P=NP

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Abstract

This paper investigates an extremely classic NP-complete problem: How to determine if a graph G, where each vertex has a degree of at most 4, can be 3-colorable(The research in this paper focuses on graphs G that satisfy the condition where the degree of each vertex does not exceed 4. To conserve space, it is assumed throughout the paper that graph G meets this condition by default.). The author has meticulously observed the relationship between the coloring problem and semidefinite programming, and has creatively constructed the corresponding semidefinite programming problem $R(G)$ for a given graph G. The construction method of $R(G)$ refers to Theorem 1.1 in the paper. I have obtained and proven the conclusion: A graph G is 3-colorable if and only if the objective function of its corresponding optimization problem $R(G)$ is bounded, and when the objective function is bounded, its minimum value is 0.

1 Introduction

The question of whether every problem that can be verified in polynomial time can also be solved in polynomial time was proposed by the computer scientist Stephen Cook in 1971, which is the famous P vs NP problem[1]. Among the 21 NP-complete problems Karp listed[2], the Graph Coloring problem is included: to color all vertices of a graph such that no two adjacent vertices have the same color. In fact, even when the graph is restricted to very specific conditions, the Graph Coloring problem remains NP-complete. In 1981, Holyer concluded that the edge coloring problem for 3-regular graphs is NP-complete[3]. This problem is actually equivalent to the vertex coloring problem for 4-regular graphs (subsequent papers refer to k-coloring as vertex coloring). This paper successfully proves that for any graph G where the degree of each vertex does not exceed 4, there exists a polynomial-time algorithm for the 3-colorability decision problem.

In the late 20th century, mathematicians embarked on an in-depth study of the theory of semi-definite programming. It began with Rajendra Karmarkar's introduction of the interior-point algorithm for linear programming [4]. Following this, many mathematicians extended the interior-point algorithm to semi-definite programming, including Michael J. Todd [5][6], Goemans, M. X., Williamson, D. P. [7], Nesterov, Y., Nemirovskii, A. [8], among others. Nesterov and Nemirovskii systematically discussed the theories related to semi-definite programming in their paper and demonstrated that their algorithm possesses polynomial-time convergence.

A completely positive matrix: For an n-th order matrix A, if A can be expressed as $A = C^tC$, and all elements of matrix C are non-negative, then A is said to be a completely positive matrix. This can be denoted as $A \in CP_n$. A copositive matrix: For an n-th order matrix A, if for any n-dimensional non-negative vector x, the quadratic form x^tAx is greater than or equal to 0, then A is said to be a copositive matrix. This can be denoted as $A \in COP_n$. Regarding the theory of copositive and completely positive matrices, Abraham Berman and Naomi Shaked-Monderer discuss these topics in detail in their book [9]. Here, we will not elaborate further.

This paper obtains the following beautiful theorem:

Theorem 1.1. For a given graph $G = (V, E)$ of order n(The degree of each vertex in graph G is at most 4.) , the following semi-definite program is constructed((1)-(17)):

min_{d_*,p_*}

$$f(G) \tag{1}$$

$s.t.$

$$D(G) + P(G) \succeq 0 \tag{2}$$

$$D(G) = D(G)^T = \begin{bmatrix} D_{1,1} & D_{1,2} & D_{1,3} & \cdots \\ D_{2,1} & D_{2,2} & D_{2,3} & \cdots \\ D_{3,1} & D_{3,2} & D_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = (D_{i,j})_{(n+1)\times(n+1)} \tag{3}$$

$$D_{i,i} = D_{i,i}^T = \begin{bmatrix} d_{i,i,1} & d_{i,i,4} & d_{i,i,5} \\ d_{i,i,4} & d_{i,i,2} & d_{i,i,6} \\ d_{i,i,5} & d_{i,i,6} & d_{i,i,3} \end{bmatrix} (1 \le i \le n) \tag{4}$$

$$D_{i,n+1} = D_{n+1,i}^T = \begin{bmatrix} d_{i,n+1,1} \\ d_{i,n+1,2} \\ d_{i,n+1,3} \end{bmatrix} (1 \le i \le n), D_{n+1,n+1} = \begin{bmatrix} d_{n+1,n+1} \end{bmatrix} \tag{5}$$

If vertices v_i and v_j in graph G are adjacent($i \neq j$), then ((6)):

$$D_{i,j} = D_{j,i}^T = \begin{bmatrix} d_{i,j,1} & d_{i,j,2} & d_{i,j,3} \\ d_{i,j,4} & d_{i,j,5} & d_{i,j,6} \\ d_{i,j,7} & d_{i,j,8} & d_{i,j,9} \end{bmatrix} (1 \le i \le n, 1 \le j \le n) \tag{6}$$

