together with Theorem 2, this implies that

$$\Omega\left(n^{r-\frac{r(r-1)}{a^{r-2}}}\right) \le \mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(1,a,\ldots,a)) \le o(n^{r-\frac{1}{a^{r-2}}}).$$

It would be interesting to improve the gaps in any of the above cases.

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