# COMPACT BILINEAR OPERATORS AND PARAPRODUCTS REVISITED 

ÁRPÁD BÉNYI, GUOPENG LI, TADAHIRO OH, AND RODOLFO H. TORRES


#### Abstract

We present a new proof of the compactness of bilinear paraproducts with CMO symbols. By drawing an analogy to compact linear operators, we first explore further properties of compact bilinear operators on Banach spaces and present examples. We then prove compactness of bilinear paraproducts with $C M O$ symbols by combining one of the properties of compact bilinear operators thus obtained with vanishing Carleson measure estimates and interpolation of bilinear compactness.


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## 1. Introduction

The concept of compactness in the context of general multilinear operators was defined in Calderón's seminal work on interpolation [5]. However, outside interpolation theory, the first manifestation of this concept in harmonic analysis appeared much later in the work [2] by the first and fourth authors who proved that commutators of bilinear Calderón-Zygmund operators with $C M O$ functions are compact from $L^{p}\left(\mathbb{R}^{d}\right) \times L^{q}\left(\mathbb{R}^{d}\right)$ into $L^{r}\left(\mathbb{R}^{d}\right)$ for appropriate exponents $p, q, r$, thus extending the classical result of Uchiyama [26] to the bilinear setting; see also [1] in the context of bilinear pseudodifferential operators. Various generalizations and variations have followed, and the concept of bilinear compactness has taken on a life of its own within this area of research. For an overview (certainly not exhaustive) of recent results on commutators of several classes of bilinear operators in harmonic analysis, see the survey paper (4).

A fundamental result in the theory of linear Calderón-Zygmund operators is the celebrated $T(1)$ theorem due to David and Journé [13, which states that a singular integral operator $T$ with a Calderón-Zygmund kernel is bounded if and only if it satisfies a certain weak boundedness property (WBP) and $T(1)$ and $T^{*}(1)$ are functions in $B M O$ (when properly defined). Here, $T^{*}$ denotes the formal transpose of $T$. In the same paper (see [13, p. 380]), David and Journé [13] presented another equivalent and extremely elegant statement that

[^0]avoids mentioning the WBP, based on controlling the action of $T$ on the omnipresent character functions in harmonic analysis, $x \mapsto e^{i x \cdot \xi}$ for all $\xi \in \mathbb{R}^{d}$. A simplified proof was then presented by Coifman and Meyer [11, which was followed by several wavelet-based proofs. Finally, Stein [25] provided a quantitative statement of the $T(1)$ theorem only in terms of appropriate $L^{2}$-estimates, which completely avoids the mentioning of the WBP and BMO. Nonetheless, in his proof, both the WBP and BMO conditions are still used in some form. This version of a $T(1)$ theorem by Stein is based on controlling the action of $T$ and $T^{*}$ on normalized bump functions, which can be more directly verified in some applications. It is important to mention that all these different arguments employ in one way or another the construction of paraproduct operators which reduce the matter to the particular case of a simpler operator $T$ satisfying $T(1)=T^{*}(1)=0$. The proof of boundedness of paraproduct operators by a direct method without using the $T(1)$ theorem is then key.

In the multilinear setting, the first partial version of the $T(1)$ theorem was obtained by Christ and Journé [8], while the full result [15] is due to Grafakos and the last named author of this article. In [15], the result was proved using the multilinear version of the control on exponentials and through an iterative process, relying on Stein's $T(1)$ theorem in the linear setting. In particular, the formulation in [15] was not in a truly multilinear analogue of the original formulation in [13]. A version of the bilinear $T(1)$ theorem closest to that in [13] is due to Hart [18.

Interestingly, the study of compactness of commutators in the multilinear setting brought back a lot of attention to results involving the notion of compactness even in the linear setting. The literature nowadays has an abundance of harmonic analysis results related to compactness of commutators in a plethora of different settings such as compact weighted estimates, compact extrapolation, and compact wavelet representations, in both the linear and multilinear cases, and also numerous extensions of the classical Kolmogorov-Riesz compactness theorem (a main tool for proving compactness; see, for example, 17]). See again [4] for a survey on these extensions.

Compact Calderón-Zygmund operators exist, but most examples are provided by operators arising in the context of layer potential techniques on smooth bounded domains and by those artificially constructed, and their compactness can be easily established directly. Perhaps, one notable exception is the class of pseudodifferential operators introduced by Cordes [12] and revisited recently in the weighted setting in [7]. The other important exception is provided by paraproduct operators with appropriate symbols, which we will revisit here in the bilinear setting.