If vertices v_i and v_j in graph G are not adjacent($i \neq j$), then((7)):

$$D_{i,j} = D_{j,i}^T = d_{i,j} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} (1 \le i \le n, 1 \le j \le n) \tag{7}$$

$$P(G) = P(G)^T = \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & \cdots \\ P_{2,1} & P_{2,2} & P_{2,3} & \cdots \\ P_{3,1} & P_{3,2} & P_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = (P_{i,j})_{(n+1)\times(n+1)} \tag{8}$$

$$P_{i,i} = P_{i,i}^T = (0)_{3\times 3}(1 \le i \le n) \tag{9}$$

$$P_{i,n+1} = P_{n+1,i}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} (1 \le i \le n), P_{n+1,n+1} = \begin{bmatrix} 0 \end{bmatrix} \tag{10}$$

If vertices v_i and v_j in graph G are adjacent($i \neq j$), then ((11)):

$$P_{i,j} = P_{j,i}^T = (0)_{3\times 3} \tag{11}$$

If vertices v_i and v_j in graph G are not adjacent($i \neq j$), then((12)(13)):

$$P_{i,j} = P_{j,i}^T = \begin{bmatrix} p_{i,j,1} & p_{i,j,2} & p_{i,j,3} \\ p_{i,j,4} & p_{i,j,5} & p_{i,j,6} \\ p_{i,j,7} & p_{i,j,8} & p_{i,j,9} \end{bmatrix} (1 \le i \le n, 1 \le j \le n) \tag{12}$$

$$p_{i,j,k} \le 0(1 \le i \le n, 1 \le j \le n, 1 \le k \le 9) \tag{13}$$

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n} f_{i,j}(G) + 4\sum_{i=1}^{n}(d_{i,n+1,1} + d_{i,n+1,2} + d_{i,n+1,3}) + 6d_{n+1,n+1} \\ = f(G) \end{aligned} \tag{14}$$

If $i = j$, then:

$$f_{i,j}(G) = 2tr(D_{i,i}), (1 \le i \le n) \tag{15}$$

If vertices v_i and v_j in graph G are adjacent($i \neq j$), then:

$$f_{i,j}(G) = d_{i,j,2} + d_{i,j,3} + d_{i,j,6} + d_{i,j,4} + d_{i,j,7} + d_{i,j,8} \tag{16}$$

If vertices v_i and v_j in graph G are not adjacent($i \neq j$), then:

$$f_{i,j}(G) = 6d_{i,j} \tag{17}$$

(Note that d_* are variables.), We refer to the above semi-definite programming problem as the Deng-semi-definite programming problem for graph G, denoted as $R(G)$.For a given graph G, the following conclusions can be drawn: If G is 3-colorable, then the minimum value of the objective function $f(G)$ for its corresponding Deng-semi-definite programming $R(G)$ is 0. If G is not 3-colorable, then the objective function $f(G)$ for its corresponding Deng-semi-definite programming $R(G)$ is unbounded.

Next, we will prove the correctness of this statement.

2 Example

Since I am just an undergraduate majoring in Information and Computational Science from the School of Mathematical Sciences at Tiangong University (TGU) in China, I believe it is necessary to provide examples to summarize the main ideas of this paper to prevent it from being overlooked by the mathematical community. Firstly, for K_2 , which is 3-colorable, referring to the statement of Theorem 1.1 in the paper, the minimum value of the objective function $f(K_2)$for the corresponding semi-definite programming problem $R(K_2)$ is 0:

min_{d_*}

$$\begin{aligned} f(K_2) = 2(d_{1,2,2} + d_{1,2,3} + d_{1,2,6} + d_{1,2,4} + d_{1,2,7} + d_{1,2,8}) \\ +4\sum_{i=1}^{2}(d_{i,3,1} + d_{i,3,2} + d_{i,3,3}) + 4d_{3,3} + 2tr(D(K_2)) \end{aligned} \tag{18}$$

$s.t.$

$$D(K_2) = \begin{bmatrix} d_{1,1,1} & d_{1,1,4} & d_{1,1,5} & d_{1,2,1} & d_{1,2,2} & d_{1,2,3} & d_{1,3,1} \\ d_{1,1,4} & d_{1,1,2} & d_{1,1,6} & d_{1,2,4} & d_{1,2,5} & d_{1,2,6} & d_{1,3,2} \\ d_{1,1,5} & d_{1,1,6} & d_{1,1,3} & d_{1,2,7} & d_{1,2,8} & d_{1,2,9} & d_{1,3,3} \\ d_{1,2,1} & d_{1,2,4} & d_{1,2,7} & d_{2,2,1} & d_{2,2,4} & d_{2,2,5} & d_{2,3,1} \\ d_{1,2,2} & d_{1,2,5} & d_{1,2,8} & d_{2,2,4} & d_{2,2,2} & d_{2,2,6} & d_{2,3,2} \\ d_{1,2,3} & d_{1,2,6} & d_{1,2,9} & d_{2,2,5} & d_{2,2,6} & d_{2,2,3} & d_{2,3,3} \\ d_{1,3,1} & d_{1,3,2} & d_{1,3,3} & d_{2,3,1} & d_{2,3,2} & d_{2,3,3} & d_{3,3} \end{bmatrix} \succeq 0 \tag{19}$$

