COMPACT MODULI OF CALABI-YAU CONES AND SASAKI-EINSTEIN SPACES

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Abstract. We construct proper moduli algebraic spaces of K-polystable $\mathbb{Q}$-Fano cones (a.k.a. Calabi-Yau cones) or equivalently their links i.e., Sasaki-Einstein manifolds with singularities.

As a byproduct, it gives alternative algebraic construction of proper K-moduli of $\mathbb{Q}$-Fano varieties. In contrast to the previous algebraic proof of its properness ([BHLLX21, LXZ22]), we do not use the δ-invariants ([FO18, BJ22]) nor the L^2-normalized Donaldson-Futaki invariants. We use the local normalized volume of [Li18] and the higher Θ-stable reduction [Od24b] instead.

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1. Introduction

It follows from the recent breakthrough [DS14], combined with the Gromov's precompactness theorem and the theory of Gromov-Hausdorff distance [Gro99], that there should be compactifying topological space of moduli spaces of Kähler-Einstein Fano manifolds, letting the boundary parametrizes certain singular Kähler-Einstein Fano varieties. Indeed, [DS14] uses Cheeger-Colding theory to prove that for Kähler-Einstein smooth Fano manifolds of fixed dimensions, the Gromov-Hausdorff limits, which always exists for subsequences, admit the structure of log terminal Kähler-Einstein Fano varieties, which are $\mathbb{Q}$-Gorenstein smoothable (cf., [OSS16, §2]). In particular, they admit the structure of algebraic varieties. This also matches the general K-moduli conjecture then ([Od10, Conjecture 5.2]), a purely algebro-geometric conjecture, which was later refined in the course of developments. Indeed, the obtained compact moduli topological spaces were enhanced to be proper (compact) algebraic spaces by using K-stability in the series of works [OSS16, LWX19, SSY16, Od15a] (cf., also [MM93] which predates K-stability). By more birational algebro-geometric discussions after that, recently we witnessed a celebrated algebraic (re-)proof and generalization of the facts that the moduli (stack) of K-polystable Fano varieties satisfies the valuative criterion of properness by [BHLLX21, LXZ22], and the coarse moduli is projective ([CP21, XZ22]). Their algebraic proof depends on the theory of δ-invariants [FO18, BJ22] of Fano varieties. This reproves and extends the original construction [LWX19, Od15a] (cf., also closely related results in [SSY16]) to generally singular $\mathbb{Q}$-Fano varieties.

In this paper, we extend this K-moduli construction to that of Fano *cones*, e.g., contracted (pluri-)anticanonical line bundles over Kähler-Einstein Fano manifolds, by a different method. What underlies historically behind this theory of cone type varieties is the real odd-dimensional version of Kähler-Einstein manifolds, the structure of Sasaki-Einstein manifolds on their links (see the textbook [BG07]) as we review in the second section. One benefit of our approach is that it also gives alternative (partial) proof of the algebraic construction of the K-moduli of Fano varieties, without depending on the δ-invariant nor L^2-normalized Donaldson-Futaki invariants, but rather we only use the normalized volume or the volume density of the Einstein metrics, to match more easily with theory of log terminal singularities. We work over an algebraically closed field $\Bbbk$ of characteristic 0.

Main Theorem (cf., Definitions 2.9, 2.15, 2.21, Theorem 3.17)**.** *For each algebraically closed field* $\Bbbk$ *of characteristic* 0*,* $n \in \mathbb{Z}_{>0}$ *and* $V \in$

$\mathbb{R}_{>0}$, there is a proper moduli algebraic $\Bbbk$-space of n-dimensional K-polystable Fano cones over $\Bbbk$ (a.k.a., Calabi-Yau cones[1]) of the volume density V.

If $\Bbbk = \mathbb{C}$, equivalently, real $(2n-1)$-dimensional compact Sasaki-Einstein spaces of volume density V form a compact Moishezon analytic space for each fixed n and V.

More precise version, logarithmic generalization and a corollary are stated and proved as Theorem 3.17, 3.20 and 3.21 respectively. We conjecture (Conjecture 3.23) that these coarse moduli spaces are also projective schemes with Weil-Petersson type (singular) Kähler metrics.

2. Preparation

We give a brief review of some preparatory materials. Most of (but perhaps not all of) the materials in this section are known before as we give references to each.

2.1. **Sasakian geometry.**

A classical differential geometric (or contact geometric) counterpart of the Ricci-flat Kähler cone is the geometry of Sasakian manifolds, which we briefly recall. This subsection is intended to be more differential geometric review, and only works over $\Bbbk = \mathbb{C}$ whenever some varieties appear.

Definition 2.1 (Sasakian manifolds cf., [Sas60, BG14])**.** A Riemannian manifold (S, g) of (real) odd dimension $2n-1$ with $n \in \mathbb{Z}_{>0}$ is said to underlie a *Sasakian manifold* if there is a complex structure J on $C^o(S) := S \times \mathbb{R}_{>0}$ with repsect to which $g_{C(S)}$ is a Kähler metric, so that J further extends to the cone $C(S) := C^o(S) \cup 0$. The corresponding Kähler form is, by the definition of $g_{C(S)}$, $\omega_{C(S)} := \sqrt{-1}\partial\overline{\partial} r^2$.

Then, the corresponding $\xi := J\left(r\frac{\partial}{\partial r}\right)$ is called the Reeb vector field. To avoid confusion with more algebraic flexible variant (positive vector field in Definition 2.9), we call it the *metric Reeb vector field*. We also have a contact 1-form $\eta := \iota_\xi g = \iota_{\frac{\partial}{\partial r}} \omega_{C(S)}$ associated, where ι stands for the interior product. The actual Sasaki structure on M is in addition to g, further we encode ξ (or η equivalently). [2]

There are also associated structures such as $(1,1)$ type real tensor field $\Phi \in \Gamma(M, \mathrm{End}(T_S))$, which satisfies $\Phi \circ \Phi = -\mathrm{id} + \xi \otimes \eta$, where $\mathrm{End}(T_S)$ means the endomorphism bundle of the (real) tangent bundle, id means the identity map and the contact form η is defined as

[1] in some literature, it requires slightly more constraints for this terminology e.g., [CH22]

[2] although, in most of quasi-regular cases, this is redundant by [BGN03, 8.4], [BG05, 20, 21]

$\eta(v) = g(\xi, v)$. The datum is again equivalent to the (almost) complex structure J on $C(S)$ by the formula $\Phi(v) = J(v - g(\xi, v)\xi) = \nabla^{\mathrm{LC}}_v \xi$ by the Levi-Civita connection ∇^{LC}. Below, we assume S is compact throughout, unless otherwise stated.

Definition 2.2 (Quasi-regular, irregular)**.** $\overline{\mathbb{R}\xi} \subset \mathrm{Isom}(S, g)$ is known to be a $r(\xi)$-dimensional compact torus. We denote its complexification as $T(\mathbb{C}) \simeq (\mathbb{C}^*)^{r(\xi)}$. If $r(\xi) = 1$ (resp., $r(\xi) > 1$), we call (S, g) or corresponding Fano cone is quasi-regular (resp., irregular). If (S, g) is irregular and further the action of $T(\mathbb{C}) \simeq \mathbb{C}^*$ on $C(S)$ is free, we call (S, g) is regular.

The following is also well-known (cf., e.g., [BG07]).

Proposition 2.3. *For a compact Sasakian manifold (S, g, ξ), the following are equivalent:*

- (i) *$(C(S), g_{C(S)})$ is Ricci-flat.*
- (ii) *(S, g) satisfies $\mathrm{Ric}_g = (\dim(X) - 1)g$, hence Einstein manifold in particular (called Sasaki-Einstein manifold).*
- (iii) *In quasi-regular case, $S/(T(\mathbb{C}) \simeq \mathbb{C}^*)$ admits a natural branch divisor with standard coefficients Δ (which is 0 in regular case) so that $(S/(T(\mathbb{C}) \simeq \mathbb{C}^*), \Delta)$ is a log K-polystable $\mathbb{Q}$-Fano variety with conical weak Kähler-Einstein metric.*

In the above case, $C(S) \cup 0$ has only log terminal singularity at the vertex.

As our highlight, we consider the limiting behaviour.

Definition 2.4. We consider all the (real) $2n - 1$-dimensional compact Sasaki-Einstein manifolds (S, g) of the volume V and denote their isomorphic classes as $M_{n,V}$.

Corollary 2.5. *$M_{n,V}$ is precompact with respect to the Gromov-Hausdorff topology. Furthermore, $M_{n,V}(i = 1, 2, \cdots)$ satisfies the non-collapsing condition (compare* [DS14]*):*

> *For any $0 < r \leq \mathrm{diam}(S_i, g_i)$, there is a positive real number c such that we have $\mathrm{vol}(B_r(p, (S_i, g_i)) \geq cr^n$. Here, $\mathrm{diam}(-)$ means the diameter, vol means the volume and $(B_r(p, (S_i, g_i))$ means the geodesic ball of radius r and the center p in (S_i, g_i).*

Proof. The former claim follows from the Myers theorem and Gromov precompactness theorem. The latter claim follows from the Bishop-Gromov comparison theorem. ∎

From this claim, we naturally expect some compactness theorem and the main aim of our paper is to give a proof to its algebraic analogue.

Martelli-Sparks-Yau [MSY08] studied necessary condition of the existence of Sasaki-Einstein metric (using the case study when $(C(S),J)$ in [MSY06]) and showed that the volume is an algebraic number.

Definition 2.6. For a compact Sasaki-Einstein manifold (S,g,ξ), define its volume as

$$\begin{aligned}\mathrm{vol}^{\mathrm{DG}}(\xi) &:= \frac{1}{(2\pi)^n n!}\int_{C(S)} e^{-r^2}\omega_{C(S)}^n\\ &= \frac{\mathrm{vol}(S,g)}{\mathrm{vol}\left(S^{2n-1}(1)\right)}.\end{aligned}$$

The denominator is with respect to the standard round metric on the unit sphere. Note that the above invariant $\mathrm{vol}^{\mathrm{DG}}$ is invariant under the rescale of r by $cr(c \in \mathbb{R}_{>0})$.

The above volume has a relatively algebraic nature and is determined only by a positive vector field ξ as follows. We leave the details of the proof of the following to [CS18, §6], [MSY08] or [Li18].

Proposition 2.7 ([MSY08],[CS18], [CS19, Proposition 6.6], and Lemma 2.14 later)**.** *For any compact Sasaki manifold (S,g,ξ), the cone $(C(S),J)$ is a complex affine variety, which we write as* $\mathrm{Spec}(R)$*, acted by the algebraic $\Bbbk$-torus $T(\mathbb{C})$ which induces the T-eigenspaces decomposition $R = \oplus_{\vec{m}\in \mathrm{Hom}(T(\mathbb{C}),\mathbb{C}^*)}$ and we regard ξ as an element of* $\mathrm{Hom}(\mathbb{C}^*, T(\mathbb{C}))$*. Then, we can define and write the index character F (*[MSY08]*) as*

$$\begin{aligned}F(\xi,t) &:= \sum_{\vec{m}\in M} e^{-t\langle \vec{m},\xi\rangle}\dim R_{\vec{m}}\\ &= \frac{A_0(\xi)}{t^n} - \frac{A_1(\xi)}{t^{n-1}} + O(t^{2-n})\end{aligned}$$

and further, if ξ is the metric Reeb vector field for the Ricci-flat Kähler metric, we have

$$A_0(\xi) = \mathrm{vol}^{\mathrm{DG}}(\xi) \tag{1}$$
$$= \widehat{\mathrm{vol}}(\mathrm{val}_\xi). \tag{2}$$

Here (2) *is in the sense of C.Li* [Li18]*, which algebraizes the theory and is the topic of the next subsection. Actually the equality $A_0(\xi) = \widehat{\mathrm{vol}}(\mathrm{val}_\xi)$ makes sense and holds true for more general cases (abstract Reeb vector field in Definition 2.9) as we see in Lemma 2.14.*

In particular the above volume $\mathrm{vol}^{\mathrm{DG}}(\xi)$ *is determined only by the multi-Hilbert function of the decomposition* $\Gamma(\mathcal{O}_X) = R = \oplus_{\vec{m}} R_{\vec{m}}$.

Theorem 2.8. *If* (S, g) *is a compact Sasaki-Einstein manifold, the following hold.*

(i) *(*[MSY06, MSY08]*)* $\mathrm{vol}^{\mathrm{DG}}(-)$ *extends to whole Reeb cone (cf., Definition 2.9) and is minimized at* ξ *which is determined by the Sasakian structure.*

(ii) *(*[Li18, CS19]*) the metric Reeb vector field* ξ *of the metric tangent cone* $C(S)$ *satisfies the normalization (or gauge-fixing) condition* $A_{C(S)}(\xi) = n$*. Here,* $A_{C(S)}(-)$ *denotes the log discrepancy function in the theory of the minimal model program and* val_ξ *denotes the naturally corresponding valuation of* $\mathcal{O}_{C(S),0}$ *to* ξ.

The above results show some algebro-geometric natures of the Sasakian geometry. From the next section, we mainly work by such algebro-geometric framework.

2.2. **Affine cone.** Henceforth, we work on more algebro-geometric side, over an algebraically closed field $\Bbbk$ of characteristic 0 unless otherwise stated. Take an arbitrary algebraic torus T, we set $N := \mathrm{Hom}(\mathbb{G}_m, T)$, $M := N^\vee = \mathrm{Hom}(T, \mathbb{G}_m)$ throughout. Before going further, we promise the further toric notation throughout the paper (which follows [Od24b]).

Notations 1. For any rational polyhedral cone $\tau \subset N \otimes \mathbb{R}$, we denote an affine toric variety corresponding to it as $U_\tau(\supset T)$ and its T-invariant vertex (closed point) as p_τ. We denote the quotient stack $[U_\tau/T]$ as Θ_τ. We often consider non-zero irrational element $\xi \in (\tau \setminus N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q})$ i.e., of $\mathbb{Q}$-rank r.

On the dual side, we set $M := \mathrm{Hom}(N, \mathbb{Z})$ as the dual lattice, $\tau^\vee := \{x \in M \otimes \mathbb{R} \mid \langle \tau, x\rangle \subset \mathbb{R}_{\geq 0}\}$, $\mathcal{S}_\tau := \tau^\vee \cap M$. If we regard M as the the character lattice of T, then the character of T which corresponds to $\vec{m} \in M$ is denoted by $\chi_{\vec{m}}$.

Definition 2.9 (cf., [LS13, CS18, CS19])**.** (i) Consider a normal affine variety X with the vertex x and an action of an algebraic torus T, which is *good* in the sense of [LS13], i.e., the action is effective and x is contained in the closure of any T-orbit which characterizes x. [3] We consider the decomposition

$$R := \Gamma(X, \mathcal{O}_X) = \oplus_{\vec{m} \in M} R_{\vec{m}}$$

[3]When $r = 1$, $T \curvearrowright X \ni x$ is also sometimes said to be a quasicone in the literature.

and define the *moment monoid* as

$$\sigma_R := \{\vec{m} \in M \mid R_{\vec{m}} \neq 0\},$$

the *moment cone* as

$$\mathbb{R}_{\geq 0}\sigma_R \subset M \otimes \mathbb{R},$$

and the *Reeb cone* as

$$C_R := \{\xi \in N \otimes \mathbb{R} \mid \langle \vec{m}, \xi \rangle > 0 \quad \forall \vec{m} \in \sigma_R \setminus \{0\}\}.$$

We call an element ξ of C_R an *abstract Reeb vector field* or *positive vector field*. An important observation is that, for each ξ, we can associate a valuation val_ξ with the center $x \in X$ (Definition 2.10). See also there is an interpretation of C_R as an analogue to the Kähler cone (cf., e.g., [BvC18, §2]), where it is called the *Sasaki cone*.

(ii) An *affine cone* [4] is simply a triplet $(X, T \curvearrowright X, \xi)$ where X is an affine algebraic $\mathbb{k}$-scheme, $T \curvearrowright X$ is a good action, and ξ is its abstract Reeb vector field. We denote the dimension of X as n.

ξ is called regular (resp., quasi-regular) if $r = 1$ and T acts freely on $X \setminus \{x\}$ (resp., $r = 1$). In the case when ξ is regular, we denote the corresponding polarized projective scheme as (V, L) so that $X = \mathrm{Spec} \oplus_{m \geq 0} H^0(V, L^{\otimes m})$. If ξ is quasi-regular, i.e., in the general $r(\xi) = 1$ case, we can similarly construct the quotient $[(X \setminus x)/T] \to (X \setminus x)//T =: V$ as a projective scheme (the GIT quotient). If X is normal, V is also automatically normal (cf., [Mum65, Chapter I]) and we can consider the ramification index $m(D)$ for each prime divisor D on V and form the *standard ramification divisor* as a Weil $\mathbb{Q}$-divisor $\sum_D \frac{m(D)-1}{m(D)} D$ on V. This morphism $(X \setminus x) \to (V, \sum_D \frac{m(D)-1}{m(D)} D)$ is called the *(algebraic) Seifert $\mathbb{G}_m$-bundle* in [Kol04, Kol13b] (cf., also [LL19, §3.1]), It globally realizes the so-called transverse Kähler structure on the locally orthogonal direction to the Reeb foliation in $r = 1$ case (in the irregular case, one can only locally realize it cf., [BG07]).

(iii) A $\mathbb{Q}$-*Gorenstein (affine) cone*[5] is (generalized) affine cone where X is reduced which is normal crossing in codimension 1, satisfying the Serre condition S_2, K_X is $\mathbb{Q}$-Cartier, $T \curvearrowright X$

[4] or *generalized affine cone*, to clarify that it is in the broader sense than the classical case i.e., when $r = 1$ and ξ is regular as [Kol13b, (3.8)]

[5] temporary name in this paper

is a good action, and ξ is its abstract Reeb vector field. It is called a *Fano cone* if X is also log terminal. We also call it a *(kawamata-)log-terminal cone* in this paper. Similarly, if X is log canonical (resp., semi-log-canonical), we call it a *log canonical cone* (resp., *semi-log-canonical cone*).

There is a valuative interpretation of positive vector field.

Definition 2.10 (Monomial valuations cf., [BFJ08], [Li18, §2.2])**.** If $x \in X \curvearrowleft T$ is an irreducible affine cone, then for each positive vector field $\xi \in C_R$ (see Definition 2.9), we can associate a valuation val_ξ of X centered at the vertex x as follows:

$$\mathrm{val}_\xi(f) := \min_{\vec{m} \in \sigma_R} \{\langle \vec{m}, \xi \rangle \mid f_{\vec{m}} \neq 0\}, \tag{3}$$

where $f = \sum f_{\vec{m}}$ is the decomposition for $R = \oplus R_{\vec{m}}$. If we take log resolution of $(X, \sum_i \mathrm{div}(f_i))$ where f_i are T-homogeneous generators of R, it follows that the above val_ξ is quasi-monomial (compare [BFJ08]).

The following should be known to experts.