proof:The 6 group 3-coloring methods for K_2 are: v_1 is red, v_2 is yellow; v_1 is red, v_2 is blue; v_1 is yellow, v_2 is red; v_1 is yellow, v_2 is blue; v_1 is blue, v_2 is red; v_1 is blue, v_2 is yellow. Essentially, these 6 coloring methods are equivalent under the permutation of the three colors red, yellow, and blue. For each vertex v_i, coloring it red corresponds to vector $x_1 = [1 , 0, 0]$, coloring it blue corresponds to vector $x_2 = [0 , 1 , 0]$, and coloring it yellow corresponds to vector $x_3 = [0 , 0 , 1]$. The 6 colorings of K_2 correspond to vectors $X^{(1)} = [x_1 , x_2 , 1]$, $X^{(2)} = [x_1 , x_3 , 1]$, $X^{(3)} = [x_2 , x_1 , 1]$, $X^{(4)} = [x_2 , x_3 , 1]$,$X^{(5)} = [x_3 , x_1 , 1]$,and $X^{(6)} = [x_3 , x_2 , 1]$, respectively. It is not difficult to calculate that :

$$f(K_2) = \sum_{i=1}^{6} X^{(i)} D(K_2) {X^{(i)}}^T \geq 0 \tag{20}$$

When $D(K_2)$is a zero matrix, the value of the objective function $f(K_2)$ becomes zero.

In fact, we can explore more 3-colorable graphs such as K_3, and obtain inequalities similar to Equation (20) through similar discussions. However, for graphs that are not 3-colorable, since they do not have a 3-coloring scheme, it is not possible to derive inequalities similar to (20). Furthermore, if the objective function of their corresponding $R(G)$ is bounded, then the dual problem $R^*(G)$ will have a solution (as seen in Theorem 3.1 of the paper). Through some delicate construction, this solution can lead to a completely positive matrix, which will be the solution to another dual problem $K^*(G)$ (as seen in Theorem 3.3 of the paper). This would contradict Theorem 3.3 of the paper. Hence, for graphs that are not 3-colorable, the objective function of their corresponding $R(G)$ must be unbounded. The previous discussion outlines the central thesis of the paper, and we will now engage in a detailed demonstration of the argument.

3 Some important theorems

Theorem 3.1. For a given n-th order graph G(The degree of each vertex in graph G is at most 4.) , the dual problem of the corresponding semi-definite program $R(G)$ is $R^*(G)$:

max

$$0$$

s.t.

$$Z(G) \succeq 0 \tag{21}$$

$$\langle (1)_{3\times 3}, Z_{i,j} \rangle = 6, (1 \le i, j \le n) \tag{22}$$

$$Z(G) = Z(G)^T = \begin{bmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} & \cdots \\ Z_{2,1} & Z_{2,2} & Z_{2,3} & \cdots \\ Z_{3,1} & Z_{3,2} & Z_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = (Z_{i,j})_{(n+1)\times(n+1)} \tag{23}$$

$$Z_{i,i} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} (1 \le i \le n) \tag{24}$$

$$Z_{i,n+1} = Z_{n+1,i}^T = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} (1 \le i \le n), Z_{n+1,n+1} = \begin{bmatrix} 6 \end{bmatrix} \tag{25}$$

If vertices v_i and v_j in graph G are adjacent($i \neq j$), then:

$$Z_{i,j} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} (1 \le i \le n, 1 \le j \le n) \tag{26}$$

If vertices v_i and v_j in graph G are not adjacent($i \neq j$), then:

$$Z_{i,j} = Z_{j,i}^T = \begin{bmatrix} z_{i,j,1} & z_{i,j,2} & z_{i,j,3} \\ z_{i,j,4} & z_{i,j,5} & z_{i,j,6} \\ z_{i,j,7} & z_{i,j,8} & z_{i,j,9} \end{bmatrix} \tag{27}$$

$$z_{i,j,k} \geq 0, (1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq 9) \tag{28}$$

A graph G is 3-colorable if and only if there exists a matrix $Z(G)$ that satisfies equations (21) to (28).

Theorem 3.2. A D-graph is defined as a graph that is not 3-colorable, and in which each vertex has a degree of at most 4, and the graph becomes 3-colorable after removing any single edge. We refer to such a graph as a D-graph. For a D-graph G, construct its copositive programming $K(G)$(Let t be any given positive constant.):

min_{s_*}

$$g(G) + t \sum_{1 \leq i < j \leq n} p_{i,j}(G) \tag{29}$$

$s.t.$

$$S(G) \in COP_{3n+1} \tag{30}$$

$$S(G) = S(G)^t = \begin{bmatrix} S_{1,1} & S_{1,2} & S_{1,3} & \cdots \\ S_{2,1} & S_{2,2} & S_{2,3} & \cdots \\ S_{3,1} & S_{3,2} & S_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = (S_{i,j})_{(n+1)\times(n+1)} \tag{31}$$

$$S_{i,i} = S_{i,i}^t = \begin{bmatrix} s_{i,i,1} & s_{i,i,4} & s_{i,i,5} \\ s_{i,i,4} & s_{i,i,2} & s_{i,i,6} \\ s_{i,i,5} & s_{i,i,6} & s_{i,i,3} \end{bmatrix} (1 \leq i \leq n) \tag{32}$$

$$S_{i,n+1} = S_{n+1,i}^t = \begin{bmatrix} s_{i,n+1,1} \\ s_{i,n+1,2} \\ s_{i,n+1,3} \end{bmatrix} (1 \leq i \leq n), S_{n+1,n+1} = \begin{bmatrix} s_{n+1,n+1} \end{bmatrix} \tag{33}$$

If vertices v_i and v_j in graph G are adjacent($i \neq j$), then:

$$S_{i,j} = S_{j,i}^t = \begin{bmatrix} s_{i,j,1} & s_{i,j,2} & s_{i,j,3} \\ s_{i,j,4} & s_{i,j,5} & s_{i,j,6} \\ s_{i,j,7} & s_{i,j,8} & s_{i,j,9} \end{bmatrix} (1 \leq i \leq n, 1 \leq j \leq n) \tag{34}$$

If vertices v_i and v_j in graph G are not adjacent($i \neq j$), then:

$$S_{i,j} = (0)_{3\times 3}(1 \leq i \leq n, 1 \leq j \leq n) \tag{35}$$

$$\begin{split}\sum_{i=1}^{n}\sum_{j=1}^{n} g_{i,j}(G) + 4\sum_{i=1}^{n}(s_{i,n+1,1} + s_{i,n+1,2} + s_{i,n+1,3}) + 6s_{n+1,n+1} \\ = g(G)\end{split} \tag{36}$$

If $i = j$, then:

$$g_{i,j}(G) = 2tr(S_{i,i}), (1 \leq i \leq n) \tag{37}$$

If vertices v_i and v_j in graph G are not adjacent$(i \neq j)$, then:

$$g_{i,j}(G) = p_{i,j}(G) = 0(1 \leq i \leq n, 1 \leq j \leq n) \tag{38}$$

If vertices v_i and v_j in graph G are adjacent$(i \neq j)$, then:

$$g_{i,j}(G) = s_{i,j,2} + s_{i,j,3} + s_{i,j,6} + s_{i,j,4} + s_{i,j,7} + s_{i,j,8} \tag{39}$$

$$p_{i,j}(G) = g(G) - 2g_{i,j}(G) + 4tr(S_{i,j})(1 \leq i \leq n, 1 \leq j \leq n) \tag{40}$$

For the D-graph G, the objective function of the corresponding completely positive programming $K(G)$ is unbounded.

Theorem 3.3. For a given D-graph G of order n, the dual problem of its corresponding copositive programming $K(G)$ is the completely positive programming $K^*(G)$(Let t be any given positive constant.):
max

$$0$$

s.t.

$$B(G) + t \sum_{1 \leq i < j \leq n} E_{i,j}(G) \in CP_{3n+1} \tag{41}$$

$$B(G) = B(G)^t = \begin{bmatrix} B_{1,1} & B_{1,2} & B_{1,3} & \cdots \\ B_{2,1} & B_{2,2} & B_{2,3} & \cdots \\ B_{3,1} & B_{3,2} & B_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = (B_{i,j})_{(n+1)\times(n+1)} \tag{42}$$

$$B_{i,i} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} (1 \leq i \leq n) \tag{43}$$

$$B_{i,n+1} = B_{n+1,i}^t = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} (1 \leq i \leq n), B_{n+1,n+1} = \begin{bmatrix} 6 \end{bmatrix} \tag{44}$$

If vertices v_i and v_j in graph G are adjacent, then:

$$B_{i,j} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} (1 \leq i \leq n, 1 \leq j \leq n) \tag{45}$$