Lemma 2.11. (i) *In the setup of Definition 2.9* (i)*, if mK_X is Cartier for some positive integer m, then it is automatically linearly trivial.*

(ii) *For a $\mathbb{Q}$-Gorenstein affine cone $T \curvearrowright X \ni x$ with the abstract Reeb vector field ξ, there is a positive integer l and a nowhere vanishing holomorphic section $\Omega \in \Gamma(\mathcal{O}(lK_X))$ and some real number λ so that*

$$L_\xi \Omega = \sqrt{-1}\lambda\Omega.$$

Here, L_ξ means the Lie derivative. Moreover, $T \curvearrowright X \ni x$ with ξ is a Fano cone if and only if $\lambda > 0$.

Proof. For (i), by taking the index 1-cover with respect to K_X, we can reduce to the quasi-Gorenstein case. By the complete reducibility of $T \curvearrowright \Gamma(mK_X)$ we can write K_X as a T-invariant divisor which does not contain the vertex, but then it is easy to see the support is empty. See [PS11, LS13] for discussions in more general case. The classical case i.e., for the regular action of T with $r = 1$, see [Kol13b, 3.14] for the proof of this (i). For (ii) is proved in [CS19, Lemma 6.1, 6.2] for the normal case. For non-normal case, the same proof works by combining with [GZ16, 16.45, 16.47 and the proof]. □

The last statement also follows from the Seifert $\mathbb{G}_m$-bundle interpretation by [Kol13b, §9.3], [LL19, §3.3.1] (cf., also [LLX18, §3.1]). λ of the above is interestingly interpretted as log discrepancy of X by C.Li

[Li18]. For instance, its posivitiy was known to be equivalent to log terminality of X as observed in [CS19, §6].

Definition 2.12. (i) The *multi-Hilbert function* of an affine cone $T \curvearrowright X \ni x$ with the positive vector field $\xi \in N_{\mathbb{R}}$ is a map $M \to \mathbb{Z}_{\geq 0}$ defined by $\vec{m} \mapsto \dim R_{\vec{m}} =: \chi_X(\vec{m})$.

(ii) (cf., [MSY08, CS18, CS19]) The *multi-Hilbert series* of an affine cone $T \curvearrowright X \ni x$ with the positive vector field $\xi \in N_{\mathbb{R}}$ [6] is defined as

$$F(\xi, t) = \sum_{\vec{m} \in M} e^{-t\langle \vec{m}, \xi\rangle} \dim R_{\vec{m}}.$$

A priori one can regard it only as a formal function but we review in Lemma 2.14 (from [MSY06, CS18]) later that as far as R is finitely generated, the series is meromorphic around 0 so that it encodes the multi-Hilbert function $\{\dim R_{\vec{m}}\}_{\vec{m}}$ as the Fourier coefficients. Obviously, the moment monoid, the moment cone and the Reeb cone are determined only by the multi-Hilbert function.

(iii) (cf., [CS18]) Take an affine cone $\tilde{T} \curvearrowright X \ni x$ with an algebraic $\Bbbk$-torus $\tilde{T}(\supset T)$, its character lattice $\tilde{M} := \mathrm{Hom}(T, \mathbb{G}_m)$ (resp., $\tilde{N} := \mathrm{Hom}(\mathbb{G}_m, T)$) [7] and $\eta \in \tilde{N} \otimes \mathbb{R}$. We denote the decomposition for the $\tilde{T}$-action of $R := \Gamma(\mathcal{O}_X)$ as $\oplus_{\tilde{m}} R_{\tilde{m}}$.

The *weighted (multi-)Hilbert series*[8] of the affine cone $\tilde{T} \curvearrowright X \ni x$ with the positive vector field $\xi \in N_{\mathbb{R}}$, for the η-direction, is defined as

$$C_\eta(\xi, t) := \sum_{\tilde{m} \in \tilde{M}} e^{-t\langle \vec{m}, \xi\rangle} \langle \tilde{m}, \eta\rangle \dim R_{\tilde{m}}.$$

(iv) (cf., [HS17]) Consider any faithfully flat affine family between algebraic $\Bbbk$-schemes $\pi\colon \mathcal{X} = \mathrm{Spec}_S(\mathcal{R}) \to S$ with connected S, with $\mathcal{O}_S$-algebra $\mathcal{R}$, where an algebraic $\Bbbk$-torus acts on $\mathcal{X}$ fiberwise. We apply the complete reducibility of T to $\mathcal{R}$ to obtain the decomposition $\mathcal{R} = \oplus_{\vec{m} \in M} \mathcal{R}_{\vec{m}}$ where $\mathcal{R}_{\vec{m}}$ denotes the $\mathcal{O}_S$-module corresponding to the character $\vec{m}$. If $\mathcal{R}_{\vec{m}}$ for any $\vec{m}$ is faithfully flat over S, we call the family $\pi\colon \mathcal{X} \to S$ is *T-equivariantly faithfully flat over S*, or T-fppf for short (if no

[6] Originally called *index character* by [MSY08, CS18, CS19] in the context of equivariant index theory but this term may sound more familiar to algebraic geometers

[7] This notation is set in this way as later we often apply to the central fiber of the test configuration (Definition 2.15 (i)) so that $\tilde{T}$ is often $T \times \mathbb{G}_m$, unless it is a product test configuration.

[8] called *weighted character* in [CS18]

confusion), in this paper. Note that this condition is said to be admissibility of the deformation in [HS17] and is stronger than the faithful flatness of π. See also the following Lemma 2.13 (ii).

Here is some useful general lemma.

Lemma 2.13. (i) *(Characterization of good action) Take an affine algebraic* $\Bbbk$*-scheme* $X = \mathrm{Spec}(R)$ *on which an algebraic* $\Bbbk$*-torus* T *acts. It is a good action if and only if the multi-Hilbert functions are all finite i.e.,* $\dim(R_{\vec{m}}) < \infty$ *for any* $\vec{m} \in M$*, and the moment cone* $\mathbb{R}_{\geq 0}\sigma_R$ *is strictly convex i.e., does not contain any line. It is further equivalent to the non-triviality of the Reeb cone* $C_R \neq \emptyset$*.*

(ii) *(Constancy of multi-Hilbert function) As Definition 2.12* (iv)*, consider any flat affine family (resp.,* T*-equivariantly faithfully flat) between algebraic* $\Bbbk$*-schemes* $\pi\colon \mathcal{X} \to S$ *with irreducible* S *whose generic point is* η*, where an algebraic* $\Bbbk$*-torus acts on* $\mathcal{X}$ *fiberwise. If we compare the multi-Hilbert function of the generic fiber* X_η *and any fiber* X_s *for* $s \in S$*, we have that* $\chi_{X_s}(\vec{m})$ *is either* 0 *or* $\chi_{X_\eta}(\vec{m})$*. Furthermore,* $\chi_{X_s}(\vec{m}) = \chi_{X_\eta}(\vec{m})$ *for any* $\vec{m}$ *if and only if* π *is* T*-equivariantly faithfully flat.*

(iii) *In the* T*-equivariantly faithfully flat situation of the above* (ii)*, if* $T \curvearrowright \mathcal{X}_s$ *for some* $s \in S$ *is a good action, it holds for any* $s \in S$*.*

Proof. We first prove (i). The only if direction is discussed in [LLX18, §3.1] (cf., also [LS13]). For the converse i.e., the if direction, note that $R_{\vec{0}} = \Bbbk$ since it is a finite extension of $\Bbbk$ and is an integral domain, while we assume $\Bbbk$ is algebraically closed. Since $R_{\vec{0}} = R^T$ i.e., coincides with the T-invariant subring of R, it follows that the polystable locus X^{ps} of X consists of finite T-orbits. Furthermore, by the standard arguments using the Raynold operator, it follows that X^{ps} is connected. Hence it consists of a single T-orbit, say Tx for some closed point $x \in X$. Denote the identity component of the stabilizer $\mathrm{stab}(x)$ of x as $\mathrm{stab}^0(x)$. Then the character lattice of $T/\mathrm{stab}^0(x)$ should be trivial since otherwise it would contradicts with the strict convexity of $\mathbb{R}_{\geq 0}\sigma_R$.

Next we prove (ii). We can and do assume S is affine and write the family as $\mathcal{X} = \mathrm{Spec}_R(\mathcal{R})$ by a $(R :=)\Gamma(\mathcal{O}_S)$-algebra. Apply the complete reducibility of T to $\mathcal{R}$ to obtain $\mathcal{R} = \oplus_{\vec{m}\in M}\mathcal{R}_{\vec{m}}$, with R-modules $\mathcal{R}_{\vec{m}}$. Since $\mathcal{R}$ is R-flat, each $\mathcal{R}_{\vec{m}}$ is also R-flat hence locally free module because of the finitely generatedness assumption. Therefore, the assertion follows. Finally, (iii) follows from (i) and (ii). □

We note that a certain embedded version of (i) (resp., (ii)) is partially proved in literature as [CS18, 4.5], [KR00, 4.1.19] (resp., [CS18, after Definition 5.1]). We also review the following for the next section.

Lemma 2.14 ([CS18, CS19])**.** *Recall that we set $n := \dim(X)$.*

(i) *The multi-Hilbert series $F(\xi,t)$ can be written as*

$$F(\xi,t)=\frac{A_0(\xi)}{t^n}-\frac{A_1(\xi)}{2t^{n-1}}+O(t^{2-n}),$$

with t around $0\in\mathbb{C}$ where $A_i(\xi)(i=0,1)$ are C^∞-function on the Reeb cone C_R (Definition 2.9). In general, we have

$$A_0(\xi)=\widehat{\mathrm{vol}}(\mathrm{val}_\xi), \tag{4}$$

where the latter $\widehat{\mathrm{vol}}(-)$ means the normalized volume in the sense of [Li18] *(Definition 2.27). Note that this holds for general affine cone as proved in* [CS18, §4].

Furthermore, $r(\xi)=1$, T is regular and ξ is the generator of N, in the notation of Definition 2.9, $A_0(\xi)=(L^{n-1})$, $A_1(\xi)=(L^{n-2}.K_V)$. Note the obvious homogeneity $A_i(c\xi)=c^{-n+i}A_i(\xi)$ for $i=0,1$.

(ii) *The weighted multi-Hilbert series $C_\eta(\xi,t)$ can be written as*

$$C_\eta(\xi,t)=\frac{B_0(\xi)}{t^{n+1}}-\frac{B_1(\xi)}{2t^n}+O(t^{1-n}),$$

where $B_i(\xi)(i=0,1)$ are C^∞-function on the Reeb cone C_R.

B functions are directional derivative of A functions in the sense that $B_i(\xi)=-D_\eta A_i(\xi)$ for $i=0,1$ where D_η denotes the directional derivative along the direction η. Hence, we again have the homogeneity $B_i(c\xi)=c^{-n-1+i}A_i(\xi)$ for $i=0,1$.

Proof. See [CS18, Theorem 3] (which depends on [KR00, §5.8]) for the proof of (i). (ii) follows from the derivation by terms of $\sum_{\tilde{m}\in\tilde{M}}e^{-t\langle\tilde{m},\xi-s\eta\rangle}\dim R_{\tilde{m}}$. See the details in the proof of [CS18, Theorem 4]. ☐

2.3. K-stability of affine cones.

Definition 2.15. (i) (cf., [CS18, CS19]) A (affine T-equivariant) *test configuration* of an affine cone $(T\curvearrowright X,\xi)$ means $T\times\mathbb{G}_m$-equivariant affine T-equivariantly faithfully [9] flat morphism $\pi\colon\mathcal{X}\to\mathbb{A}^1$ from a normal affine scheme $\mathcal{X}$, where T acts only fiberwise (while acting trivially on the base $\mathbb{A}^1$) and $\mathbb{G}_m$

[9] this T-equivariant faithful flatness is necessary e.g. to avoid $X\times(\mathbb{A}^1\setminus\{0\})$ and some other pathological examples, which is missed in some literature

acts multiplicatively on the base $\mathbb{A}^1$ such that its restriction to $\pi^{-1}(\mathbb{A}^1\setminus\{0\})$ is the product $X\times(\mathbb{A}^1\setminus\{0\})$. A test configuration is called product test configuration if there is a T-equivariant isomorphism $\mathcal{X}\simeq X\times\mathbb{A}^1$. For simplicity, we sometimes abbreviate the whole data of a test configuration just as $\mathcal{X}$ if it does not make confusion.

(ii) (cf., [CS18]) The *Donaldson-Futaki* [10] *invariant* $\mathrm{DF}(\mathcal{X},\xi)$ of a test configuration $(T\times\mathbb{G}_m)\curvearrowright\mathcal{X}\xrightarrow{\pi}\mathbb{A}^1$ is defined (up to a dimensional constant $2((n+1)!(n-1)!)$) as

$$\mathrm{DF}(\mathcal{X},\xi):=(n+1)A_1(\xi)B_0(\xi)-nA_0(\xi)B_1(\xi), \tag{5}$$

where $A_i(\xi)$ are that of any fibers $\pi^{-1}(s)$ for $s\in\mathbb{A}^1$ (see Lemma 2.13 (ii) for the independence on s) $B_i(\xi)$ are that of $\mathcal{X}_0=\pi^{-1}(0)$ defined in Definition 2.12 (ii).

If $r=1$, T is regular and ξ is the generator of N, in the notation of Definition 2.9, we have

$$A_0(\xi)=(L^{n-1}), \qquad A_1(\xi)=(L^{n-2}.K_V), \tag{6}$$

$$B_0(\xi)=(\mathcal{L}^n), \qquad B_1(\xi)=(\mathcal{L}^{n-1}.K_{\mathcal{V}/\mathbb{P}^1}), \tag{7}$$

where $(\mathcal{V},\mathcal{L})$ denotes the (polarized projective) test configuration in the sense of [Don02] which arise as $((\mathcal{X}\setminus\overline{\mathbb{G}_m\cdot(x,1)})/\mathbb{G}_m)\to\mathbb{A}^1$ by [Wan12, Od13a] so that it fits to the original intersection number formula in *loc.cit* for the polarized projective setup.

Then, the (generalized) affine cone $(T\curvearrowright X,\xi)$ is *K-stable (resp., K-semistable)* if and only if $\mathrm{DF}(\mathcal{X},\xi)>0$ (resp., $\mathrm{DF}(\mathcal{X},\xi)\geq 0$) for any non-trivial affine test configuration $\pi\colon\mathcal{X}\to\mathbb{A}^1$. It is said to be *K-polystable* if it is K-semistable and $\mathrm{DF}(\mathcal{X},\xi)=0$ only occurs when $\mathcal{X}$ is a product test configuration.

(iii) (cf., [LX14]) A test configuration of a Fano cone is called a *special test configuration* if $(\mathcal{X},\mathcal{X}_0)$ is purely log terminal.

(iv) (cf., [CS18, CS19]) For a special test configuration $\mathcal{X}$ of a Fano cone, its *Donaldson-Futaki invariant* is equivalently defined as

$$\mathrm{DF}(\mathcal{X},\xi)=\frac{d}{dt}|_{t=0}\widehat{\mathrm{vol}}_{\mathcal{X}_0}(\mathrm{val}_{\xi-t\eta}),$$

up to a dimensional constant $2((n+1)!(n-1)!)$. where η is the holomorphic vector field induced by the $\mathbb{G}_m$-action on $\mathcal{X}_0$. Note also that if we normalize ξ and η i.e., to multiply suitable positive real number constants, so that $\xi-\epsilon\eta$ for $\epsilon\ll 1$ all satisfy

[10] defined and coined the name in [CS18], after [Don02] which generalizes [Fut83]

the gauge fixing condition ([CS19, Definition 6.3, Proposition 6.4]) i.e., replace η by $T_\xi(\eta) := \frac{A(\xi)\eta - A(\eta)\xi}{n}$ (cf., [LLX18, 3.9]), we can replace the above $\widehat{\mathrm{vol}}$ by the (unnormalized) volume function $\mathrm{vol}(-)$ i.e., consider instead

$$D_{-T_\xi(\eta)}\,\mathrm{vol}_{\mathcal{X}_0}(\xi)$$

as it has the same sign as $\mathrm{DF}(\mathcal{X})$ where $D_{-T_\xi}(-)$ again denotes the directional derivative. One of its benefits is that we know the *strict convexity* of $\mathrm{vol}(-)$ on C_R by [MSY08] (later generalized by [LX18, §3.2.2] to possibly singular T-varieties using the Okounkov body [Oko96, LM09]).

Example 2.16 (From one parameter subgroup to test configuration). If we consider a multi-Hilbert scheme H for affine closed subscheme of $\mathbb{A}^N$ of positive weights for an algebraic $\Bbbk$-torus T with fixed multi-Hilbert function, the centralizer of T in $\mathrm{GL}(N)$ which we denote as G, naturally acts on H. If we take one parameter subgroup $\rho\colon \mathbb{G}_m \to G$ and a point $[X] \in H$, the family over $\overline{\rho(\mathbb{G}_m)\cdot[X]}$ is automatically a test configuration. This is essentially the way [CS18, Definition 5.1] first defined the test configurations. Indeed, it is also easy to see that all test configuration, in our above more abstract sense, arises in this manner.

Note that more Donaldson-type (i.e., [Don02]) definition is also proved to be equivalent, as an analogue of [LX14], by Wu [Wu23]. There is also a partial related result in [LWX21, 4.3], which we extend in the proof of Theorem 3.6.

We also prepare the Duistermaat-Heckman measures and norm functionals analogous to the global projective setup, following [DH82, Od12a, His16, BHJ17, Der16, Wu22].

Definition 2.17. (i) ([DH82, His16]) For a $\mathbb{Q}$-Gorenstein affine cone $T \curvearrowright X \ni x$ with an additional commuting action of $\mathbb{G}_m$, we define the *Duistermaat-Heckman measure* $DH(X)$ for the $\mathbb{G}_m$-action as the probability measure

$$\lim_{c\to\infty}\left(\sum_{\substack{\lambda\in\mathbb{Z},\\ \vec{m}\in M, \langle\vec{m},\xi\rangle<c}} \frac{\dim(R_{\vec{m}})_\lambda}{\dim(R_{\vec{m}})}\quad \delta_{\frac{\lambda}{\langle m,\xi\rangle}}\right).$$

Here, $(R_{\vec{m}})_\lambda$ is the $\Bbbk$-linear subspace of $R_{\vec{m}}$ with the $\mathbb{G}_m$-weight λ and δ_a of $a \in \mathbb{R}$ denotes the Dirac measure supported on $a \in \mathbb{R}$. Note that the existence of the limit measure (convergence) easily follows from approximating ξ by rational vectors, to which [His16, BHJ17] applies.

The Duistermaat-Heckman measure of a (T- equivariant affine) test configuration $\pi\colon \mathcal{X} \to \mathbb{A}^1$ refers to that of $\mathbb{G}_m \curvearrowright \mathcal{X}_0 = \pi^{-1}(0)$.

(ii) ([Wu22, Chapter V] cf., also [BHJ17, Der16]) For a (T-equivariant affine) test configuration $(T \times \mathbb{G}_m) \curvearrowright \mathcal{X} \xrightarrow{\pi} \mathbb{A}^1$ of the Fano cone $T \curvearrowright X$, we define $(I^{\mathrm{NA}} - J^{\mathrm{NA}})(\mathcal{X})$ as $I^{\mathrm{NA}}(\varphi_{\mathcal{X}}) - J^{\mathrm{NA}}(\varphi_{\mathcal{X}})$ of [Wu22, Definition V.8], where $\varphi_{\mathcal{X}}$ is the Fubini-Study function ([Wu22, IV.2.1], compare [BJ22]) induced by $\mathcal{X}$ (or $\pi_*\mathcal{O}_{\mathcal{X}}$).

Lemma 2.18 (cf., [Od12a, 2.7(i), p.2283 l2], [BHJ17, §7.2] [Der16, 3.10, 3.11, §4])**.** *The strict positivity $(I^{\mathrm{NA}} - J^{\mathrm{NA}})(\mathcal{X}) > 0$ holds unless $\mathcal{X}$ is the trivial test configuration i.e., $(T \times \mathbb{G}_m)$-equivariantly isomorphic to $X \times \mathbb{A}^1$.*

Proof. By approximating ξ by a sequence of rational elements in $N_{\mathbb{Q}}$, due to [BHJ17, 7.8], it is enough to show that $J^{\mathrm{NA}})(\mathcal{X}) > 0$ unless $\mathcal{X}$ is the trivial test configuration. Furthermore, again by the same approximation, [BHJ17, 7.8] implies that $J^{\mathrm{NA}}(\varphi_{\mathcal{X}}) = \sup \operatorname{supp} DH(\mathcal{X}) - \int_{\mathbb{R}} \lambda DH_{\mathcal{X}}$. If this attains 0, it follows that $DH_{\mathcal{X}}$ is a Dirac measure whose support is a single point. Since we assume $\mathcal{X}$ is normal, it is then trivial test configuration. See also [Der16, 4.7]. $\square$

Problem 2.19 (cf., [Od13b])**.** *Can we prove that any K-semistable $\mathbb{Q}$-Gorenstein affine cone $(T \curvearrowright X, \xi)$ is semi-log-canonical by extending the method to* [Od13b]*? What is the differential geometric meaning behind it if it is true? (cf., e.g.* [BG14]*).*

The following is the Yau-Tian-Donaldson type correspondence, which generalizes the case of Fano manifolds mainly after [Ber16, DaSz16, CDS15, Tia15].

Theorem 2.20. *(i) (*[CS18, CS19]*) A (smooth) Fano cone $T \curvearrowright X$ admits a structure of Ricci-flat Kähler metric, which is a cone metric of some Sasaki metric on the link, with metric Reeb vector field ξ if and only if $(T \curvearrowright X \ni x, \xi)$ is K-polystable in the sense of Definition 2.15.*

(ii) ([Li21]*) A (possibly klt) Fano cone $T \curvearrowright X$ admits a structure of Ricci-flat (weak) Kähler metric, which is a cone metric of some Sasaki metric on the smooth locus of the link, with metric Reeb vector field ξ if and only if $(T \curvearrowright X \ni x, \xi)$ is K-polystable in the sense of Definition 2.15.*

Such a cone type complete Ricci-flat Kähler metric g on $(T \curvearrowright X \ni x, J, \xi)$ is often called *Ricci-flat Kähler cone metric* and such a Fano cone with Ricci-flat Kähler cone metric (cf., Prop 2.3) is often called *Calabi-Yau cone* in the literatures under some additional smoothness assumptions, which is in particular Fano cones. We call $\widehat{\mathrm{vol}}(x \in X)$ its *volume density* (cf., Proposition 2.7 for the motivation).

We need to be careful not to mix up with log canonical cone we define in Definition 2.9, which are often the affine cone over polarized projective (log canonical) Calabi-Yau varieties (see [Kol13b, 3.1]). We also note the latter singular version (ii) relies on the recent theory of "weighted" framework by using the moment maps (cf., [Ino19a, Lah19, Ino19b, HaL23, AJL23, Li21]).

Motivated by the above Definition 2.20 (ii), we introduce the following terminology, allowing singularities to extend the notions reviewed in §2.1.

Definition 2.21. A (compact) *Sasaki-Einstein space* (or *Sasaki-Einstein manifold with singularities*) is a metric space which is decomposed as $S = S^{\mathrm{reg}} \sqcup S^{\mathrm{sing}}$ (an open dense subset which is a manifold and closed singular locus) with an Einstein (Riemannian) metric g on S^{reg} which induces the metric structure, together with a Reeb vector field $\xi \in \Gamma(T_{S^{\mathrm{reg}}})$, such that the following holds:

> corresponding complex structure extends to the cone $C(S) := (S \times \mathbb{R}_{>0}) \cup 0$ which underlies a Ricci-flat weak Kähler cone (in the sense of [Li18] and the previous Theorem 2.20 (ii)).

We hope for more intrinsic equivalent definition.

2.3.1. *CM* $\mathbb{R}$*-line bundle.* The CM line bundle, first introduced by [FS90, §10, 10.3-5] and generalized by [PT06, FR06] to singular setup, gives a canonical ample line bundle on the K-moduli of canonically polarized varieties ([Fjn18, KP17, PX17]), polarized Calabi-Yau varieties ([Vie95, Od21]), Fano varieties ([CP21, XZ20]) as we expect the same for more general polarized K-stable polarized varieties (cf., [FS90, Od10]).

As in the setup for flat families of the polarized varieties, we first introduce CM $\mathbb{R}$-line bundle on the moduli stack and then discuss its descended $\mathbb{R}$-line bundle on the good moduli space. Note that it is originally motivated by the generalization theory of Weil-Petersson metrics [FS90], which was originally obtained as a Quillen-type metric for virtual vector bundles à la Donaldson.

Definition 2.22. Consider an arbitrary flat family of n-dimensional Fano cones $T \curvearrowright \mathcal{X} \to S$, whose relative coordinate ring we decompose as $\mathcal{X}=\mathrm{Spec}_S \oplus_{\vec{m}} \mathcal{R}_{\vec{m}}$. Here, $\vec{m}$ runs over the character lattice $M:=\mathrm{Hom}(T,\mathbb{G}_m)$. Then, we consider the following series

$$L(\xi,t):=\bigotimes_{\vec{m}\in M}(\det \mathcal{R}_{\vec{m}})^{\otimes e^{-t\langle \vec{m},\xi\rangle}}, \tag{8}$$

which give $\mathbb{R}$-line bundle i.e., element of $\mathrm{Pic}(S)\otimes_{\mathbb{Z}}\mathbb{R}$ for each complex number t with $\mathrm{Re}(t)>0$ (the same proof as [CS18, 4.2]). Then, by the same arguments as [CS18, 4.3], it has following expansion of meromorphic type:

$$L(\xi,t)=\frac{\mathcal{B}_0 n!}{t^{n+1}}+\frac{\mathcal{B}_1(n-1)!}{t^n}+O(t^{1-n}), \tag{9}$$

if we write in the additive notation (note that $\mathrm{Pic}(S)\otimes\mathbb{R}$ is a finite dimensional real vector space). After the following normalization

$$\mathcal{C}_0(\xi):=n!\mathcal{B}_0, \tag{10}$$

$$\mathcal{C}_1(\xi):=-\frac{(n-1)\cdot(n!)}{2}\mathcal{B}_0-(n-1)!\mathcal{B}_1, \tag{11}$$

we define the *CM* $\mathbb{R}$*-line bundle* $\lambda_{\mathrm{CM}}(T\curvearrowright\mathcal{X})$ as $-(n-1)A_1(\xi)\mathcal{C}_0+nA_0(\xi)\mathcal{C}_1$ as an element of $\mathrm{Pic}(S)\otimes\mathbb{R}$. Note that the above normalization is suitable in the sense that if ξ is regular and S is a smooth proper curve,

$$\deg\mathcal{C}_0(\xi)=(\mathcal{L}^n), \tag{12}$$

$$\deg\mathcal{C}_1(\xi)=(\mathcal{L}^{n-1}.K_{\mathcal{Y}/S}), \tag{13}$$

where $\mathcal{Y}\to S$ denotes the ξ quotient of $\mathcal{X}\setminus\sigma(S)$ where σ denote the vertex section, and $\mathcal{L}$ is the corresponding ample line bundle.

See also a related general construction in [Ino20].

2.4. **Donaldson-Sun theory and normalized volume.**

2.4.1. *Original work of Donaldson-Sun.* In [DS17], for log terminal singularities with weak Kähler-Einstein metrics, the analytic and even algebraic nature of the metric tangent cone is started to explore. This is somewhat a local analogue of [DS14]. As a technical assumption, [DS17] assumes the metrized log terminal singularities appear in the non-collapsed Gromov-Hausdorff limits (polarized limit space) of Kähler-Einstein manifolds. After [DS17], more algebraic works by [Li18, LL19, CS18, CS19] appear, which refines and more algebraize the conjectural picture which is now a theorem by [LX20, LX18, LWX21, BLQ22, XZ22, XZ21] (cf., the surveys [LLX18, Zhu23b]).

We briefly review this whole story in this subsection. In the work of [DS14, DS17], the following setup is mainly considered (although being somewhat more general).

Setup 1. Consider a flat projective family of n-dimensional smooth polarized varieties $f\colon (\mathcal{X},\mathcal{L}) \to B$ over a connected base algebraic $\Bbbk$-scheme B with a section $\sigma\colon B \to \mathcal{X}$ which satisfies $K_{\mathcal{X}} \equiv_B \mathcal{L}^{\otimes k}$ with a constant $k \in \mathbb{Q}$ [11] Take a sequence of points $b_i \in B(i = 1, 2, \cdots)$ whose f-fibers are $(p_i = \sigma(b_i) \in X_i, L_i)$ and admit Kähler-Einstein metrics with the Kähler class in $2\pi c_1(L_i)$.

We take a sequence of compact Kähler-Einstein manifolds $(p_i = \sigma(b_i) \in X_i, L_i, g_i, \omega_i)_{i=1,2,\cdots}$ which satisfies the *non-collapsing condition*:

there is a positive constant c such that if we consider the geodesic ball $B_r(p_i)$ in X_i with center $p_i = \sigma(b_i)$ of radius $r \in [0, 1]$ we have [12]

$$\mathrm{vol}(B_r(p_i)) \geq cr^{2n} \text{ for } i \gg 0. \tag{14}$$

Following may be worth noted.

Lemma 2.23 (non-collapsing condition as log-terminality)**.** *If f extends to a locally stable projective family $\overline{\mathcal{X}}(\supset \mathcal{X}) \to \overline{B}$ and b_i converges to a point $b \in \overline{B}$ and further that $f^{-1}(b)$ is K-polystable. Then, the above non-collapsing condition* (14) *holds if and only if $\sigma(b) \in f^{-1}(b)$ is log terminal.*

Proof. The only if direction follows from [DS14] and the characterization of log terminality as local volume boundedness of the adapted measure in [EGZ09]. If $k < 0$, then both conditions automatically hold. In the case $k = 0$, the if direction also follows from [EGZ09, DGG23] (see also [Od21]). In the case $k > 0$, it follows from [BG14, Son17, DGG23]. □

Theorem 2.24 ([DS17])**.**
(i) *(loc.cit 1.1) Passing to a subsequence of i, there is a polarized limit space*[13] *$Z \ni p$ with a singular Kähler metric g as a complex analytic space.*
(ii) *(loc.cit 1.3) Suppose we take a sequence of $(p \in Z, ag)$ with $a \to \infty$, then there is a unique polarized limit space as a log terminal affine variety C with Ricci-flat Kähler cone (singular)*

[11] [DS14, DS17] only requires to fix n and the constancy of the volume of the fiber $\mathcal{L}_b^n$ for each $b \in B$.

[12] (or equivalently the same for r in $[0, \epsilon]$ for small fixed ϵ independent of i)

[13] the enhanced notion of pointed Gromov-Hausdorff limit to consider complex structure as defined in [DS14, DS17]

metric on which a positive dimensional algebraic $\mathbb{C}$-torus $T(\mathbb{C})$-acts.

(iii) *(loc.cit §2.3, §3) The above construction of C from $Z \ni p$ can be separated into 2-steps i.e., $(Z \ni p) \rightsquigarrow W \rightsquigarrow C$, where W is a Fano cone.*

The above C is called *(local) metric tangent cone* of $p \in Z$ with respect to g. There are also related results in [VdC11]. The above item (ii) above inspired the later developments in particular by the following conjecture.

Conjecture 2.25 ([DS17, 3.22])**.** *For any log terminal analytic space $x \in X$ with a weak Kähler-Einstein metric, the metric tangent cone only depends on the complex analytic germ of $x \in X$ (or $\widehat{\mathcal{O}_{X,x}}$, equivalently).*

In the next subsection, we also review the developments after the above conjecture.

2.4.2. *Normalized volume* [Li18]. Motivated by the above Donaldson-Sun theory, there were developments of algebro-geometric machinery which in particular partially solved the above conjecture under Setup 1. Here we review the developments. Henceforth, we fix a closed point $x \in X$ as a n-dimensional klt singularities over k, on which we put the trivial valuation. We write the space of valuations of X with the center x, as $\mathrm{Val}_x(X)$. Note that there is a natural bijection:

$$\mathrm{Val}_x(X) \simeq \mathrm{Val}_k(\mathcal{O}_{x,X}) \simeq \mathrm{Val}_k(\widehat{\mathcal{O}_{x,X}}) \tag{15}$$

For each v, we define the following.

Definition 2.26 (Local volume)**.** ([ELS03], [LM09])

$$\mathrm{vol}_x(v) := \limsup_{m\to\infty} \frac{\dim_k(\mathcal{O}_{X,x}/\{f \mid v(f) \geq m\})}{m^n/n!}$$

This lim sup is known to be lim (cf., *loc.cit*).

In the case X has a structure of affine cone and v comes from a positive vector field, recall that Lemma 2.14 (4) (from [MSY06, CS18]) gives another expression of this (unnormalized) volume function $\mathrm{vol}(-)$ in terms of multi-Hilbert function.

On the other hand, we take the subspace of the space (15) defined by $A_X(v) = n$ (cf., [MSY08, CS19]) and write as

$$\mathrm{Val}^n_x(X) \simeq \mathrm{Val}^n_k(\mathcal{O}_{x,X}) \simeq \mathrm{Val}^n_k(\widehat{\mathcal{O}_{x,X}}). \tag{16}$$

Definition 2.27 ([Li18])**.** Define the *normalized volume* of $x \in X$ for v as

$$\widehat{\mathrm{vol}}_x(v) := A_X(v)^n \mathrm{vol}_x(v). \tag{17}$$

Here, if $A_X(v) = +\infty$, then we also set $\widehat{\mathrm{vol}}_x(v) := +\infty$. Equivalently, we consider $\mathrm{vol}_x(v)$ on the normalized section $\mathrm{Val}_x^n(X)$.

By the effect of normalization using the log discrepancy A_X, for any positive real number λ,

$$\widehat{\mathrm{vol}}(\lambda v) = \widehat{\mathrm{vol}}(v) \tag{18}$$

so that the normalized volume gives a function

$$\widehat{\mathrm{vol}} \colon (\mathrm{Val}_x(X) \setminus \{v_{\mathrm{triv}}\})/\mathbb{R}_{>0} \to \mathbb{R} \cup \{+\infty\}. \tag{19}$$

Here, v_{triv} is the trivial valuation and the action of $\mathbb{R}_{>0}$ on $(\mathrm{Val}_x(X) \setminus \{v_{\mathrm{triv}}\})$ is simply given by the multiplication on the real valuations. The quotient

$$(\mathrm{Val}_x(X) \setminus \{v_{\mathrm{triv}}\})/\mathbb{R}_{>0} \tag{20}$$

or homeomorphic $\mathrm{Val}_x^n(X)$ is often called *non-archimedean link* after the topological theory of links of complex algebraic singularities ([Sas60, Bri66, Mil69, BG07]). We note that clearly the latter normalization convention is motivated by the Theorem 2.8 (ii).

Remark 2.28. We also note that if v comes from the plt blow up $\pi \colon X' \to X$ with the exceptional divisor (Kollár component[14]) E, $\widehat{\mathrm{vol}}(v)$ equals the generalized log Donaldson-Futaki invariant $\mathrm{DF}(X' \to X, -(K_{X'/X} + E))$ defined in [Od15b]. In the theory, blow up π is regarded as an analogue of test configuration in the sense of [Don02].

2.5. **Donaldson-Sun 2-step degeneration and its algebraization.**

2.5.1. *General procedure.* By making use of the theory of normalized volumes in the previous section, the Donaldson-Sun 2-step degeneration theory is now put in a format of algebraic geometry and developped, due to the work of [Li18, LL19, Blu18, CS18, CS19, LX18, LWX21, Xu20, XZ22, XZ21, BLQ22]. We briefly summarized the conclusion as follows.

[14] in the sense of [LX18] henceforth, see also the original [Sho96, 3.1], [Pro00, 2.1]

Theorem 2.29. *Suppose $\Bbbk$ is algebraically closed field of characteristic 0. We fix a pointed n-dimensional log-terminal $\Bbbk$-variety $x \in X$, consider the stalk $\mathcal{O}_{X,x}$ and completed stalk $\widehat{\mathcal{O}_{X,x}}$ (Only the formal germ of x matters below, hence X is not assumed to be proper in general). Then we have the following.*

(i) *(*[Blu18, Xu20, XZ21, BLQ22]*) There is a unique (necessarily quasi-monomial* [Xu20]*) valuation $v = v_X \in \mathrm{Val}_x(X)$ up to the $\mathbb{R}_{>0}$-action, which minimizes* [15] *the normalized volume $\widehat{\mathrm{vol}}(-)$. We denote the rank of (the groupification M of) $\Gamma := \mathrm{Im}(v_X)$ as r and the dual lattice as $N := \mathrm{Hom}(M, \mathbb{Z})$.*

(ii) *(*[LX20, LX18, XZ22]*) $\mathrm{gr}_v(\mathcal{O}_{X,x}) := \oplus_{s\in\Gamma}(\{f \in \mathcal{O}_{X,x} \mid v(f) \geq s\}/\{f \in \mathcal{O}_{X,x} \mid v(f) > s\})$ is of finite type over $\Bbbk$ (*[XZ22] *for general $r \geq 1$ case). There is a natural action of the alglebraic $\Bbbk$-torus $T := N \otimes \mathbb{G}_m$ on $W := \mathrm{Spec}(\mathrm{Gr}_v)$ and this provides K-semistable Fano cone structure to W (*[LX20, LX18]*). We denote its vertex as x_W.*

(iii) *(*[DS14, CS18, LWX21, Li21]*) If $\Bbbk = \mathbb{C}$, C has a (weak) Ricci-flat Kähler cone metric. Further, if X is compactified to the (non-collapsed) polarized limit space of Kähler-Einstein manifolds in the sense of* [DS14], [DS17, §2.1]*, then C is a metric tangent cone of $X \ni x$ and is unique.*

(iv) *(*[LWX21]*) There is a (non-canonical) affine test configuration of W to a unique K-polystable Fano cone C. In particular, C is uniquely determined by germ of $x \in X$ (i.e., Conjecture 2.25 by Donaldson-Sun* [DS17] *holds in the case of Setup 1).*

2.5.2. *Local vs global volume.* The following type of inequalities are very important to study K-stability of singular $\mathbb{Q}$-Fano varieties. In the dimension 2, [OSS16] originally used the Bishop-Gromov type inequality to deduce a similar (weaker) result which we also review.

Theorem 2.30 ([Liu18])**.** *For any K-semistable $\mathbb{Q}$-Fano variety X and x its closed point, we have*

$$(-K_X)^n \leq \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(v). \tag{21}$$

Here is a weaker version, which in turn is a straightforward consequence of the Bishop-Gromov inequality for orbifolds.