If vertices v_i and v_j in graph G are not adjacent$(1 \leq i \leq n, 1 \leq j \leq n)$, then:

$$B_{i,j} = B_{j,i}^t = \begin{bmatrix} b_{i,j,1} & b_{i,j,2} & b_{i,j,3} \\ b_{i,j,4} & b_{i,j,5} & b_{i,j,6} \\ b_{i,j,7} & b_{i,j,8} & b_{i,j,9} \end{bmatrix} \tag{46}$$

If vertices v_i and v_j in graph G are adjacent, then:

$$e_{i,i} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} (1 \le i \le n) \tag{47}$$

$$E_{i,j}(G) = (e_{k,l})_{(n+1)\times(n+1)}, (1 \le k \le n, 1 \le l \le n), \tag{48}$$

$$e_{i,n+1} = e^t_{n+1,i} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, (1 \le i \le n), e_{n+1,n+1} = \begin{bmatrix} 6 \end{bmatrix} \tag{49}$$

If vertices v_k and v_l in graph G are adjacent, then:
when $(k,l) = (i,j)$:

$$e_{k,l} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{50}$$

when $(k,l) \ne (i,j)$:

$$e_{k,l} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \tag{51}$$

If vertices v_k and v_l in graph G are not adjacent, then:

$$e_{k,l} = (0)_{3\times3} \tag{52}$$

If vertices v_i and v_j in graph G are not adjacent, then:

$$E_{i,j}(G) = (0)_{(3n+1)\times(3n+1)} (1 \le i \le n, 1 \le j \le n) \tag{53}$$

For the D-graph G, there does not exist a matrix $B(G)$ that satisfies equations (41) to (53).

4 Proof of Theorem 3.2 and Theorem 3.3

Based on the duality theory of copositive programming [9], we know that for a given D-graph G, the objective function of its corresponding copositive programming $K(G)$ is bounded if and only if its dual problem $K^*(G)$ has a feasible solution that satisfies the constraints. This means that we only need to prove Theorem 3.2. We first prove several lemmas, and then present the final proof of Theorem 3.2.

Lemma 1. If U is a positive definite matrix, and the minimum solution of $XUX^t + vX^t + u$ is X_0, and $X_0UX_0^t + vX_0^t + u > 0, X_0HX_0^t + hX_0^t + w = 0$, then there exists a positive constant C_1 such that : $XHX^t + hX^t + w + C_1(XUX^t + vX^t + u) > 0 (For\, any\, X)$.

proof:Write out the matrices A and B corresponding to the given quadratic form:

$$A = \begin{bmatrix} H & h^t/2 \\ h/2 & w \end{bmatrix}, B = \begin{bmatrix} U & v^t/2 \\ v/2 & u \end{bmatrix} \tag{54}$$

From the conditions of the problem, we know that matrix B is positive definite. It is not difficult to know from the related knowledge of positive definite matrices that there exists a positive constant C_1 such that $A + C_1B$ is also a positive definite matrix. This proves the statement.

Lemma 2. $T_{i,j}(G) = a(x_ix_j + y_iy_j + z_iz_j) + b(x_iy_j + x_iz_j + y_ix_j + y_iz_j + z_ix_j + z_iy_j)$,

When $x_i + y_i + z_i = 1$,x_i=0or1,y_i=0or1,z_i=0or1,

then :

$$T(G) = \sum_{i=1}^{n}\sum_{j=1}^{n} k_{i,j}T_{i,j}(G) > 0 \tag{55}$$

($k_{i,j}$are given constants, taking the value of 0 or 1.).There exist positive constants C_3 and C_4 such that the matrix corresponding to the quadratic form

$$T(G) + C_3\sum_{i=1}^{n}(x_i + y_i + z_i - 1)^2 + C_4\sum_{i=1}^{n}(x_iy_i + x_iz_i + y_iz_i) \tag{56}$$

is a copositive matrix.

proof:Let's first discuss the 3^n cases where $x_i = t_i, y_i = 0, z_i = 0$ or $x_i = 0, y_i = t_i, z_i = 0$ or $x_i = 0, y_i = 0, z_i = t_i$ (for $i = 1$ to n). By applying Lemma 1, we analyze each of these cases and know that there exists a positive constant C_5 such that the quadratic form

$$T(G) + C_5\sum_{i=1}^{n}(x_i + y_i + z_i - 1)^2 > 0 \tag{57}$$

for each of these 3^n cases.