Theorem 2.31 (cf., [OSS16])**.** *If there is a quotient singularity of the type $\mathbb{C}^n/\Gamma$ (Γ is a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$) on a Kähler-Einstein*

[15] analogue and the generalization of the volume minimization in [MSY06, MSY08] (called "Z-minimization" in *loc.cit*)

$\mathbb{Q}$-Fano variety X, we have

$$(-K_X)^n \leq \frac{1}{|\Gamma|} \cdot \frac{2(2n-1)^n \cdot n!}{(2n-1)!! = (2n-1)\cdot(2n-3)\cdots 3\cdot 1}. \tag{22}$$

The estimation comes from the comparison of the (real) space form, rather than the complex space form $\mathbb{P}^n$, on which we can not expect (complex) algebraic variety structure in general.

Proof. The proof in the 2-dimensional case ([OSS16, 2.7] cf., also [Tia90][16]) naturally extends. Apply (orbifold version of) the Bishop-Gromov comparision theorem to X to obtain

$$\frac{(2\pi)^n(-K_X)^n/n!}{\mathrm{vol}(S^{2n}(2n-1))} \leq \frac{1}{|\Gamma|}.$$

Hence the assertion follows from

$$\begin{aligned}\mathrm{vol}(S^{2n}(2n-1)) &= (2n-1)^n \cdot \frac{(2n+1)\pi^{n+\frac{1}{2}}}{\Gamma(n+\frac{3}{2}) = (n+\frac{1}{2})(n-\frac{1}{2})\cdots\frac{1}{2}\times\sqrt{\pi}} \\ &= (2n-1)^n \cdot \frac{2\pi^n}{(n-\frac{1}{2})(n-\frac{3}{2})\cdots\frac{1}{2}}.\end{aligned}$$

$\square$

2.5.3. *Toric case.* We can see the strength of the K-stability theory and Donaldson-Sun theory by one of simplest example - toric case. As first pointed out by [CS19, §1], it readily re-proves a somewhat weaker version of the following celebrated result by Futaki-Ono-Wang [FOW09] and its generalization to singular case [Ber20]. Note that the condition in *loc.cit* Theorem 1.2, the meaning of the assumptions in *loc.cit* Theorem 1.2 (on the vanishing of the de Rham cohomology class of the contact bundle and the positivity basic first Chern class of the normal bundle of the Reeb foliation) is later clarified to be equivalent to the ($\mathbb{Q}$-)K-triviality which matches Definition 2.9 and Lemma 2.11. Hence, we restate their result in that manner, although in a somewhat weak statements (on the stability side).

Theorem 2.32 (cf., [FOW09], [Ber20])**.** *Suppose $T \curvearrowright X = C(S)$ is a toric Fano cone i.e., a (log terminal) Fano cone which admits an action of algebraic $\Bbbk$-torus T' which preserves the structure. We denote $N' := \mathrm{Hom}(\mathbb{G}_m, T')$ and $M' := \mathrm{Hom}(T', \mathbb{G}_m)$. Then X is K-semistable and T'-equivariantly K-polystable as Fano cone with respect to a (unique) Reeb vector field $\xi \in N' \otimes \mathbb{R}$.*

[16] cf., Remark 2.8 of [OSS16]

Proof. The (unique) $\widehat{\mathrm{vol}}$-minimizing valuation $v = v_X$ of $\mathcal{O}_{X,x}$ is T'-invariant hence is a toric valuation in the sense of [Blu18, §8]. Thus, if we write $C(S) = \operatorname{Spec} R = \oplus_{\vec{m}\in M'} R_{\vec{m}}$ for $M' := \operatorname{Hom}(T', \mathbb{G}_m)$ as the decomposition of the coordinate ring by the T-action, it is easy to see $\mathrm{gr}_v(R) = R$ so that $W = X$ in the Theorem 2.29. In particular, X is K-semistable with respect to a Reeb vector field in $N \otimes \mathbb{R}$ and $T \subset T'$. Further, if there is a T'-equivariant special test configuration $(\mathcal{X}, \xi, \eta)$, then since $\dim(X) = \dim(T')$, $T \times \mathbb{G}_m$ can not act effectively on any component of $\mathcal{X}_0$ so that $\mathcal{X}$ is a T'-equivariant product test configuration. Hence, because $\mathcal{X}_0$ has vanishing Futaki invariant with respect to ξ by its K-semistability, it follows that X is T'-equivariantly K-polystable with respect to ξ'. □

This contrasts with the fact that toric Fano manifolds do not necessarily have Kähler-Einstein metrics, but this is due to the flexibility of the (abstract) Reeb vector fields.

2.6. **Affine generalized test configurations vs filtered blow ups.** As a preparatory general material, we excerpt a part of [Od24b] on the theory of generalized test configurations and give an alternative description of affine generalized test configuration for general affine varieties (cf., also [Ino22, BJ21]). In short, generalized test configurations correspond to certain ideals sequences or their (filtered) blow ups. This generalizes the perspective of [Od13a].

Firstly, we recall the definition of generalized test configurations.

Definition 2.33. For an affine toric variety U_τ which corresponds to a rational polyhedral cone $\tau \in N \otimes \mathbb{R}$ as in Notation 1, a *generalized test configuration* of affine variety X is a faithfully flat T-equivariant affine morphism $p\colon \mathcal{Y} \twoheadrightarrow U_\tau$.

If general fibers are affine cones in the sense of Definition 2.15 with respect to a p-fiberwise action of an additional algebraic $\mathbb{k}$-torus T', we further require the following:

> $\mathcal{Y} = \operatorname{Spec} \mathcal{R}$ with a finite type $\Gamma(U_\tau)$-algebra $\mathcal{R}$ that decomposes as $\mathcal{R} = \oplus_{\vec{m}\in M} \mathcal{R}_{\vec{m}}$, to the T'-eigen-$\Gamma(U_\tau)$-submodules, then $\mathcal{R}_{\vec{m}}$ for each $\vec{m}$ is a locally free module.

Example 2.34. For a klt affine variety $X \ni x$, take the valuation v_X of minimizing normalized volume ([Li18, Blu18]) and set W as $\operatorname{Spec}(\mathrm{gr}_{v_X}(\mathcal{O}_{X,x}))$. Denote the groupificiation N of $\mathrm{Im}(v_X)$ and a rational polyhedral cone $\tau \subset N$ which includes ξ.

Naturally, there is a canonical generalized test configuration $\pi_\tau\colon \mathcal{X}_\tau \twoheadrightarrow U_\tau$ of X whose fiber over p_τ is W ([Tei03, LX18]). This gives a positive weight deformation of W in the sense of [Od24b]. See more details in [Od24b, §2] (cf., also [LX18]).

Lemma 2.35. *Fix an affine* $\Bbbk$*-variety* $\mathrm{Spec}(R) = X$ *and a rational polyhedral cone* $\tau \subset N \otimes \mathbb{R}$ *as before. Then, the following two sets have natural bijective correspondence with each other:*

(i) *the set of isomorphic classes of (affine) generalized test configuration over* U_τ *which dominates* [17] $X \times U_\tau$*, denoted by* $\pi\colon \mathcal{X} = \mathrm{Spec}(\mathcal{R}) \twoheadrightarrow \mathrm{Spec}(\Bbbk[\mathcal{S}_\tau])$ *(here,* $\mathcal{S}_\tau := \{x \in M \otimes \mathbb{R} \mid \langle x, \tau\rangle \subset \mathbb{R}_{\geq 0}\} \cap M$*)*

(ii) *the set of sequence of ideals of* R *which we write* $\{I_{\vec{m}} \subset R\}_{\vec{m}\in M}$ *(or* $\vec{m} \mapsto I_{\vec{m}}$*) that satisfies the following:*
- $I_{\vec{m}} \cdot I_{\vec{m}'} \subset I_{\vec{m}+\vec{m}'}$ *for any* $\vec{m}, \vec{m}' \in M$*,*
- $I_{\vec{m}} \subset I_{\vec{m}'}$ *for any* $\vec{m}, \vec{m}' \in M$ *with* $\vec{m}' - \vec{m} \in \mathcal{S}_\tau$*,*
- *and* $\oplus_{\vec{m}} I_{\vec{m}} \cdot \chi(\vec{m})(= \mathcal{R})$ *is of finite type over* $\Bbbk$ *(or equivalently, over* $\Bbbk[\mathcal{S}_\tau]$*),*
- $I_{\vec{m}} \neq R$ *for (some or any)* $\vec{m} \in -\mathcal{S}_\tau^o$*.*

(ii) *is analogous to the description of equivariant toric vector bundles by family of filtrations in* [Kl90] *(see also* [Ino19a, §2.2], [BJ21, A3]*). Indeed, for each* $\xi \in \tau$ *and each* $k \in \mathbb{R}_{\geq 0}$*, one can define graded sequence of ideals as*

$$\mathfrak{a}_{\xi,k} := \sum_{\vec{m}, \langle \vec{m}, \xi\rangle \leq -k} I_{\vec{m}}. \tag{23}$$

Furthermore, below we assume X *is normal and consider the following sets (3). Then, the above set* (i) *with normal* $\mathcal{X}$ *(or equivalently,* (ii) *with normal* $\mathcal{R}$*) has a natural map to the set (3) for each* $\xi \in \tau^o$*.*

(3) *the relative isomorphism class (over* X*) of a birational projective morphism* $h\colon Y_\xi \to \mathrm{Spec}(R)$ *with an effective* h*-antinef* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* E_ξ*.*

Proof. We first prove the bijection between the first two i.e., (i) and (ii). Since $\mathcal{R}$ is flat over $\Bbbk[\mathcal{S}_\tau]$, $\mathcal{R} \subset \mathcal{R} \otimes_{\Bbbk[\mathcal{S}_\tau]} \Bbbk[M] \simeq R[M]$. We put $I_{\vec{m}} := \{f \in R \mid \chi(\vec{m})f \in \mathcal{R}\}$ for each $\vec{m} \in M$. The condition that $\mathcal{X}$ dominates $X \times U_\tau$ ensures that $I_{\vec{m}}$ are ideals of R and the last condition of $I_{\vec{m}}$ in the item (ii) is equivalent to the surjectivity of π i.e., $\mathcal{R} \neq (\oplus_{\vec{m}\in\mathcal{S}_\tau\setminus 0} R)\mathcal{R}$.

Next we discuss the map from (ii) with normal $\mathcal{R}$ to the set (3). Take and fix ξ. As it may be even irrational, we take a rational approximation $\xi'_q (q = 1, 2, \cdots)$ i.e., $(0 \neq)\xi'_q \in N_{\mathbb{Q}} \cap \tau$ and $\xi'_q \to \xi$ for $q \to \infty$.

[17] analogous to the blow up type (semi) test configurations in [Od13b]

We take a shriking sequence of rational polyhedral cones $\tau_i(i=1,2,\cdots)$ of $N_{\mathbb{R}}$ such that $\tau_i \supset \tau_{i+1}$ for each i, $\xi \in \tau_i^o$ and $\cap_i \tau_i = \mathbb{R}_{\geq 0}\xi$. We consider the sequence of localization

$$\begin{aligned}\mathcal{R} \subset & \quad \mathcal{R} \otimes_{\Bbbk[\mathcal{S}_\tau]} \Bbbk[\mathcal{S}_{\tau_1}] \\ \subset & \quad \mathcal{R} \otimes_{\Bbbk[\mathcal{S}_\tau]} \Bbbk[\mathcal{S}_{\tau_2}] \\ \cdots & \\ \subset & \quad \mathcal{R} \otimes_{\Bbbk[\mathcal{S}_\tau]} \Bbbk[\Gamma_{\geq 0}] (=: \mathcal{R}_\xi),\end{aligned}$$

where the last $\Bbbk[\Gamma_{\geq 0}]$ is the global N-ring ([Od24b]) for ξ. Recall that $\operatorname{Spec} \mathcal{R}_\xi \to \Delta_\xi$ is the associated isotrivial degeneration to the generalized test configuration $\mathcal{X}$ (compare [Od24b]). With respect to the T-action, we similarly decompose $\mathcal{R}_\xi = \oplus R_{\vec{m},\xi}$ as $\mathcal{R}$. Then we claim its following simple structure.

Claim 2.36. *There is a strictly increasing sequence of positive real numbers $r_i(i=1,2,\cdots)$ and ideals $I^{(i)}(i=1,2,\cdots)$ of R with $I^{(i)} \subset I^{(i+1)}$ for each i, which satisfies the following: For any $\vec{m} \in M$ if $\langle \vec{m}, -\xi \rangle \in [r_i, r_{i+1})$, we have $R_{\vec{m},\xi} = I^{(i)}$.*

proof of Claim 2.36. The claim follows straightforwardly from the Noetherian property of R and the fact that $R_{\vec{m},\xi} \subset R_{\vec{m}',\xi}$ if $\langle \vec{m}' - \vec{m}, \xi \rangle \geq 0$. Note that the latter follows from the $\Bbbk[\Gamma_{\geq 0}]$-algebra structure of $\mathcal{R}_\xi$. $\square$

For each q, since $\langle \xi'_q, M \rangle \simeq \mathbb{Z}$, one can define the projective spectrum $h_q \colon Y_q(:= \operatorname{Proj}_R \oplus_{k \in \langle \xi'_q, M \rangle \cap \mathbb{R}_{\leq 0}} \mathfrak{a}_{\xi'_q,k}) \to X$, birational projective over X. Further this construction naturally associates an effective h_q-antiample $\mathbb{Q}$-divisor $E_{\xi'_q}$. Now we analyze how this $(Y_{\xi'_q}, E_{\xi'_q})$ behaves as q increases.

We now consider $(\mathcal{R} \otimes_{\Bbbk[\mathcal{S}_\tau]} \Bbbk[\mathcal{S}_{\tau_i}])|_{-\mathcal{S}_\tau} \subset (\mathcal{R}_\xi)|_{-\mathcal{S}_\tau}$. Note that the latter is a $(-)\mathcal{S}_\tau$-graded finite type R-algebra and the former enlarges as i increases. Therefore, there is some $I \in \mathbb{Z}_{>0}$ such that $(\mathcal{R} \otimes_{\Bbbk[\mathcal{S}_\tau]} \Bbbk[\mathcal{S}_{\tau_i}])|_{-\mathcal{S}_\tau} = (\mathcal{R}_\xi)|_{-\mathcal{S}_\tau}$ for any $i \geq I$. We take its finite generators $f_j \in R_{\vec{m}_j,\xi}(j=1,\cdots,l)$ and write them as $f_j = \sum_{k=1,\cdots,k_i} g_{j,k} \cdot \chi(m_{j,k})$ with $g_{j,k} \in R_{\vec{m}_j - m_{j,k}}$ and $m_{j,k} \in \Gamma_{\geq 0}$. By replacing $\{\xi'_q\}$ by its subsequence $\{\xi'_q\}_{q>Q}$ for large enough Q if necessary, we can and do assume $\langle m_{j,k}, \xi'_q \rangle \geq 0$ for any q. Therefore, it follows that $\{\mathfrak{a}_{\xi'_q,k}\}_{k \in \mathbb{R}_{\geq 0}}$ (as a set) does not depend on q and is equal to $\{I^{(i)}\}_i$ and further that jumping numbers of k only continuously changes for ξ'_q. This implies that Y_q are all isomorphic and $E_{\xi'_q}$ converge to a $\mathbb{R}$-divisor (after the replacement of $\{q\}$ to its subsequence to $q > Q$ as done above), which we denote as E_ξ. From the construction, E_ξ is relatively nef. We complete the proof. $\square$

Under the above correspondence in Lemma 2.35, further we take a rational approximation $\xi'_q (q = 1, 2, \cdots)$ i.e., $(0 \neq)\xi'_q \in N_{\mathbb{Q}} \cap \tau$ such that

$$\langle \xi - \xi'_q, \tau^\vee \rangle > 0 \tag{24}$$

and $\xi'_q \to \xi$ for $q \to \infty$. Then, for each q, take a sufficiently divisible positive integers d_q so that $d_q \xi'_q \in N$, consider the toric morphism $\mathbb{A}^1 \to U_\tau$ which correpsonds to $d_q \xi'_q$, and the pullback of $\mathcal{X}$ as $\mathcal{X}_{d_q \xi'_q} := \mathcal{X}_{U_\tau} \times \mathbb{A}^1 \to \mathbb{A}^1$. We denote its central fiber as $(\mathcal{X}_{d_q \xi'_q})_0$. Then, the following holds.

Lemma 2.37. *$(\mathcal{X}_{d_q \xi'_q}, (\mathcal{X}_{d_q \xi'_q})_0)$ is log canonical for $q \gg 0$ if and only if (Y_ξ, E_ξ) is log canonical.*

Proof. Since $E_{\xi'_q}$ increases (under the assumption (24)) and converge to E_ξ, the latter is equivalent to the log canonicity of $(Y_q, E_{\xi'_q})$ for $q \gg 0$. Then, the assertion follows from [LWX21, 2.21] (whose proof extends to the non-cone situation). □

Note that the above two lemmas generalize [LWX21, 2.20, 2.21] in two directions: to allow non-cones for general fibers, and to allow *generalized* test configurations with higher dimensional case.

2.7. **(Higher) Θ-stratification.** One of the main tools in the algebraic construction of K-moduli of Fano varieties (cf., [Xu20, BHLLX21, LXZ22]) is stack-theoretic incarnation and generalization of the theory of Harder-Narasimhan filtration, which is introduced by [HL14] and called the Θ-*stratification*. Roughly put, it allows a family-wise optimal destabilization along one parameter family. Correspondingly, the properness-type criterion after Langton [Lan75] is established in [AHLH23]. Because of the irrational nature of $\widehat{\mathrm{vol}}$-minimizing valuations (associated to the Reeb vector fields), it is more convenient and suitable to use a generalization of [HL14, AHLH23] to multi-variable parameter family setup and degenerations in its irrational direction, which is done as a part of a theory of [Od24b, §2, Ex 2.10]. We briefly review some of the main definitions and theorems below, in a simplified weaker form, following the toric Notations 1.

Definition 2.38 ([HL14, Od24b]). Generalizing [HL14] ($r = 1$ case), a *higher* Θ-*stratum* of rank r and type τ in $\mathcal{M}$ consists of a union of connected components $\mathcal{Z}^+ \subset \mathrm{Map}(\Theta_\tau, \mathcal{M})$, where $\mathrm{Map}(-)$ denotes the mapping stack (cf., [AHR20, AHR19]), such that the natural evaluation morphism $\mathrm{ev}_{(1,\cdots,1)} \colon \mathcal{Z}^+ \to \mathcal{M}$ is a closed immersion.

Definition 2.39 ([HL14, Od24b]). For a quotient algebraic $\Bbbk$-stack by a linear algebraic $\Bbbk$-group, a (finite) *higher* Θ-*stratification* in $\mathcal{M}$ consists of

(i) a finite set of real numbers $1 \in \Gamma \subset (-\infty, 1] \subset \mathbb{R}$,
(ii) closed substacks $\mathcal{M}_{<c}$ of $\mathcal{M}$ for $c \in \Gamma$ which are monotonely enlarging i.e., $\mathcal{M}_{<c} \supset \mathcal{M}_{<c'}$ if $c > c'$ in Γ. Naturally, we can define substacks of $\mathcal{M}$ as $\mathcal{M}_{>c}, \mathcal{M}_{=c}, \mathcal{M}_{\geq c}, \mathcal{M}_{\leq c}$.
(iii) higher Θ-strata structure of type N, τ (Def 2.38) on each $\mathcal{M}_{=c}$ for each $c \in \Gamma$. We do not require N, τ to be identified for different c.

For Definition 2.38, we have the following Langton type theorem, for instance. Clearly, we can use it iteratively to obtain a generalization of [AHLH23, 6.12] for higher Θ-stratification.

Theorem 2.40 (Higher Θ-stable reduction [AHLH23, Od24b]). *For any quotient algebraic stack $\mathcal{M} = [H/G]$ with finite type scheme H and linear algebraic group G over $\Bbbk$, consider any morphism $f \colon \Delta \to \mathcal{M}$ from $\Delta = \operatorname{Spec} \Bbbk[[t]]$, to $\mathcal{M}$ whose closed point c maps into a closed substack $\mathcal{Z}^+$ corresponding higher Θ-strata for the cone $\tau \ni \xi$, while the generic point maps outside $\mathcal{Z}^+$. We denote the restriction of f to $\operatorname{Spec}(K)$ as f^o.*

Then, after a finite extension of R and shrinking τ, there is a toric morphism $e \colon U_\tau \to \mathbb{A}^1$ (we denote its complete localization $\operatorname{Spec} \widehat{\mathcal{O}}_{U_\tau, p_\tau} \to \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathbb{A}^1,0} = \Bbbk[[t]])$ also as e) and a modification of f as

$$f|_\xi \colon \operatorname{Spec} \widehat{\mathcal{O}}_{U_\tau, p_\tau} \to \mathcal{M},$$

which extends $f^o \circ e$ and sends $p_\tau(\kappa)$ to a point outside $\mathcal{Z}^+$, but still isotrivially degenerates to a point in $\mathcal{Z}^+$.

The original statements in [Od24b] are somewhat stronger, more general and canonical, for some irrational element $\xi \in \tau$. We use the above theorem 2.40 both for the following construction of K-moduli of Calabi-Yau cones in the next section §3 and some other later work.

3. Moduli of Calabi-Yau cones

Now we consider the moduli of K-polystable Fano cones, i.e., Calabi-Yau cones, depending on the various preparations in the previuos section.

3.1. **Boundedness.** In $n = 2$ ([HLQ23, LMS23]) and $n = 3$ ([LMS23, Zhu23a]) case, the following boundedness result is proved, and is now generalized to any n by [XZ24] [18] after the corresponding results and arguments in log Fano varieties case [Jia20].

Theorem 3.1 (boundedness [XZ24])**.** *For a fixed positive integer n, n-dimensional K-semistable $\mathbb{Q}$-Fano cone X of the normalized volume at least V are bounded.*

Corollary 3.2. *In particular, for each n and V, there are only finitely many choices of the (metric) Reeb vector fields with minimum normalized volume.*

Recall that multi-Hilbert schemes [HM04] parametrizes affine closed schemes inside $\mathbb{A}^N$ with a linear action of an algebraic torus T with fixed multi-Hilbert series. Using this, one can rephrase the above theorem:

Corollary 3.3. *For fixed n, V, n-dimensional K-semistable $\mathbb{Q}$-Fano cone X of the normalized volume at least V (for fixed n, V) are parametrized inside a finite union of certain projective multi-Hilbert schemes for all positive weights.*

proof of Corollaries. We write the proof for the convenience. For each n-dimensional K-semistable $\mathbb{Q}$-Fano cone $X = \operatorname{Spec} \oplus_{\vec{m}} R_{\vec{m}}$ of the normalized volume at least V, take a positive vector field ξ i.e., with $\langle \vec{m}, \xi \rangle > 0$ unless $R_{\vec{m}} = 0$. By Theorem 3.1, combined with the latter half arguments of the proof of [Od24b, Lemma 3.14], we can take a uniform finitely generated regular submonoid $\Gamma_{\geq 0} \subset M$ generated by the set of the extremal integral vectors $S \subset \Gamma_{\geq 0}$ which contains all the moment monoids of X and their $\{R_{\vec{m}}\}_{\vec{m} \in S}$ generate the coordinates ring R of X. S gives uniform embedding of $X \to \mathbb{A}^N$ and for small enough rational polyhedral cone $\tau \subset N \otimes \mathbb{R}$ which includes ξ, we have strict positivity $\langle \tau, S \rangle \subset \mathbb{R}_{>0}$. This gives an isomorphism $T \simeq \mathbb{G}_m^r$ and so that the weights on $\mathbb{A}^r$ of each $\mathbb{G}_m$ are all positive. In particular, by [HS17, Theorem 1.1, Corollary 1.2], there is a *projective* multi-Hilbert scheme (possibly non-connected) which parametrizes all n-dimensional K-semistable Fano cone X of the normalized volume at least V and the Reeb vector field ξ. The finiteness claim of the Reeb vector fields follows from the characterization of the K-semistability [CS18] in terms of the minimized normalized volumes [LX18, 1.1], combined with the constructible semi-continuity of the minimized normalized volumes [BL21, Xu20] (See also related arguments later during the proof of Theorem 3.6). □

We consider the obtained finite union of projective multi-Hilbert schemes and take its union of components which parametrize those with

[18] we learnt this result after the completion of our manuscript

normalized volume exactly V, and denote by H_V in our paper (though it also depends on n in general, just for simplicity). We sometimes fix ξ among the finite choices. Note that the leading coefficient $a_0(\xi)$ of $F(\xi, t)$ is V because of Theorem 2.7. Hence, in particular, for any pointed log terminal variety $x \in X$ parametrized in H_V, the normalized volume $\widehat{\mathrm{vol}}(x \in X)$ is at most V from the definition.

Below, due to the finiteness of ξ, we can and do fix it in addition to the fixing of volume V. We then replace T by the minimum algebraic subtorus which contains ξ in its Lie algebra, if necessary and set G to be is the commutator of $T \subset \mathrm{GL}(N)$, which is reductive. We then consider locus $H_V^{\mathrm{ss}}(\xi) := \{b \in H_V \mid 0 \in X_b \text{ is K-semistable with respect to } T, \xi\}$ and replace H_V by its closure. Note that naturally the reductive group G preserves $H_V^{\mathrm{ss}}(\xi) \subset H_V$ (see e.g., [HM04, DS17]).

From now on, we want to prove that the locus of K-semistable Fano cones of the normalized volume V and the Reeb vector field ξ, which we later denote as $B_V^{\mathrm{ss}}(\xi)$ makes sense and the corresponding quotient stack $[B_V^{\mathrm{ss}}(\xi)/G]$ admits a coarse moduli algebraic space which is proper.

3.2. **Locally closedness.** We consider the universal family over H_V which we denote as $(\mathbb{A}^N \times H_V) \supset \mathcal{U}_V \twoheadrightarrow H_V$ with the fixed multi-Hilbert series. Since the weights of the action of $T = \mathbb{G}_m^r \curvearrowright \mathbb{A}^N$ are all positive, the fibers contain the origin i.e., $\mathcal{U}_V \supset 0 \times H_V$ and the action are good in the sense of [LS13] (if the fibers are normal).

We take the locus B'_V of H_V which parametrizes normal and $\mathbb{Q}$-Gorenstein fibers by Kollár's hull and husk [Kol08, Kol22]. (There will be also remarks later on the subtleties on definition of corresponding families and their logarithmic generalizations: Remarks 3.18, 3.19.) Further, inside B'_V, we take the open locus B_V where the vertices are log terminal, due to its openness.

By the standard fact (cf., e.g., [Kol13b, Lemma 3.1]), we see that for any algebraic subtorus $T' \subset T$ of rank 1, the T'-quotients of the geometric fiber of $(\mathcal{U}_V \setminus \mathcal{H}_V) \to \mathcal{H}_V \ni b$ are log $\mathbb{Q}$-Fano varieties precisely when $b \in B_V$, with respect to the natural boundary branch $\mathbb{Q}$-divisors. Hence, this B_V is exactly the locus which parametrizes the Fano cones. We denote the restriction of the G-equivariant universal family to B_V as $X_V \to B_V$. We denote an arbitrary geometric $\Bbbk$-point $b \in B_V$ and the fiber as X_b on which T acts which commutes with the G-action.

Note that for any $b \in B_V$, $\widehat{\mathrm{vol}}(X_b)$ is at most V as we explained in the previous subsection. Therefore, if we apply the semicontinuity of the normalized volume [BL21] combined with [Xu20, 1.3] to the universal family over B_V and the Noetherian arguments, it follows that $\{\widehat{\mathrm{vol}}(X_b) \mid b \in B_V(\Bbbk)\}$ is a finite set which we denote as $V = V_0 >$

$V_1 > \cdots > V_m > \cdots > V_{m'}$. Correspondinglly, we have a filtration of G-invariant open subschemes

$$B_V(v = V) \subset B_V(v \geq V_1) \subset \cdots \subset \cdots \subset B_V(v \geq V_{m'}) = B_V,$$

where we mean

$$B_V(v = V) := \{b \in B_V \mid \widehat{\mathrm{vol}}(0 \in X_b) = V\},$$
$$B_V(v \geq V_i) := \{b \in B_V \mid \widehat{\mathrm{vol}}(0 \in X_b) \geq V_i\}.$$

Clearly we have $B_V(v \geq V_0) = B_V(v = V)$ though. We sometimes simply denote $B_V(v \geq V_i)$ as $B_V(i)$.

Claim 3.4. $B_V(0) = B_V \cap H_V^{\mathrm{ss}}(\xi)$.

Proof. For a geometric point $b \in B_V$, the Reeb vector field ξ gives a valuation v_ξ of X_b with $\widehat{\mathrm{vol}}(v_\xi, X_b) = V$. Therefore, if $\widehat{\mathrm{vol}}(0 \in X_b) = V$ i.e., $b \in B_V(0)$ if and only if the $\widehat{\mathrm{vol}}$-minimizing valuation of $X_b \ni 0$ is exactly v_ξ but it is characterized by the K-semistability of $(0 \in X_b \curvearrowleft T, \xi)$ in the sense of [CS18, CS19], by [LX18, 1.1, 1.3] (cf., also [CS18, 6.1]). $\square$

Hence, we can consider the algebraic stack of K-semistable Fano cones of fixed normalized volume V and the Reeb vector field ξ, which we denote as

$$\mathcal{M}_V^{\mathrm{ss}}(\xi) := [B_V^{\mathrm{ss}}(\xi)/G]. \tag{25}$$

We also write $\mathcal{M}_V$ for the bigger quotient stack $[B_V/G]$, $\mathcal{M}_V(i) := [B_V(v \geq V_i)/G]$ for each $i = 1, 2, \cdots, m'$.

By [Xu20] (cf., also [BL21]), more precisely by its third paragraph of the proof of Theorem 1.3 and its Theorem 2.18, the $\widehat{\mathrm{vol}}$-minimizing valuations is uniformly taken i.e., obtained as the restriction of the same quasi-monomial valuation of $\mathcal{O}_{0,\mathbb{A}^N}$, on each strata of some finite stratification $\{B_V((j))\}_j$ by some locally closed connected subsets $B_V((j))$ of B_V. Note that *loc.cit* crucially depends on the bounded complements by Birkar [Bir19] as well as an analogue of the invariance of local plurigenera (cf., [HMX13]). By *loc.cit* for instance, the normalized volume function on each strata $B_V((j))$ constant, hence is a subset of $B_V(i)$ for some i. We give more details by Theorem 3.6 and its Step (i) of the proof. Furthermore, that next Theorem 3.6 further implies that the inclusion $B_V((j)) \hookrightarrow B_V(i) \setminus B_V(i-1)$ is proper i.e., closed immersion. Because both $\{B_V((j))\}_j$ and $\{B_V(i)\}_i$ are finite, it follows that each $B_V((j))$ is a connected component of some $B_V(i) \setminus B_V(i-1)$. We summarize:

Claim 3.5. *On each connected component of $B_V(i) \setminus B_V(i-1)$, the $\widehat{\mathrm{vol}}$-minimizing valuations of $0 \in X_b$ for $b \in B_V(i) \setminus B_V(i-1)$ is uniformly taken i.e., obtained as the restriction of the same quasi-monomial valuation of $\mathcal{O}_{0,\mathbb{A}^N}$ for large enough simultaneous uniform embedding $0 \in X_b \in \mathbb{A}^N$.*

Note that in general, valuation of a ring does not necessarily restrict to a valuation on its quotient ring.

3.3. **Higher Θ-reductivity-type theorem.** Now we discuss Θ-reductivity (cf., [HL14, AHLH23]) of the moduli stack of Fano cones in a somewhat generalized form, to include non-cone affine varieties, as we also later use it in other works as well. This is a partial generalization of [Xu20, 1.3], [ABHLX20, §5] (cf., also [BL21]). The statement is compatible with the framework of [Od24b], as an irrational and family analogue of the theory of the Harder-Narasimhan filtration or the Θ-strata [AHLH23].

Theorem 3.6. *Consider an arbitrary faithfully-flat affine klt morphism $\pi\colon Y \to S$ with a section $\sigma\colon S \to Y$ for a reduced algebraic $\mathbb{k}$-scheme S over $\mathbb{k}$ of characteristic 0, suppose there is a constant V such that for any geometric point $s \in \mathrm{Spec}(R)$, the geometric fiber $Y_s \ni \sigma(s)$ satisfies $\widehat{\mathrm{vol}}(\sigma(s) \in Y_s) = V$.*

Then there is an algebraic torus $T = N \otimes \mathbb{G}_m$, $\xi \in N_{\mathbb{R}} \setminus N_{\mathbb{Q}}$ and a rational polyhedral cone $\tau \ni \xi$, all independent of s, such that π extends to a faithfully flat affine klt family $\tilde{\mathcal{Y}} = [\mathcal{Y}/T]$ over the quotient algebraic S-stack $\Theta_\tau \times_{\mathbb{k}} S = [U_\tau(S)/T(S)]$, such that for any s, $\tilde{\mathcal{Y}}$ restricts to the positive weight deformations of $Y_s \ni \sigma(s)$ to the K-semistable Fano cones W_s discussed in §2.4 and [Od24b, §2].

If we restrict out attention to the case of Fano *cones*, our proof of the above theorem implies the following. It further provides the structure of a higher Θ-strata to their parameter space of with constant normalized volumes, as Claim 3.16 shows later.

Corollary 3.7 (of the *proof* of Theorem 3.6)**.** *The moduli stack $\mathcal{M}_V^{\mathrm{ss}}$ of n-dimensional K-semistable Fano cones is Θ-reductive in the sense of* [HL14].

For the construction of moduli of Fano cones, we only need the above corollary, but the theorem 3.6 will be also used in later work.

Simpler proof of Theorem 3.6 for $n = 2$. We first note that $n = 2$ case can be checked by the following standard arguments, under the assumption that S is a smooth $\mathbb{k}$-curve. In this case, we take a closed point c and suppose that $\sigma(c)$ has $\mathbb{Q}$-Gorenstein index N. Now we take the canonical cover of Y as $\tilde{Y} \to Y$ of the Galois group $\mu_N(\overline{\mathbb{k}})$. Suppose that the generic fiber $Y_\eta \ni \sigma(\eta)$ (resp., special fiber $Y_c \ni \sigma(c)$ for π is the (quasi-étale) quotient by a finite group G_η (resp., G_c) of the order a_η (resp., a_c). If the preimage of $\sigma(S)$ in $\tilde{Y}$, denoted as $\tilde{S}$, has degree bigger than 1 over S, we make base change of $\tilde{Y} \to S$ by $\tilde{S} \to S$ and denote the obtained family (resp., section) as $\tilde{\pi}\colon \tilde{Y} \to \tilde{S}$ (resp., $\tilde{\sigma}$). Then consider the pointed $\tilde{\pi}$-generic fiber which we suppose to be the quotient by a finite group $\tilde{G}_\eta$ (resp., $\tilde{G}_c$) of the order b_η (resp., b_c). Note that $a_c \geq a_\eta$, $b_c \geq b_\eta$, $a_c = Nb_c$, $a_\eta \leq Nb_\eta$ from the construction. From our assumption, we have $a_c = a_\eta = \frac{4}{V}$ i.e., the local volume does not decrease at c. Combining them, we obtain $b_0 = b_\eta$ i.e., we can assume that $\tilde{\pi}$ is a flat family of ADE singularities whose local fundamental groups orders do not change. Then it is a classical fact $\tilde{\pi}$ is formally trivial so that the (divisorial) $\widehat{\mathrm{vol}}$-minimizing valuations v_s of $\sigma(s) \in Y_s = \pi^{-1}(s)$ for closed points of $s \in S$ do not jump in the sense that it can be realized as a S-flat coherent ideal on Y supported on $\sigma(S)$ (whose blow up gives a plt blow up for each $s \in S$, which corresponds to v_s). Hence, Theorem 3.6 for $n = 2$ follows. □

proof of Theorem 3.6. Now we work for the general case for any n. The proof consists of the following three steps:

(i) To prove a weaker version of the statement of Theorem 3.6, which allows finite stratification of S. It uses arguments similar to [Xu20, proof of Theorem 1.3] (also [BL21, BLX22]), which in turn crucially uses the bounded complements by Birkar [Bir19] and [BCHM10].
(ii) Apply a similar method to the proof of the Θ-reductivity of [ABHLX20, §4], originally for families of K-semistable Fano varieties, to obtain an affine faithfully flat finite type family i.e., to prove Claim 3.9.
(iii) Prove that the obtained family over $\Theta_\tau \times_{\mathbb{k}} S$ is klt morphism of the desired kind.

We first prove Step (i). By [Xu20, 4.2] (cf., also [BL21, BLX22]), which uses the bounded complements [Bir19, Theorem 1.8], after possible replacement of S by a larger smooth (a priori *non*-connected) quasi-compact algebraic $\mathbb{k}$-scheme $\tilde{S}$ with a surjective morphism $\varphi\colon \tilde{S} \to S$, we can and do take a relative m-complement $\mathcal{D}$ for $\pi \times_S \tilde{S}$, so that the following holds. Below, the sub-indices mean the base change to the geometric points.

Condition 3.8. *The geometric fibers $(Y_s, \mathcal{D}_{s'}/m)$, for any geometric point s' of $\tilde{S}$ with $s = \varphi(s)$, are log canonical Calabi-Yau pairs and all the Kollár components (regarded as valuations) v of some fiber $\mathcal{Y}_s$ centered at $\sigma(s)$ and $\widehat{\mathrm{vol}}(v) < n^n + 1$ are log canonical valuations for some s' with $\varphi(s') = s$.*

By taking fiberwise log resolutions of $\pi \times_S \tilde{S}$, possibly after replacement of $\tilde{S}$ by the union of (resolutions of) its stratifications and considering associated base changes, we can and do assume there is a fiberwise log resolution $\mu\colon \tilde{Y} \to Y$ and its certain μ-exceptional $\mathbb{Q}$-divisor $\mathcal{E}$ such that $(\tilde{Y}_{s'}, (\mathcal{E} + (\mu^*\mathcal{D})/m)_{s'})$ are log smooth sub log canonical Calabi-Yau pairs for all geometric point s' of $\tilde{S}$. We denote the connected components of $\tilde{S}$ as $\{\tilde{S}_i\}_i$. From our construction, the set of ($\mathbb{R}_{>0}$-rescaling equivalence classes of) log canonical valuations $v_{s'}$ for $(\tilde{Y}_{s'}, (\mathcal{E} + (\mu^*\mathcal{D})/m)_{s'})$ can be naturally identified with $\Delta(\lfloor(\mathcal{E} + (\mu^*\mathcal{D}/m))_{s'}\rfloor)$ which does not depend on s', as far as s' stays inside a fixed connected component $\tilde{S}_i$ of $\tilde{S}$. Here, $\Delta(-)$ denotes the (projectivized) dual intersection cone complex of a normal crossing divisor.