Then we can decompose the original problem into the 3^n cases where $x_i = t_i, y_i = a_it_i, z_i = b_it_i$ or $x_i = a_it_i, y_i = t_i, z_i = b_it_i$ or $x_i = a_it_i, y_i = b_it_i, z_i = t_i$ (for $i = 1$ to n, where a_i and b_i are positive constants less than or equal to 1). We discuss these cases using the theory of positive definite matrices. For each case, we need to further classify the discussions:

1.From the previous discussions, combined with the properties of positive definite matrices, we can know that there exists positive constants q and C_5 such that when a_i, b_i (for $i = 1$ to n) $< q$, in each of these cases we have:

$$T(G) + C_5\sum_{i=1}^{n}(x_i + y_i + z_i - 1)^2 > 0. \tag{58}$$

2.When there is an a_i or b_i greater than or equal to q, it can be proven that there exists a $p > 0$ such that in each case we have:

$$\sum_{i=1}^{n}(x_i + y_i + z_i - 1)^2 + \sum_{i=1}^{n}(x_iy_i + x_iz_i + y_iz_i) > p. \tag{59}$$

Then, by Lemma 1, we know that there exists a $C_4 > 0$ such that when there is an a_i or b_i greater than or equal to p,in each of these cases we have: :

$$T(G) + (C_5 + C_4) \sum_{i=1}^{n} (x_i + y_i + z_i - 1)^2 + C_4 \sum_{i=1}^{n} (x_i y_i + x_i z_i + y_i z_i) > 0 \quad (60)$$

, which proves Lemma 2.

***Proof of Theorem*3.2 :** For the D-graph G, we construct the following quadratic form:

$$S^{(1)}(G) = \sum_{1 \le i < j \le n} S^{(1)}(G)_{i,j} \quad (61)$$

If vertex v_i in graph G is adjacent to vertex v_j : $S^{(1)}(G)_{i,j} = a(x_i x_j + y_i y_j + z_i z_j) + b(x_i y_j + x_i z_j + y_i x_j + y_i z_j + z_i x_j + z_i y_j)$. If vertex v_i in graph G is not adjacent to vertex v_j: $S^{(1)}(G)_{i,j} = 0$.

If vertex v_i is colored red, then $[x_i\,, y_i\,, z_i] = [1\,, 0\,, 0]$; if it is colored yellow, then $[x_i\, y_i\, z_i] = [0\,, 1\,, 0]$; if it is colored blue, then $[x_i\, y_i\, z_i] = [0\,, 0\,, 1]$. Then, if a set of colorings is applied to the D-graph G, for adjacent vertices v_i and v_j , if they have the same color, then $S^{(1)}(G)_{i,j} = a$; if they have different colors, then $S^{(1)}(G)_{i,j} = b$. It satisfies: $mb < 0, a + (m-1)b > 0$ (where m is the number of edges in graph G). Due to the properties of the D-graph G, it is not difficult to obtain: when $x_i + y_i + z_i = 1$, where x_i, y_i, z_i are 0 or 1, then $S^{(1)}(G) \ge a + (m-1)b > 0$. From Lemma 2, there exist positive constants C_3 and C_4 such that the matrix $S^{(2)}(G)$ corresponding to

$$S^{(1)}(G) + C_3 \sum_{i=1}^{n} (x_i + y_i + z_i - 1)^2 + C_4 \sum_{i=1}^{n} (x_i y_i + x_i z_i + y_i z_i) \quad (62)$$

is a copositive matrix. Substituting $S^{(2)}(G)$ into the copositive programming $K(G)$ corresponding to graph G, at this point,

$$g(G) + t \sum_{1 \le i < j \le n} p_{i,j}(G) = 6(mb + m(a + (m-1)b)t) \quad (63)$$

, and $mb + m(a + (m-1)b)t$ can take any negative value. This is how I prove Theorem 3.2.

5 Proof of Theorem 1.1 and Theorem 3.1

From the duality theory of semi-definite programming, we know that Theorem 1.1 is equivalent to Theorem 3.1. Therefore, we only need to prove Theorem 1.1.

Lemma 3. Any graph G(with each vertex of degree at most 4) that cannot be 3-colored has a subgraph K (Assuming the subgraph K has indices $i_1, i_2, ..., i_s$)

that is a D-graph, and if the semi-definite programming $R^*(K)$ corresponding to this D-graph K does not have a solution that satisfies the constraints, then the semi-definite programming $R^*(G)$ corresponding to the graph G also does not have a solution that satisfies the constraints.

proof:By contradiction, If the graph G(with each vertex of degree at most 4) has a matrix A that satisfies the constraints of $R^*(G)$, then there necessarily exists a principal submatrix of A(with rows and columns indexed by: $3i_1 - 2, 3i_2 - 2, ..., 3i_s - 2, 3i_1 - 1, 3i_2 - 1, ..., 3i_s - 1, 3i_1, 3i_2, ..., 3i_s, 3n+1$) that serves as a solution satisfying the constraints of $R^*(K)$.

Lemma 4. If both A and B are positive semi-definite matrices,and the quadratic forms xAx^T and xBx^T satisfy $kerB \in kerA$,and $A = E^TE$, $B = F^TF$, then the set of row vectors of E can necessarily be expressed linearly in terms of the set of row vectors of F.

proof:From the given conditions, we have $ker(F) \in ker(E)$, which means F can be augmented to form F^0 such that $ker(E) = ker(F^0)$. Therefore, the rank of E is equal to the rank of F^0, that is, $r(E) = r(F^0)$. It follows that $xE = 0$ is equivalent to $x(E\ F^0) = 0$, which is in turn equivalent to $xF^0 = 0$. This implies that the rank of E is equal to the rank of $(E\ F^0)$, which is also equal to the rank of F^0. Hence, the sets of row vectors of E and F^0 are equivalent.

Lemma 5. For the graph G, the kernel of the matrix that satisfies the semi-definite programming constraint $R^*(G)$ contains the following set of vectors:

$$\begin{bmatrix} Z_1 & Z_2 & \cdots & Z_n & x_0 \end{bmatrix} Z_i = \begin{bmatrix} Q_i & Q_i & Q_i \end{bmatrix} \tag{64}$$

$$\sum_{i=1}^{n} Q_i + x_0 = 0 \tag{65}$$

proof:From Equation (21)-(28), the proof can be immediately derived.

***Proof of Theorem* 1.1 :**

$\Rightarrow$:

In the context of graph G and its corresponding $D(G)$ in Theorem 1.1,expressed as a quadratic form:

$$XD(G)X^T = \sum_{j=1}^{n}\sum_{i=1}^{n} X_i D_{i,j} X_j^T + 2\sum_{i=1}^{n} D_{i,n+1} X_i^T + d_{n+1,n+1} \tag{66}$$

$$X = \begin{bmatrix} X_1 & X_2 & \cdots & X_n & 1 \end{bmatrix} X_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix} \tag{67}$$

For the vertex v_i, coloring it red corresponds to $X_i = [1, 0, 0]$,coloring it blue corresponds to $X_i = [0, 1, 0]$, and coloring it yellow corresponds to $X_i = [0, 0, 1]$. For a set of colorings of graph G, there are six permutations of red, yellow, and blue: red-yellow-blue, red-blue-yellow, blue-red-yellow, blue-yellow-red, yellow-red-blue, and yellow-blue-red. These colorings can be represented by six vectors, denoted as$X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}, X^{(5)}, X^{(6)}$. If this set of colorings ensures that no two adjacent vertices have the same color, it is easy to see that :

If vertices v_i and v_j in graph G are adjacent:

$$\sum_{k=1}^{6} X_i^{(k)} D_{i,j} {X_j^{(k)}}^T = d_{i,j,2} + d_{i,j,3} + d_{i,j,6} + d_{i,j,4} + d_{i,j,7} + d_{i,j,8} \tag{68}$$

If vertices v_i and v_j in graph G are not adjacent:

$$\sum_{k=1}^{6} X_i^{(k)} D_{i,j} {X_j^{(k)}}^T = 6d_{i,j} \tag{69}$$

, and consequently

$$f(G) = \sum_{k=1}^{6} X^{(k)} D(G) {X^{(k)}}^T \geq -\sum_{k=1}^{6} X^{(k)} P(G) {X^{(k)}}^T \geq 0 \tag{70}$$

. The sufficiency is thus proven.

$\Leftarrow$:

Let's introduce two definitions:

1.D-Coloring: For a D-graph, there exists a set of colorings such that only one edge $e_{i,j}$ has associated vertices v_i and v_j that are colored the same. We refer to this set of colorings as the D-coloring with respect to the edge $e_{i,j}$. According to the definition of a D-graph, every edge has a corresponding D-coloring.

2.D-Coloring Matrix: For a set of colorings in a D-graph, for each vertex v_i, coloring red corresponds to $e_i = 1, f_i = 0, g_i = 0$; coloring yellow corresponds to $e_i = 0, f_i = 1, g_i = 0$; and coloring blue corresponds to $e_i = 0, f_i = 0, g_i = 1$. This set of colorings can be permuted in six ways: red-yellow-blue, red-blue-yellow, blue-red-yellow, blue-yellow-red, yellow-red-blue, and yellow-blue-red. The corresponding assignments are denoted as $e_i^{(k)}, f_i^{(k)}, g_i^{(k)} (k = 1, ..., 6)$. Construct a quadratic form:

$$\sum_{k=1}^{6} (1 + \sum_{i=1}^{n} (e_i^{(k)} x_i + f_i^{(k)} y_i + g_i^{(k)} z_i))^2 = XLX^T \tag{71}$$

$$X = \begin{bmatrix} X_1 & X_2 & \cdots & X_n & 1 \end{bmatrix} \; X_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix} \tag{72}$$

. If this set of colorings is a D-coloring, then the matrix L corresponding to the quadratic form XLX^t of this set of D-colorings is referred to as a D-coloring matrix of graph G with respect to the edge $e_{i,j}$,It can also be called the D-coloring matrix of this set of D-colorings.