Moreover, using that identification, as far as their discrepancies of $v \in \mathrm{Val}_{\sigma(s)}(Y_s)$ over X_s are negative (which are the case if $\widehat{\mathrm{vol}}(v, Y_s) < n^n + 1$), the corresponding normalized volume $\widehat{\mathrm{vol}}(v, Y_s)$ only depends on the connected component $\tilde{S}_i$ in which s' sits i.e., constant on $\tilde{S}_i$. This follows from [Xu20, Theorem 2.18] (cf., also [BCHM10], [HMX13, Theorem 4.2]). Note that when a Kollár component v approximates enough the $\widehat{\mathrm{vol}}(\sigma(s) \in Y_s)$-minimizing valuation (cf., [LX20, 1.3] for such approximability result), then it automatically follows $\widehat{\mathrm{vol}}(v, Y_s) < n^n + 1$. Thus, we conclude that the $\widehat{\mathrm{vol}}$-minimizing valuation are all identified in $\Delta(\lfloor(\mathcal{E}+(\mu^*\mathcal{D}/m))_{s'}\rfloor)$. Note that $s \in \varphi(\tilde{S}_i)$ is constructible by the Chevalley's lemma.

Therefore, there is a finite stratification $\{S_\alpha\}_\alpha$ by connected locally closed subsets S_α such that replacing S by $\sqcup_\alpha S_\alpha$, we can and do assume that

- π factors through $Y \hookrightarrow \mathbb{A}^N_S \to S$ for some large enough N,
- σ is the 0-section,
- for any α, there is a linear action of T on $\mathbb{A}^N$ and $\widehat{\mathrm{vol}}$-minimizing valuations v_α for $0 \in Y_s$ for any $s \in S_\alpha$ are achived by the same Reeb vector field $\xi_\alpha \in N \otimes \mathbb{R}$.

We prove it as follows. Following [DS17], we consider the finitely generated $\mathrm{Im}(v_\alpha)$- graded $\mathcal{O}_{S_\alpha}$-algebra $\mathcal{R} := \mathrm{gr}_{v_\alpha}(\mathcal{O}_{Y_{S_\alpha}})$. Then, we consider the family $W_\alpha := Spec_{S_\alpha}\mathcal{R}$ which incorpolates $W_s := \mathrm{Spec}\,\mathrm{gr}_{v_\alpha}(\mathcal{O}_{Y_s,0})$ for each geometric point $s \in S_\alpha$. We take finite generators of $\mathcal{R}$ possibly after taking open coverings of S_αs (for simplicity, we do not change the notation S_αs). Then, we further take their lifts of such generators sections to $\pi_*\mathcal{O}_{Y_{S_\alpha}}$ to embed $Y_{S_\alpha} \hookrightarrow \mathbb{A}^N \times S_\alpha$ over S_α and consider the T-action with the weights (holomorphic spectrum in [DS17]) as their degrees, we obtain the above data.

We now proceed to Step (ii) i.e., to prove the following claim.

Claim 3.9. *π extends to a faithfully flat affine pointed family $\overline{\pi}\colon \tilde{\mathcal{Y}} \to \Theta_\tau \times_{\Bbbk} S$ such that it restricts to the degeneration of $Y_s \ni \sigma(s)$ to the K-semistable Fano cones W_s discussed in §2.4, for generic s.*

Note that the last claim is not asserted for when s is the closed point, but we will improve this point in the next step (iii). By the previuos Step (i), at least $\overline{\pi}$ exists over the generic point of S (this can be reproved using the Galois descent after [XZ21]), which we denote as $\eta = \mathrm{Spec}(K)$. We write the obtained degeneration as $Y_K \rightsquigarrow W_K = \mathrm{Spec}\,\mathrm{gr}_v(\mathcal{O}_{Y_K})$.

Also, given the previous Step (i), we can and do assume S is $\mathrm{Spec}(R)$ with a DVR R of essentially finite type over $\Bbbk$ and we are reduced to prove the remained Step (ii) and Step (iii). This reduction is analogous to the valuative criterion of properness, and easily follows from [Od24b, 3.14] combined with [AHLH23, Appendix A3]. We denote the closed point as $c = \mathfrak{m} \in S$ and its residue field as κ. Now, we follow the method of [ABHLX20, §5], (cf., also [BLZ22, Theorem 5.3]).

Set the finitely generated R-algebra $R_Y := \Gamma(\mathcal{O}_Y)$ and consider the $\widehat{\mathrm{vol}}$-minimizing valuation $v = v_{Y_K}$ of the generic fiber Y_K of Y, consider the corresponding algebraic $\Bbbk$-split torus $T = N \otimes \mathbb{G}_m$, and take the positive vector field $\xi \in N \otimes \mathbb{R}$. Take its corresponding generalized test configuration (positive weights deformations) as in [Od24b, Ex 2.10].

In what follows, we refine the approximation of v by the Kollár components [LX20, Theorem 1.3]. The arguments are analogous to [ABHLX20, §5].

Consider a positive vector field ξ' for W_K in $N \otimes \mathbb{R}$, which is close enough to the $\widehat{\mathrm{vol}}(-, Y_K)$-minimizer v. Then the associated graded ring $\mathrm{gr}_{\xi'}(\mathcal{O}_{Y_K})$ is isomorphic to $\mathrm{gr}_\xi(\mathcal{O}_{Y_K})$ with different grading (cf., e.g., [LXZ22]), hence in particular induces valuation v' of Y_K. We consider a set of such v' as a neighborhood of v and we denote whose immersion by $\iota\colon U(\subset N_{\mathbb{R}}) \hookrightarrow \mathrm{Val}_{Y_K,0}$. Then it is easy to take a sequence of such $v' \in \iota(U)$ as $v'_q = \iota(\xi'_q)(q = 1, 2, \cdots) \in \iota(U \cap \frac{1}{q}N)$ which is induced by divisors $E_{K,q}(q=1,2,\cdots)$ over Y_K so that $v'=v'_q=q^{-1}\cdot \mathrm{ord}_{E_{K,q}}\to v$, $A_{Y_K}(E_{K,q})=O(q)$ for $q\to\infty$.

Recall that the restriction of the normalized volume function $\widehat{\mathrm{vol}}(-)$ to each small open face of $\mathrm{Val}_{Y_K,0}$ with respect to a fixed log smooth model, is Lipschitz continuous by combining [JM12, 5.7] and [BFJ12, Cor D] (cf., also [MSY08, Appendix C], [LX18, §3.2.2] for stronger results in special case) hence so is the case along $\iota(U)$ for small enough U. We consider more on the local behaviour around the $\widehat{\mathrm{vol}}(-,Y_K)$-minimizer v. In general, for a quasi-monomial valuation v' close enough to v in $\mathrm{Val}_{Y_K,0}$ which is associated to some positive vector field of W_K, $\widehat{\mathrm{vol}}(v',Y_K)=\widehat{\mathrm{vol}}(v',W_K)$ because we have $\mathrm{gr}_{v'}(\mathcal{O}_{Y_K})=\mathrm{gr}_{v'}(\mathcal{O}_{W_K})$.

On the other hand, take the graded valuation ideals $\mathfrak{a}_\bullet(\mathrm{ord}_{E_q})$ of $\mathcal{O}_Y$ with respect to E_q and $\mathfrak{b}_{q,\bullet}$ of $\mathcal{O}_{Y_\kappa}$ at the reduction Y_κ as in [ABHLX20, §5]. Here, E_q is the closure of $E_{K,q}$ over (a large enough normalized blow up of) Y. Then, analogously to Eqn (20) in the proof of 5.3 of *loc.cit*, we have

$$
\begin{aligned}
V &\le \mathrm{lct}(Y_\kappa;\mathfrak{b}_{q,\bullet})^{n+1}\,\mathrm{mult}(\mathfrak{b}_{q,\bullet}) && (26)\\
&\le \mathrm{lct}((Y,Y_\kappa);\mathfrak{b}_{q,\bullet})^{n+1}\,\mathrm{mult}(\mathfrak{b}_{q,\bullet}) && (27)\\
&\le A_{Y_K}(E_{K,q})^{n+1}\,\mathrm{mult}(\mathfrak{a}_\bullet(\mathrm{ord}_{E_q})) && (28)\\
&= A_{(Y,Y_K)}(E_q)^{n+1}\,\mathrm{mult}(\mathfrak{a}_\bullet(\mathrm{ord}_{E_q})) && (29)\\
&= V+o\left(\frac{1}{q}\right). && (30)
\end{aligned}
$$

(26) follows from the definition of the (minimum) normalized volume and our assumption in Theorem 3.6. (27) follows from the inversion of adjunction. (28) follows from [Blu21, 2.7], [BX19, (3)]. Finally, (30) follows from the fact that $\widehat{\mathrm{vol}}(-,Y_K)=\widehat{\mathrm{vol}}(-,W_K)$ on U and the same discussion as [ABHLX20, the proof of Claim 1], if we use a Fano cone analogue of Martelli-Sparks-Yau-Li's derivative formula ([Hu23, Lemma 3.3.1] cf., also [Li17, §4.1]).

From the above (26), (27), (28), (29), (30), we obtain the following Blum-type invariants' convergence:

$$
\epsilon_q := \lim_{q\to\infty}\left(A_{(Y,Y_\kappa)}(E_q)-\mathrm{lct}((Y,Y_\kappa);\mathfrak{b}_{q,\bullet})\right)=0. \tag{31}
$$

The above quantity - the difference of log discrepancy and the log canonical thresholds for the valuative ideals, is introduced and systematically studied by Blum [Blu21]. By the formula in *loc.cit* Proposition 2.8, one can apply [BCHM10, 1.2 or 1.4.3] to blow up Y to extract only E_q for $q \gg 0$ which we denote as $\mu\colon Y_q \to Y$ with $\mathbb{Q}$-factorial Y_q with μ-antiample E_q (which automatically characterize μ), with $(Y_q, (1-\epsilon_q)E_q + Y_\kappa)$. Applying [HMX13, 1.1] to the pair for $q \gg 0$, we obtain that $(Y_q, E_q + Y_\kappa)$ is log canonical so that in particular, Y_κ is semi-log-canonical (and irreducible) by the adjunction. Note that, in particular, we obtain the finite typeness of the $\mathcal{O}_Y$-algebra $\oplus_{p\in\mathbb{Z}_{\geq 0}}\mu_*\mathcal{O}_{Y_q}(-pE_q)$. as $E_q \subset Y_q$ is relatively anti-ample. In other words, we obtain the extended Rees construction

$$\begin{aligned}\mathrm{Gr}_{\xi'_q}(R_Y) &:= \oplus_{m\in M\cap(\xi'_q\geq 0)}\{f \in R_Y \mid v'_q(f) \geq \langle \xi'_q, m\rangle\}\\ &\subset R_Y[M\cap(\xi'_q \geq 0)]\end{aligned}$$

gives a finitely generated R-algebra. Here, the notation Gr is taken in contrast with gr for its quotient as we used before (e.g., Theorem 2.29 (ii)). Now, we fix a rational polyhedral cone $\tau \subset N_\mathbb{R}$ which contains ξ and consider the restriction of the above graded ring as $\mathrm{Gr}_{\xi'_q}(R_Y)|_{\tau^\vee} := \oplus_{m\in M\cap(\tau^\vee)}\{f \in R_Y \mid v'_q(f) \geq \langle\xi'_q, m\rangle\} \subset R_Y[M\cap(\tau^\vee)]$ which is again of finite type over R. Here, $\tau^\vee$ denotes the dual cone of τ in $M_\mathbb{R} = \mathrm{Hom}(N_\mathbb{R}, \mathbb{R})$. We can and do assume that ξ'_q satisfies that $\langle \xi - \xi'_q, \tau^\vee\rangle \subset \mathbb{R}_{\geq 0}$, so that for each q, we have $\langle \xi'_{q'} - \xi'_q, \tau^\vee\rangle \subset \mathbb{R}_{\geq 0}$ for $q' \gg q$. Then, it follows that $\mathrm{Gr}_{\xi'_q}(R_Y)|_{\tau^\vee}$ is a sub R-algebra of $\mathrm{Gr}_\xi(R_Y)|_{\tau^\vee} := \oplus_{m\in M\cap(\tau^\vee)}\{f \in R_Y \mid v'_q(f) \geq \langle\xi, m\rangle\} \subset R_Y[M\cap(\tau^\vee)]$ and $\cup_q \mathrm{Gr}_{\xi'_q}(R_Y)|_{\tau^\vee} = \mathrm{Gr}_\xi(R_Y)|_{\tau^\vee}$. Take a generator of the monoid $\tau^\vee \cap M$ as $m_1, \cdots, m_s$. Then from the ACC (the ascending chain condition) of ideals of R_Y i.e., the Noetherian property of R_Y, the following holds: for each $i = 1, \cdots s$, we can take a large enough $q_i \in \mathbb{Z}_{>0}$ which satisfies that $\mathrm{Gr}_{\xi'_q}(R_Y)|_{\tau^\vee}(m_i) = \mathrm{Gr}_\xi(R_Y)|_{\tau^\vee}(m_i)$ for $q \geq q_i$. Here (m_i) means the degree m_i-part. For $q := \max\{q_i \mid i\}$, it thus follows that for any $q > q'$, we have $\mathrm{Gr}_{\xi'_q}(R_Y)|_{\tau^\vee} = \mathrm{Gr}_\xi(R_Y)|_{\tau^\vee}$ and in particular it is of finite type over R. Thus we obtain a family of finite type over $U_\tau \times S$ which extends π. We write the obtained isotrivially degenerating family as $\mathcal{Y} = \mathcal{Y}_\xi$, the morphism $\mathcal{Y} \to U_\tau \times S$ still as π, and its restriction over $\mathrm{Spec}(K)$ (resp., $\mathrm{Spec}(\kappa)$) as $\mathcal{Y}_K = \mathcal{Y}_{\xi,K}$ (resp., $\mathcal{Y}_c = \mathcal{Y}_\kappa = \mathcal{Y}_{\xi,\kappa}$), and its restriction over $p_\tau \times S$ as $\mathcal{Y}_{p_\tau} \to (p_\tau\times)S$.

The relative affineness of $\overline{\pi}$ follows from the above ring-theoretic construction, and the faithful flatness of $\overline{\pi}$ follows from [Tei03, Proposition 2.3]. Hence we conclude the proof of Claim 3.9 and Step (ii).

We proceed to the last Step (iii). We keep fixing τ and also take large enough q with $\mathrm{Gr}_{\xi'_q}(R_Y)|_{\tau^\vee} = \mathrm{Gr}_\xi(R_Y)|_{\tau^\vee}$. Then, we obtain a family $\mathcal{Y}'_q \to \mathbb{A}^1_t \times S$ as the basechange of $\mathcal{Y} \to (U_\tau \times S)$ through $\mathbb{A}^1_t \to U_\tau$

that corresponds to $d\xi'_q \in N$ for some divisible enough positive integer d. We apply arguments similar to [ABHLX20, §5.3] to $\mathcal{Y}'_q \to \mathbb{A}^1 \times S$.

By the relative version (over S) of [LWX21, Lemma 2.21] $\mathcal{Y}_{p_\tau} := \mathcal{Y}|_{p_\tau \times S(\simeq S)} \simeq \mathcal{Y}'_q|_{0\times S}$ is $\mathbb{Q}$-Gorenstein family over S. Indeed, from the construction of $\mathcal{Y}$, the restriction $\mathcal{Y}|_{p_\tau \times S}$ is the relative cone of $(E_q, \mathrm{Diff}_{E_q}(0))$ with the $\mathbb{Q}$-polarization $-E_q|_{E_q}$, where $\mathrm{Diff}(-)$ denotes the usual Shokurkov's different, over S which is a log canonical pair by the adjunction. Similarly, if we focus on the fiber over the closed point c of S, $(Y_{q,\kappa}, E_q|_{Y_{q,\kappa}} =: e_q)$ is log canonical by adjunction (here $Y_{q,\kappa}$ denotes the central fiber of Y_q) and hence $(e_q, \mathrm{Diff}_{e_q}(0))$ is semi-log-canonical again by the adjunction. Therefore, $\mathcal{Y}_{(p_\tau,c)} = (\mathcal{Y}'_q)_{(t,c)}$ is semi-log-canonical as the affine cone of $(e_q, \mathrm{Diff}_{e_q}(0))$ (cf., e.g., [Kol13b, Lemma 3.1]). (Alternatively, we can apply [Od24b, Lemma 2.23] to the generalized test configuration $\mathcal{Y}_c$ to confirm the same consequence, combining with the same arguments as [ABHLX20, Claim 2], using the ACC of log canonical thresholds [HMX13].) Then it follows that $(\mathcal{Y}, \pi^*\partial U_\tau(\times c))$ is a log canonical pair by the inversion of adjunction (cf., e.g., [Kaw06, OX12]). Here $\partial U_\tau(\times c)$ means the (simple normal crossing) toric boundary of U_τ.

Now, suppose $\mathcal{Y}_{(p_\tau,c)}$ is not log terminal and obtain a contradiction.

Note that from the construction $\mathcal{Y}$ restricts to a generalized test configuration of $Y_c = \mathcal{Y}_{(1,c)}$ with the degeneration to $\mathcal{Y}_{(p_\tau,c)}$.

By [Od24b, Lemma 2.21], for the small enough fixed τ, there is a sequence of ideals $\{I_{\vec{m}}\}_{\vec{m}}$ of R such that $\mathcal{Y}|_{c\in S}$ is given as $\mathrm{Spec} \oplus_{\vec{m}\in\mathcal{S}_\tau} I_{\vec{m}}$, which also gives a birational projective morphism ([Od24b, Lemma 2.21 (3)]), which we denote as $\mu''\colon Z'' \to Y_c = \mathcal{Y}_{(1,c)}$. (We use the notation with the simple prime $'$ for another model for notational compatibility with [LWX21, 4.3] as we use its arguments in this proof later). We put the total exceptional divisor of μ'' i.e., the divisorial part of μ''-exceptional locus as E'' (with all coefficients set as 1).