By contradiction, if the graph G is not 3-colorable, but the objective function value of its corresponding semi-definite program $R(G)$ is bounded, then the dual problem $R^*(G)$ of $R(G)$ must have a solution that satisfies constraints (15) to (21). By Lemma 3, it follows that there necessarily exists a subgraph K of G that is a D-graph, and the semi-definite program $R^*(K)$ corresponding to subgraph K must have a matrix $Z^{(0)}(K)$ that satisfies constraints (21) to (28).

For graph K (where the order of graph K is n^*), if we choose any vertex v_i and any edge associated with v_i, we can obtain a set of D-colorings for this edge, and consequently, we can obtain a corresponding D-coloring matrix, denoted as A_i.

Since the degree of v_i is necessarily less than or equal to 4, for this set of D-colorings, if we only change the color of v_i while keeping the colors of all other vertices the same, we can obtain the D-coloring matrix B_i for another edge associated with v_i. Performing this operation for each vertex allows us to obtain

$$W = \sum_{i=1}^{n^*}(a_i A_i + b_i B_i) \tag{73}$$

(where a_i and b_i are undetermined coefficients,and both are greater than 0.).

Then, we construct a D-coloring matrix L_{e_j} for each edge and sum them to obtain

$$Y = \sum c_{e_j} L_{e_j} \tag{74}$$

(where c_{e_j} are undetermined coefficients,and both are greater than 0.). Clearly, $W + Y$ is a completely positive matrix.

Due to the construction of A_i and B_i, it is not difficult to prove that the kernel of $W + Y$ can only be the set of vectors

$$\begin{bmatrix} Z_1 & Z_2 & \cdots & Z_{n^*} & x_0 \end{bmatrix} Z_i = \begin{bmatrix} Q_i & Q_i & Q_i \end{bmatrix} \tag{75}$$

$$\sum_{i=1}^{n} Q_i + x_0 = 0 \tag{76}$$

Therefore, by Lemma 4 and Lemma 5, the row vector group of $Z^{(0)}(K)$ can be linearly expressed by the row vector group of $W + Y$.

We can choose coefficients a_i, b_i, c_{e_j} to be sufficiently large so that $Z^{(0)}(K)+ W + Y$ is a completely positive matrix[9].

At this point, we can express $Z^{(0)}(K) + W + Y$ in the form of (Assuming that the D-coloring matrix corresponding to the edge $e_{i,j}$ is A, then $Z^{(0)}(K) + A - E_{i,j}(G)$ must satisfy equations (41) to (53).)

$$Z^{(1)}(K) + \sum_{1 \le i < j \le n^*} m_{i,j} E_{i,j}(K) \tag{77}$$

(where $Z^{(1)}(K)$ satisfies conditions (41) to (53), and the definition of $E_{i,j}(K)$ can be found in Theorem 3.3). By continuing to increase the coefficient $m_{i,j}$ until they are all equal, we obtain

$$Z^{(2)}(K) + \sum_{1 \le i < j \le n^*} t E_{i,j}(K) \tag{78}$$

(where $Z^{(2)}(K)$ satisfies conditions (41) to (53)). The above operations ensure that

$$Z^{(2)}(K) + \sum_{1 \le i < j \le n^*} t E_{i,j}(K) = O(K) \tag{79}$$

is a completely positive matrix. In fact, $O(K)$ is the solution to the completely positive programming $K^*(K)$ corresponding to graph K that satisfies the constraints (41) to (53). However, based on the already proven Theorem 3.3 and the 3-coloring intractability of graph K, we derive a contradiction. Therefore, the necessity of the theorem is established.

6 P=NP

By Theorem 1.1, the problem of determining whether a graph G (with each vertex of degree at most 4) is 3-colorable can be converted into a semi-definite programming problem. In the process of designing the code, we can add a constraint $f(G) \geq -100$ to ensure the halting criterion. Then, the minimum value output by the code is 0 when the graph G is 3-colorable, and the minimum value is -100 when the graph G is not 3-colorable. The coding is not difficult, and I have uploaded the code to my personal homepage ((1. https://b23.tv/ld3ICCG2 2. https://www.zhihu.com/people/deng-zi-58-20,Copy the link and open it in the browser)). It is easy to see from Theorem 1.1 that the semi-definite programming problem $R(G)$ corresponding to graph G, with the additional constraint $f(G) \geq -100$, can be solved in polynomial time[10][11]. Since the 3-coloring problem for graph G (with each vertex of degree at most 4) is an NP-complete problem [3], this implies P=NP.

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