By [Od24b, Lemma 2.23] and the above arguments, it follows that (Z'', E'') is log canonical. Suppose it is not purely log terminal. Then we take a (log crepant) dlt modification (see e.g., [OX12, Kol13b]) of (Z'', E'') as $f\colon Z \to Z''$ with $E := f_*^{-1}E'' + \mathrm{Exc}(f)_{\mathrm{red}}$. Then by the definition, (Z, E) is divisorially log terminal and Z is $\mathbb{Q}$-factorial. If we set $g := \mu'' \circ f$, then we can describe the generalized test configuration as

$$\begin{aligned}\mathcal{Y}|_{c\in S} &= \mathrm{Spec} \oplus_{\vec{m}\in\mathcal{S}_\tau} \mu''_*\mathcal{O}_{Z''}(-E_{\vec{m}})\\ &= \mathrm{Spec} \oplus_{\vec{m}\in\mathcal{S}_\tau} g'_*\mathcal{O}_{Z''}(-g^*E_{\vec{m}}),\end{aligned}$$

by our correspondence ([Od24b, Lemma 2.21]). Now we can apply [LX20, Lemma 3.8, Proposition 2.10] to have a plt blow up of Y_c supported on the closed point $\sigma(c)$, which we denote as $Z' \to Y_c$ with the normal exceptional divisor E'. Let us take a common resolution of Z and Z' as $Z \overset{p}{\leftarrow} \tilde{Z} \overset{q}{\rightarrow} Z'$, and set $G := p^*(K_Z + E) - q^*(K'_Z + E')$. Then since the coefficient of $q_*^{-1}E'$ is positive from the choice of ord'_E and the negativitiy lemma (cf., [KM98, 3.38, 3.39]), we know that G is effective which we denote by $\sum_i c_i E_i$ with $c_i > 0$. Now, we can apply the same arguments to [LX20, Lemma 3.7, 3.8] for the graded ideals $\mathfrak{a}_{\xi,\bullet}$ in [Od24b, Lemma 2.21 (a) (and also the corresponding $\mathbb{R}$-divisor E_ξ in (3))] to see that

$$\widehat{\mathrm{vol}}(Y_c \ni \sigma(c), \mathrm{ord}_{E'}) < \mathrm{vol}_X(Z). \tag{32}$$

Note that the left hand side can be computed by the model $Z \to Y_c$ and the right hand side is $\mathrm{vol}_X(Z) = V$, by Lemma 2.13 (ii) which identifies with the local volume over the generic point of S. This contradicts our assumption that $\widehat{\mathrm{vol}}(Y_c \ni \sigma(c)) = V$. Hence it follows that $\mathcal{Y}_{(p_\tau,c)}$ is irreducible. However, because of Lemma 2.13 (ii) and the uniqueness of $\widehat{\mathrm{vol}}$-minimizing valuation ([XZ21], [BLQ22]), it follows that v is the $\widehat{\mathrm{vol}}$-minimizing valuation of $\mathcal{Y}_\kappa \ni 0$ and the obtained generalized test configuration $\mathcal{Y}|_{c\times U_\tau}$ is the positive weight deformation for it. We complete the proof of Step (iii). □

Remark 3.10 (Other possible approaches). The author believes there are also several other possible approaches to the last Step (iii) of the above proof. For instance, we believe the above arguments of (32) is equivalently replacable by analogous arguments to [LWX21, §4, Proposition 4.3] by discussing the Donaldson-Futaki invariants $\mathrm{DF}(-,\xi)$ for generalized test configurations by similarly localizing at the central fiber (as in [Fut83, DT92], [Ino22, 2.22]) as Definition 2.15. The arguments are essentially a higher rank analogoue to [LX14, Step 3, Proposition 5] which *loc.cit* called the $\mathbb{Q}$-Fano extension process. Also recall that the final step of the proof in [LWX21, Proposition 4.3] (and [LX14, Proposition 5]) are, similarly to above [LX20, 3.8], both to boil down to the simple positivity of (local) volumes along the degenerate fibers (while other steps in [LX14] requires the Hodge index theorem as in [Od12a]).

Also we may be able to go back to obtain a generalized test configuration $\mathcal{Y}' \to U_\tau$ which corresponds to (Z', E') whose fibers over the toric boundary are all normal and $(\mathcal{Y}', \partial\mathcal{Y}')$ is plt where $\partial\mathcal{Y}'$ denotes the preimage over the toric boundary of U_τ.

proof of Corollary 3.7. The construction of the moduli stack $\mathcal{M}_V^{\mathrm{ss}}$ is done in §3 (more precisely, §3.1, §3.2). Take any family of K-semistable Fano cones $\mathcal{Y}^o$ which comes from $(\Theta_R \setminus p) \to \mathcal{M}_V^{\mathrm{ss}}$, where $\Theta_R := [\operatorname{Spec} R[t]/\mathbb{G}_m]$ and p means the closed point for the maximal ideal $(\mathfrak{m}, t)$. Then we show that it extends to $\Theta_R \to \mathcal{M}_V^{\mathrm{ss}}$ the desired assertion (Θ-reductivity) follows from the above proof of Theorem 3.6. Indeed, K-semistable Fano cones are in particular log terminal by the definition. Moreover, the test configuration over the generic point $\operatorname{Spec}(K)$ has K-semistable special fiber so that (26), (28), (30) work. Therefore, we can apply the step (ii), (iii) of the above proof of Theorem 3.6. Hence, there is an extension of $\mathcal{Y}^o$ to a faithfully flat affine klt family $\mathcal{Y}$ over Θ_R. (Step (iii) of the proof in this case is previously proved in [LWX21, §4, Proposition 4.3, Corollary 4.4]. Indeed, the Donaldson-Futaki invariant (Definition 2.15 (ii)) of the affine test configuration along $c \in S = \operatorname{Spec}(R)$ is 0 as it is the same as that along the generic point $\eta \in S = \operatorname{Spec}(R)$.) The only remained part is to show the K-semistability of the fiber over the closed point $p = (\mathfrak{m}, t)$. This follows the same arguments as [LWX21, Lemma 3.1] (a special case of the CM minimization conjecture), once we modify the proof therein verbatim by replacing the Hilbert scheme by the multi-Hilbert scheme and the Futaki invariant for the Fano varieties (corresponding to regular case) by the Futaki invariant for the fixed positive vector field ξ. This completes the proof of Corollary 3.7.

□

3.4. **S-completeness and its consequences.** In this subsection, we confirm another ingredient for the properties of the moduli stack $\mathcal{M}_V^{\mathrm{ss}}(\xi)$, after [LWX21].

First we review the following algebraic stack, which is convenient for the framework of [HL14, AHLH23].

Definition 3.11 ([HL14, §2B])**.** For a DVR R with its uniformizer π, $\overline{\mathrm{ST}}_R := [\operatorname{Spec}(R[x,y]/(xy-\pi))/\mathbb{G}_m]$. Here, $\mathbb{G}_m$ acts on x, y with weights $1, -1$ respectively. The closed point as the image of (x, y) is denoted as 0.

Then, we rephrase a theorem of [LWX21] as follows.

Theorem 3.12 (cf., [LWX21])**.** *For any fixed n, V and ξ, $\mathcal{M}_V^{\mathrm{ss}}(\xi)$ is S-complete (over $\Bbbk$) in the sense of* [AHLH23, §3.5, 3.38]*. That is for any morphism $\varphi^o \colon (\overline{\mathrm{ST}}_R \setminus 0) \to \mathcal{M}_V^{\mathrm{ss}}(\xi)$ with essentially finite type DVR R over $\Bbbk$, it extends to $\varphi \colon \overline{\mathrm{ST}}_R \to \mathcal{M}_V^{\mathrm{ss}}(\xi)$.*

Proof. This is essentially proved in [LWX21] (without the name of S-completeness in [AHLH23]). Take a $\mathbb{G}_m$-equivariant locally stable family of K-semistable Fano cone $\mathcal{X}^o \to \mathrm{Spec}(R[x,y]/(xy-\pi))$ which correpsponds to φ^o. This is equivalent to consider two special test configurations of its general fiber X. Then, [LWX21, proof of 4.1] shows that it extends to (automatically $\mathbb{G}_m$-equivariat) faithfully flat affine $\mathbb{Q}$-Gorenstein family $\mathcal{X} \to \mathrm{Spec}(R[x,y]/(xy-\pi))$. Further, [LWX21, 4.3, 4.4] (after its 4.1) shows that the fiber over the closed point (x,y) is K-semistable Fano cone, which gives rise to the desired extended morphism φ. □

See also [BX19, ABHLX20, LX18, XZ21] for related work.

Corollary 3.13. *For a K-polystable Fano cone $T \curvearrowright X, \xi$, its automorphism group* $\mathrm{Aut}(T \curvearrowright X, \xi)(\Bbbk) := \{$*T-equivariant automorphism* $X \to X\}$ *forms a reductive algebraic* $\Bbbk$*-group* $\mathrm{Aut}(T \curvearrowright X, \xi)$.

Proof. This follows from the above S-completeness theorem 3.12 by [AHLH23, 3.47]. □

Remark 3.14. The reductivity also follows from more differential geometric arguments, simply combining [DS17, Appendix] and [Li21, Theorem 2.9].

In §3.6, we also use the above S-completeness to prove the separatedness of the moduli, following [AHLH23, 1.1].

3.5. **Properness.** We are now ready to prove the following theorem, which partially generalize the results of [BHLLX21, LXZ22].

Theorem 3.15. *For any fixed n, V and ξ, $\mathcal{M}_V^{\mathrm{ss}}(\xi)$ is universally closed (i.e., satisfies the existence part of the valuative criterion).*

Proof. As a direct application of Theorem 3.6, we confirm that

Claim 3.16. *the filtration $\{\mathcal{M}_V(i)\}_i$ of §3.2 naturally holds the structure of the higher Θ-stratification (in the sense of* [Od24b, §3.3]*, after* [HL14]*) for some rational polyhedral cone $\tau \subset N \otimes \mathbb{R}$ which contains ξ.*

proof of Claim 3.16. We proceed to the proof of Claim 3.16. Take a point $b \in B_V(i) \setminus B_V(i-1)$ for $1 \le i \le m'$. From the generalization of the extended Rees construction by Teissier [Tei03, §2.1, Proposition 2.3], there is a T-equivariant morphism $\psi_b \colon \Delta_\xi^{\mathrm{gl}} \to \overline{T \cdot b}$ and induced $\overline{\psi_b} \colon [\Delta_\xi^{\mathrm{gl}}/T] \to [\overline{T \cdot b}/T]$ (see also [LX18, §2.1], [Od24b, Example 2.10, Theorem 2.12]). Now, take a connected component B of $(B_V(i) \setminus B_V(i-1))_{\mathrm{red}}$ which means the closed subset $(B_V(i) \setminus B_V(i-1))$

with the reduced scheme structure, regarded as a closed subscheme. Recall [Od24b, Lemma 3.14] which in particular says that any element of

$$\mathrm{Map}([\Delta_\xi^{\mathrm{gl}}/T], \mathcal{M}_V(i))(B) \tag{33}$$

$$= \varinjlim_{\tau \ni \xi} \mathrm{Map}([\Theta_\tau/T], \mathcal{M}_V(i))(B) \tag{34}$$

comes from $\mathrm{Map}([\Theta_\tau/T], \mathcal{M}_V(i))(B)$ for some τ possibly after replacing B by its Zariski covering. By Claim 3.5, there is an element of the left hand side (33) which induces degenerations to K-semistable Fano cones for any b simultaneously. Then, we use the above equality and represent by $\mathrm{Map}([\Theta_\tau/T], \mathcal{M}_V(i))(B)$ of the right hand side (34) for some τ. We fix such τ. Therefore, we obtain a T-equivariant morphism $\psi\colon U_\tau \times B \to B_i$ which induces $\overline{\psi}\colon \Theta_\tau \times B \to \mathcal{M}_V(i)$, where U_τ means the affine toric variety for τ. ψ localises to $(\Delta_\xi^{\mathrm{gl}} \times B) \to B_i$, which we still write as ψ. Take the connected component of the (finite type) algebraic stack $\mathrm{Map}([\Theta_\tau/T], \mathcal{M}_V(i))$ (cf., [HL14], [AHR20, 5.10, 5.11], [AHR19, 6.23]) which contains the image of B and denote as $\mathcal{B}$. Take any geometric $\Bbbk$-point $b' \in \mathcal{B}$ and consider the corresponding image of the vertex p_τ of U_τ as $[Y \subset \mathbb{A}^N]$. Then from the construction, its multi-Hilbert function remains the same as those parametrized by B. So, combined with the uniqueness of the $\widehat{\mathrm{vol}}$-minimizer [XZ21, BLQ22], it follows that the images of $p_\tau \times B$ are again inside $B_i \setminus B_{i-1}$ and the image of $\Theta_\tau \times \mathcal{B} \to \mathcal{M}_V$, which extends $\overline{\psi}$, still lies inside $\mathcal{M}_V(i)$. Denote the smallest τ among finite possibilities of B, as τ_m. Therefore, collecting all $\mathcal{B}$ for Bs, localizing to the uniform τ_m, we obtain the higher Θ_{τ_m}-stratification structure on $\mathcal{M}_V$. $\square$

Now, we proceed to the proof of Theorem 3.15 by the valuative criterion. Take any DVR R of essentially finite type over $\Bbbk$ (cf., [AHLH23, Appendix A3]), and denote its quotient field (resp., residue field) as K (resp., κ) and set $S := \mathrm{Spec}(R)$, its generic point η (resp., closed point c). Consider an arbitrary morphism $\varphi\colon S \to B_V$ which maps η into $B_V^{\mathrm{ss}}(\xi)$. We denote the corresponding family of Fano cones as $Y_K \to \eta$. Now we want to show that the induced morphism $\eta \to \mathcal{M}_V^{\mathrm{ss}}(\xi)$ can be extended to a morphism $S \to \mathcal{M}_V^{\mathrm{ss}}(\xi)$. Take any common rational positive vector field ξ' and corresponding algebraic subtorus $T' \subset T$ of rank 1. Now, we apply [LX14, Theorem 1] to a $\mathbb{Q}$-Gorenstein family of log Fano pairs $Y_K/T' \twoheadrightarrow \eta$ (cf., which underlies the associated Seifert $\mathbb{G}_m$-bundle's base cf., [Kol13b]) and take its cone. Possibly after replacement of R by its finite extension, this gives an extension $Y \twoheadrightarrow S$

of $Y_K/T' \twoheadrightarrow \eta$ as a family of Fano cones, corresponding to some morphism $S \to B_V(i)$ (after temporarily enlarging B_V if necessary). Now, by Claim 3.16, we can use the higher Θ-stable reduction theorem in [Od24b, §3.2, §3.3], in the place of [AHLH23] for the original proof of the Fano varieties case [BHLLX21, LXZ22] to decrease i, and hence finally reach $i = 0$ by its repetition thanks to the ACC of normalized volumes [XZ24, 1.2]. This completes the proof of Theorem 3.15. □

3.6. **Existence of coarse (good) moduli space.**

3.6.1. *General statements and proof.* Now we are ready to construct the moduli spaces.

Theorem 3.17 (Proper K-moduli of Fano cone)**.** *For fixed n, V, there is a moduli Artin stack of n-dimensional K-semistable Fano cones of normalized volume V and $\mathcal{M}_V^{\rm ss}(\xi)$, which admit a proper coarse (good) moduli space $M_V^{\rm ss}(\xi)$ whose closed points parametrize K-polystable Fano cones among them.*

Proof. We constructed the moduli stack of K-semistable Fano cones of normalized volume V and $\mathcal{M}_V^{\rm ss}(\xi)$ as (25). Then by the Θ-reductivity (Corollary 3.7 of Theorem 3.6) and the S-completeness theorem 3.12 (cf., [LWX21, §4]) of $\mathcal{M}_V^{\rm ss}(\xi)$, [AHLH23, Theorem A] can be applied to prove the existence of separated coarse (good) moduli space $M_V^{\rm ss}(\xi)$.

Furthermore, recall that the normalized volumes of n-dimensional log terminal singularities satisfy ACC [XZ24, 1.2]. Thus, Theorem 3.15 as an application of the higher Θ-stable reduction theorem [Od24b, §3] can be applied to show that $M_V^{\rm ss}(\xi)$ is universally closed and hence proper. We complete the proof. □

Remark 3.18 (Moduli (2-)functor)*.* The set-theoretic meaning of the moduli $\mathcal{M}_V^{\rm ss}(\xi)(\Bbbk)$ and $M_V^{\rm ss}(\xi)(\Bbbk)$ are clear i.e., n-dimensional K-semistable (resp., K-polystable) Fano cones of volume density V and the Reeb vector field ξ for T, as we defined (in the previous section), but more generally S-valued points for more general $\Bbbk$-schemes S i.e., $\mathcal{M}_V^{\rm ss}(\xi)(S)$ can be redefined more intrinsically as their "locally stable" families $T \curvearrowright \mathcal{X} \twoheadrightarrow S$ in the sense of Kollár [Kol22, Definition 3.40]. The arguments are essentially the same as [Kol08, Kol22, Xu20], so we omit the detail.

Remark 3.19 (Generalization to logarithmic setup)*.* It is also straightforward to generalize the above theorems and their proofs to kawamata-log-terminal log Fano cones (X, Δ), their families and moduli, at least when Δ are $\mathbb{Q}$-divisors and marked as $\Delta = \frac{1}{N}D$ with ($\mathbb{Z}$-)Weil divisors, over reduced[19] base. More precisely, we fix an algebraic $\Bbbk$-torus $T = N \otimes \mathbb{G}_m$ and consider the groupoid of T-faithfully flat families $\mathcal{X} \xrightarrow{\pi} S$ of normal affine cones (in the sense of Definition 2.9 (ii)) together with relative Mumford ($\mathbb{Q}$)-divisors which are marked as $\frac{1}{l}\mathcal{D}$ with $l \in \mathbb{Z}_{>0}$ and relative Mumford [20] $\mathbb{Z}$- divisors $\mathcal{D}$, which satisfy the same "$\mathbb{Q}$-Gorenstein-ness of deformation" type condition of [Kol22, 8.13 (except for 8.13.5)]. We further restrict our attention to the cases with reduced S and when π-fibers $(\mathcal{X}_s, \frac{1}{l}\mathcal{D}_s)$ for $s \in S$ are all log K-semistable with respect to ξ, and denote such groupoid as $\mathcal{M}_V^{\mathrm{lss,red}}(\xi, l)(S)$.

Then, by the log boundedness of $\mathcal{M}_V^{\mathrm{lss,red}}(\xi, l)(\Bbbk)$ ([XZ24]), we use the multi-Hilbert scheme [HS17] again of $T \curvearrowright X$ together with [Kol22, 7.3] (applied to projective compactifications of Ds with respect to a positive vector field $\xi' \in N$) to obtain a larger moduli stack of log affine cones, in the sense of Definition 2.9. Note that the divisors D are also assumed to be T-invariant which, again by the theory of [HS17], provides the reason of locally finite typeness of the obtained stack, and its finite typeness is ensured by the log boundedness [XZ24]. Then, by [Kol22, 3.22] (also cf., [Kol08]), we can take its locally closed substack generalize our previous construction of $\mathcal{M}_V^{\mathrm{ss}}(\xi)$. Then, the remained confirmation of its properties work verbatim after our discussions above in this §3 (and the materials of §2 and the references used in the proofs), using the logarithmic framework of modern birational geometry (cf., [KM98, §3]). Summarizing up, our arguments in this paper gives the following, similarly to Theorem 3.17:

Theorem 3.20 (K-moduli of log Calabi-Yau cones)**.** *We fix $n, l \in \mathbb{Z}$, and $V \in \mathbb{R}_{>0}$. Then, by the logarithmic version of the boundedness with fixed n, N, V, l. (*[Jia20, XZ24]*, cf., also Corollaries 3.2, 3.3), there are only finitely many choices of $N, T = N \otimes G_m, \xi \in N \otimes \mathbb{R}$ of n-dimensional log K-semistable log Fano cones $(T \curvearrowright X \ni x, \Delta = \frac{1}{l}D)$ where (X, Δ) are kawamata-log-terminal and are of the normalized volume V.*

Further fixing $N, T = N \otimes G_m, \xi \in N \otimes \mathbb{R}$ (just for convenience of description), they form a moduli Artin stack $\mathcal{M}_V^{\mathrm{lss,red}}(\xi, l)$ at the reduced Artin stack level, which admits a reduced proper coarse (good) moduli space $M_V^{\mathrm{lss,red}}(\xi, l)$ whose closed points parametrize n-dimensional log K-polystable log Fano cones $(X, \frac{1}{l}D)$ among them.

[19] to avoid complication, which is treated by the notion of "K-flatness" in [Kol22]

[20] which simply means being relative Cartier divisor inside the open subset of $\mathcal{X}^{\mathrm{sm}} \subset \mathcal{X}$, the relatively smooth locus, in our fiberwise normal case

In particular, if $\Bbbk = \mathbb{C}$, $M_V^{\rm lss,red}(\xi, l)(\mathbb{C})$ can be again regarded as the moduli of Sasaki-Einstein manifolds with certain singularities including "conical" type which we do not give intrinsic formulation in this paper.

3.6.2. *Reconstructing Fano varieties' K-moduli.* For the case when $\xi \in N$ and provides regular positive (Reeb) vector field, the above theorem 3.17 reproves the following.

Corollary 3.21 (K-moduli of $\mathbb{Q}$-Fano varieties). *For fixed n and fixed V, there is a moduli stack $\mathcal{M}$ of n-dimensional K-semistable $\mathbb{Q}$-Fano varieties X of the anticanonical volume $V = (-K_X)^n$, and it further admits a proper coarse (good) moduli space $M_V^{\rm ss}(\xi)$ whose closed points parametrize K-polystable $\mathbb{Q}$-Fano varieties.*

We make further discussions on the possibility of yet other (general) apporoaches to algebraic construction of K-moduli of Fano varieties, in the next section §4.

Proposition 3.22. *We fix n, V, T, ξ below. The (universal) CM $\mathbb{R}$-line bundle $\lambda_{\rm CM}(T \curvearrowright \mathcal{X})$ (Definition 2.22) on the moduli stack $\mathcal{M}_V^{\rm ss}(\xi)$ of n-dimensional Fano cones of the volume density V for the Reeb vector field ξ descends to a $\mathbb{R}$-line bundle on the coarse moduli space $M_V^{\rm ss}(\xi)$ which we denotes as $L_{\rm CM}(n, V, \xi)$ for simplicity.*

Proof. The proof is essentially the same arguments as [OSS16, §6.2, after (K-moduli) Conjecture 6.2]. Indeed, since the morphism $\mathcal{M}_V^{\rm ss}(\xi) \to M_V^{\rm ss}(\xi)$ is étale locally a GIT quotient as in the discussion of [OSS16], the descendability is equivalent to the vanishing of the Donaldson-Futaki invariants for product test configurations for every K-semistable Fano cones, which follows from the definition of K-semistability ([CS18], our Definition 2.15). □

3.6.3. *Projectivity and CM $\mathbb{R}$-line bundle.* Naturally, as conjectured in polarized setup in [FS90], [Od10], [OSS16, 6.2] and algebraic parts solved by [CP21, XZ22] for Fano varieties case, we conjecture:

Conjecture 3.23. *For $\Bbbk = \mathbb{C}$, $c_1(M_V^{\rm ss}(\xi)(\mathbb{C}), L_{\rm CM}(n, V, \xi))$ is represented by a Weil-Petersson type Kähler current on $M_V^{\rm ss}(\xi)^{\rm an} = M_V^{\rm ss}(\xi)(\mathbb{C})$. For general $\Bbbk$, $L_{\rm CM}(n, V, \xi)$ is an ample $\mathbb{R}$-line bundle so that $M_V^{\rm ss}(\xi)$ is projective.*

We omit the precise form of the natural log extension of the CM line bundle and the corresponding positivity conjecture, which the readers can guess from the polarized varieties case (cf., e.g., [OSS16, §6], [CP21, §3]).

3.7. **Examples.**

3.7.1. *Quasi-regular case.* In the quasi-regular case, as it is naturally expectable after [MSY06, MSY08, CS18, CS19], we can reduce the study to the K-moduli of (proper) log Fano varieties. This generalizes Corollary 3.21, which corresponds to the regular Reeb vector field case.

Proposition 3.24. *The moduli stack $\mathcal{M}_V^{\mathrm{ss}}(\xi)$ (resp., $M_V^{\mathrm{ss}}(\xi)$) of n-dimensional Fano cones $(T = \mathbb{G}_m) \curvearrowright X$ of volume density V for quasi-regular Reeb vector field $\xi \in N_{\mathbb{Q}}$ is isomorphic to the K-moduli stack (resp., K-moduli space) of certain log Fano varieties which appears as quotient of X.*

Proof. Take any $\mathbb{G}_m$-equivariantly faithfully flat family of affine cones $\mathbb{G}_m \curvearrowright \mathcal{X} = \cup_s X_s \subset \mathbb{A}^N \otimes S \to S \ni s$ with the weight vector $\xi = (m_1, \cdots, m_N)$, where m_i are coprime positive integers. By diagonalization, any $\mathbb{G}_m$-equivariantly faithfully flat family of quasi-regular affine cones can be localized to the above type family. As noted by [CS18, CS19] (cf., also [LL19]), the K-(semi/poly)stability of X_s is equivalent to that of $(X_s \setminus x)/\mathbb{G}_m$ with $\sum_D \frac{m_D - 1}{m_D} D$ where D runs over prime divisors inside $(z_i = 0) \subset (X_s \setminus x)/\mathbb{G}_m$ and m_D is the ramification degree of $[(X_s \setminus x)/\mathbb{G}_m] \to (X_s \setminus x)/\mathbb{G}_m$ i.e., $m_D = \gcd(m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_N)$.

If we consider the K-moduli stack $\mathcal{M}$ of log Fano varieties $((X_s \setminus x)/\mathbb{G}_m, \sum_D \frac{m_D - 1}{m_D} D)$, the above arguments give us a morphism $\varphi \colon \mathcal{M}_V^{\mathrm{ss}}(\xi) \to \mathcal{M}$. The inverse of this morphism also exists by taking the relative cones. Hence, these moduli stacks are isomorphic. Further, because of the fact that the K-polystability of X and $\varphi(X)$ are equivalent, or otherwise from the existence of proper good moduli spaces for both, we conclude that φ also induces an isomorphism at the coarse (good) moduli space level. □

Therefore, we can reduce the following to the log K-stability and log K-moduli of (usual) log Fano varieties.

Problem 3.25. *Explicitly describe the structure of K-moduli (Theorem 3.17) of quasi-regular Fano cones (Sasaki-Einstein manifolds with singularities) in the case of* [BGN03, BG05, Kol05, LST22].

3.7.2. *Irregular case.* The examples of compact *irregular* Sasaki-Einstein manifolds found in the initial stage seem to be [GMSW04a], [GMSW04b], [MS21], [FOW09] among others and they are mostly toric Fano cones, so that they do not have positive dimensional moduli spaces.

Recently, Süß[Sus21] considered an example of 3-dimensional affine T-varieties of complexity one (cf., [IS17]) which admit a positive dimensional moduli space. The example in *loc.cit* is constructed as follows.

By the correspondence of normal affine T-varieties with poloyhedral divisors [AH06], if we consider $Y = \mathbb{P}^1$ and a polyhedral divisor ([AH06]) i.e., $\mathcal{D} = \sum_{y \in \mathbb{P}^1(\Bbbk)} \mathcal{D}_y$ where $\mathcal{D}_y$ are rational polyhedra of the same tail cone σ which satisfy "proper"ness condition of [AH06, §1].

In [Sus21, §6], disjoint $y_1, \cdots, y_k \in \mathbb{P}^1(\Bbbk) \setminus \{0, \infty\}$ are fixed and we set σ, $\mathcal{D}_y$ as:

- $\sigma := \mathbb{R}_{\geq 0}(-1, 1) + \mathbb{R}_{\geq 0}(15k - 4, 8)$,
- $\mathcal{D}_0 := (\frac{2}{5}, \frac{1}{5}) + \sigma$,
- $\mathcal{D}_\infty := (\frac{-2}{3}, \frac{1}{3}) + \sigma$,
- $\mathcal{D}_{y_i}(i = 1, \cdots, k) := ([0, 1] \times \{0\}) + \sigma$,
- $\mathcal{D}_y := \sigma$ otherwise i.e., when $y \neq 0, \infty, y_i(i = 1, \cdots, k)$.

In [Sus21, §6], the corresponding T-variety X_k is proven to be K-polystable Fano cone for any positive integer k. Due to the automorphism group $\mathbb{G}_m(\Bbbk) = \Bbbk^*$ of $(\mathbb{P}^1, (0) + (\infty))$, we have the corresponding moduli

$$M_k(\Bbbk) = (\mathbb{P}^1(\Bbbk) \setminus \{0, \infty\})^k / (\mathbb{G}_m(\Bbbk) = \Bbbk^*) \simeq \Bbbk^{k-1}. \tag{35}$$

To explicitly understanding our compactification (Theorem 3.17) in this case, now we would like to consider the case when y_is can collide and can be even 0 or ∞. When the supports of some y collide, we take the Minkowski sum of the original $\mathcal{D}_y$s.

Case 1. Firstly, we consider the effect of collisions of y_i i.e., $y_i = y_j$ outside $0, \infty$ for some $1 \leq i \neq j \leq k$.

By [IS17], [Sus21, Definition 4.2, Proposition 4.3], special test configuration of the obtained T-variety X corresponds to an admissible point $y \in \mathbb{P}^1$ in the sense of *loc.cit.* Further, by [Sus21, Theorem 4.10], the K-polystability of (X, ξ) remains equivalent as far as σ_y in *loc.cit* §4 and $\mathbf{u}_y$ does not change. Thus, in particular, X is K-polystable if y_is are all in $\mathbb{P}^1 \setminus \{0, \infty\}$ even if they collides.

Case 2. Next we consider when some of y_i collide with 0. Suppose exactly $m (0 \leq m \leq k)$ of y_is attain 0. Then, $\mathcal{D}_0$ is changed to $(\frac{2}{5}, \frac{1}{5}) + ([0, m], 0)\sigma$.

Consider the case with $k = 2$ and positive m. A lengthy hand calculation (after [Sus21, 4.10, §6]), which we omit here, and confirmation by using the Sage package "TVars" created by Absar Gull, Leif Jacob, Leandro Meier and Hendrik Süß, shows the Donaldson-Futaki invariant of the special test configuration for the admissible $y = 0$ is negative.

The author thanks the creators of the Sage package, especially Hendrik Süßfor the help on this computation.

Case 3. Next we consider when some of y_i collide with ∞. Then, if $k = 2$ and m' is positive, a similar calculation as above (after [Sus21, 4.10, §6] and the same Sage program) shows the Donaldson-Futaki invariant of the special test configuration for the admissible $y = \infty$ is positive if $m' = 1$ and negative if $m' = 2$.

The conclusion is that

Proposition 3.26. *In the generalized setup of* [Sus21, §6] *(allowing y_i to be $0, \infty$ and their collision), suppose $k = 2$ and $y_i = 0$ for m is and $y_i = \infty$ for m' is. Then, the obtained Fano cone $X = X_k$ is K-polystable if and only if $m \leq 1$.*

Corollary 3.27. *For $k = 2$, the normalization of the K-moduli compactification $M_k = M_2$ (Theorem 3.17) of the above moduli 3.26 (*(35)*) of the* 3*-dimensional T-variety Fano cone of complexity* 1 *is $\mathbb{P}^1 = \mathbb{P}(1, 2)$.*

proof of Corollary 3.27. We consider $\frac{1}{y_i} =: x_i$ and take $[x_1+x_2 : x_1x_2] \in \mathbb{P}(1, 2) = (\mathbb{A}^2_{x_1+x_2,x_1x_2} \setminus (0, 0))/\mathbb{G}_m$. The natural relative version of the construction of T- variety in [AH06] gives a family of T-variety X over the above $\mathbb{A}^2_{x_1+x_2,x_1x_2} \setminus (0, 0) = (\mathbb{A}^2_{x_1,x_2} \setminus (0, 0))/S_2$. By combining with Proposition 3.26, we obtain the proof. □

By similar computations and some experiments, we expect that for at least $k = 3, 4$ and possibly other small ks, that the normalization of the K-moduli compactification of M_k is isomorphic to the log K-moduli of

$$(\mathbb{P}^1, (1 - c)[0] + c[y_1] + \cdots + c[y_k] + (1 - 3c)[\infty])$$

for distinct y_is and $0 < c \ll 1$ and their log K-polystable degenerations.

It would be also interesting to work on these moduli from quiver-gauge theoretic side on the ADS-CFT correspondence.

4. Discussions

4.1. **Via γ-invariants of Berman.** The reason of the name of "δ"-invariant in [FO18] comes from the original work of Berman [Ber13], where he introduces "γ"-invariant $\gamma(X) \in \mathbb{R}_{>0}$ for $\mathbb{Q}$-Fano varieties X from his original probabilistic (or statistical mechanical) approach to the complex Monge-Ampère measures of the Kähler-Einstein metrics. See [FO18, §2.2] for the more algebraic explanation and comparision of the two invariants (γ vs δ). We also recall that

Theorem 4.1 ([FO18, Theorem 2.5])**.** *$\delta(X) \geq \gamma(X)$ for any $\mathbb{Q}$-Fano variety X.*

See [Ber13] for the discussions on the analytic aspects of the possibility that the above can be equality in general. We point out below that the theory of $\gamma(X)$ a priori gives a direct algebraic construction of K-stable limit of Fano varieties without the iterative procedure [AHLH23, Od24b] nor some invariants a priori, as in the method via $\delta(X)$.

Proposition 4.2 (K-stable Fano via log KSBA theory)**.** *Consider any flat $\mathbb{Q}$-Gorenstein (locally stable* [Kol22]*) family of $\mathbb{Q}$-Fano varieties $\mathcal{X} \xrightarrow{\pi} \Delta$, where Δ is a smooth $\Bbbk$-curve, which satisfies $\gamma(X_t = \pi^{-1}(t)) > 1$ ("uniformly Gibbs stable") for all $t \in \Delta$. Then, $\mathcal{X}^{(m)} := \mathcal{X} \underbrace{\times_\Delta \mathcal{X} \times_\Delta \times \cdots \times_\Delta}_{m\text{-times}} \mathcal{X}$ is obtained as the relative log canonical model of $(\widetilde{\mathcal{X}}^{(m)}, (1+\epsilon)(f^{(m)})^* \mathcal{D}_m/m)$ over Δ, for any $m \gg 0$, where $f \colon \widetilde{\mathcal{X}} \to \mathcal{X}$ is any log resolution of $\mathcal{X}$, $\widetilde{\mathcal{X}}^{(m)} := \widetilde{\mathcal{X}} \underbrace{\times_\Delta \widetilde{\mathcal{X}} \times_\Delta \times \cdots \times_\Delta}_{m\text{-times}} \widetilde{\mathcal{X}}$ which contracts as $\widetilde{\mathcal{X}}^{(m)} \xrightarrow{f^{(m)}} \widetilde{\mathcal{X}}$ and $\mathcal{D}_m$ is the relative m-plurianticanonical divisor crucially constructed in* [Ber13, §6]*. In particular, $\mathcal{X}$ can be recovered as its relative diagonal.*

Recall that standard birational geometric arguments easily imply the above construction does not depend on the choice of f. Thus, the main point of the above observation is its construction of K-(semi)stable filling $\mathcal{X}_0$ does not depend on the choice of divisors etc, nor any kind of iterative procedures, since $\mathcal{D}_m$ is taken as a canonical object (for each m). Hence, this approach is close to that of the Kollár-Shepherd-Barron-Alexeev for the (log) K-ample case (cf., [Kol22]), which we now can interpret as a part of (log) K-stability theory (cf., the old survey [Od10] and references therein).

4.2. **Via δ-invariants.** Recall that the original proof in [BHLLX21, LXZ22] crucially uses the following refinement of δ-invariant:

Definition 4.3 ([BHLLX21, (1.2)])**.** For each log terminal $\mathbb{Q}$-Fano variety X,

$$M^{\mu}(X) := \left(\delta(X), \quad \inf_{\mathcal{X}} \frac{\mathrm{DF}(\mathcal{X}, -\mathrm{K}_{\mathcal{X}/\mathbb{P}^1})}{||\mathcal{X}||_{L^2}}\right),$$

as an element of $\mathbb{R}^2_{\geq 0}$ to which we put the lexicographic order. Here, $\delta(X)$ denotes the δ-invariant of $(X, -K_X)$ ([FO18, BJ22]), $\mathcal{X}$ runs over special test configurations of X, $||\mathcal{X}||_{L^2}$ is the L^2-norm of test configuration $(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1})$.

The crux of the original algebraic proof of this theorem is that we can modify the family $\mathcal{X}$ or its corresponding family $f\colon \Delta \to \mathcal{M}$ to improve $M^{\mu}(f(0))$ to make the first component $\delta(f(0))$ at least 1.

Comparison of the δ-invariant ([FO18, BJ22]) and the normalized volumes ([Li18, Li17, LL19]) are rather nontrivial problems, although they are analogous ([Liu18], [LLX18, Remark 4.5]). For instance, the following is known.

Theorem 4.4 (cf., [Liu18],[BBJ21],[CRZ19],[BJ22])**.** *For any $\mathbb{Q}$-Fano variety X, we have*

$$\delta(X, -K_X)^n(-K_X)^n \leq \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(x \in X). \tag{36}$$

A weaker version is that

$$\min\{\delta(X, -K_X), 1\}^n(-K_X)^n \leq \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(x \in X). \tag{37}$$

Proof. The weaker version (37) follows from the combination of the fact that $\min\{\delta(X), 1\}$ is the greatest Ricci lower bound ([BBJ21, §7.3], [CRZ19, Appendix]) and the logarithmic analogue [LL19, Proposition 4.6] of [Liu18, Theorem 2], applied to $(X, (1-\min\{\delta(X, -K_X), 1\})D)$. The stronger version (36) also directly follows from [BJ22, Theorem D] if we set $L = -K_X$ for X. $\square$

On the other hand, recently the theory of the δ-invariants are also generalized to the (possibly irregular) Fano cone setups in a similar manner by [Wu22, Hu23]. Note that [Hu23, 7.0.1] and the fact that minimized normalized volume can be irrational would imply that they are different in general. A natural question is

Question 4.5. *Can we use the above-mentioned generalizations of δ-invariant (*[Wu22, Hu23]*) to also give another properness proof of K-moduli of Fano cones, extending the approach in* [BHLLX21, LXZ22]*?*

We plan to discuss versions of our results for Kähler-Ricci solitons in a different paper, which can be often considered as special cases of the Sasaki-Einstein manifolds (cf., [MN15, Conjecture 1.2]).

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