It is natural to expect that compactness of paraproduct operators would play a crucial role in the proof of a $T(1)$ compactness theorem. This is in fact the case, as it was established in the first version of such a theorem by Villarroya [27], which makes some additional assumptions on the kernel of a Calderón-Zygmund operator. The recent works by Mitkovski and Stockdale [22] in the linear case (see also Remark 3.3 below) and by Fragkos, Green, and Wick [16, Theorems 1 and 2] in the multilinear case present $T(1)$ compactness results for Calderón-Zygmund operators that have a similar flavor to the original $T(1)$ theorem. Restricting ourselves to the bilinear case, the aforementioned result from [16] is as follows.

Theorem A. Let $T: \mathcal{S}\left(\mathbb{R}^{d}\right) \times \mathcal{S}\left(\mathbb{R}^{d}\right) \rightarrow \mathcal{S}^{\prime}\left(\mathbb{R}^{d}\right)$ be a bilinear singular integral operator with a standard Calderón-Zygmund kernel, and $1<p, q \leq \infty$ and $\frac{1}{2}<r<\infty$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Then, $T$ is a compact operator from $L^{p}\left(\mathbb{R}^{d}\right) \times L^{q}\left(\mathbb{R}^{d}\right)$ to $L^{r}\left(\mathbb{R}^{d}\right)$ if and only if
(i) $T$ satisfies the weak compactness property, and
(ii) $T(1,1), T^{* 1}(1,1)$, and $T^{* 2}(1,1)$ are in CMO.

In other words, as compared to the bilinear $T(1)$ theorem from [18], the weak boundedness property is replaced by an appropriate weak compactness property, while the requirement of $T$ and its transposes acting on the constant function 1 to belong to $B M O$ is now replaced by the stronger assumption of belonging to $C M O$. The appearance of $C M O$ (see Section 3 for its definition) is very natural as this space appears in other results related to compactness, starting from the result in [26].

As in the case of the $T(1)$ theorem for boundedness, a main ingredient in the proof of Theorem A (and similarly in its linear versions) is to reduce the study of the operator $T$ to that of $\widetilde{T}$ given by

$$
\widetilde{T}=T-\Pi_{T(1,1)}-\Pi_{T^{* 1}(1,1)}^{* 1}-\Pi_{T^{* 2}(1,1)}^{* 2}
$$

where $\Pi_{b}$ denotes an appropriately defined bilinear paraproduct (satisfying (3.4)), and then realize the operator $\widetilde{T}$ as a sum of compact wavelet ones. The reduction from $T$ to $\widetilde{T}$ via paraproducts is employed in [18] as well, the difference being that for the boundedness of $\widetilde{T}$ one can appeal to bilinear square function estimates. Thus, as already alluded to, the understanding of boundedness or compactness of bilinear paraproducts is of paramount importance in both the classical multilinear $T(1)$ theorem and its compact $T(1)$ counterpart; see [16, Section 5]. See also [27, Section 6] and [22, Section 4] in the linear case.

The original goals of this work were more ambitious than what we present here. However, while working on this article, we became aware of the results in [16], which address some of our initial questions about bilinear compact $T(1)$ theorems. Hence, our modest goal of this short note is to revisit only the compactness of multilinear paraproducts with $C M O$ symbols through a different lens than the one in [16, Section 5], namely, by exploring and using more delicate properties of compact bilinear operators on Banach spaces which are of interest on their own; see Section 2. Our result (Proposition 3.1) and its proof in Section 3 should be construed as a compact counterpart of [18, Lemma 5.1] on the boundedness of paraproducts; the additional ingredients in our argument will be the vanishing of the appropriate Carleson measure as well as the use of interpolation ${ }^{1}$ for compact bilinear operators from the work of Cobos, Fernández-Cabrera, and Martínez [10]. For the ease of notation, we will consider only the bilinear case but interested readers may extend the results to a more general multilinear setting.

## 2. Some subtle properties of compact bilinear operators

Given a metric space $M$, we use $B_{r}^{M}(x)$ to denote the closed ball (in $M$ ) of radius $r>0$ centered at $x \in M$. When it is centered at the origin $x=0$, we simply write $B_{r}^{M}$ for $B_{r}^{M}(0)$. When there is no confusion, we drop the superscript $M$ and simply write $B_{r}(x)$ and $B_{r}$.

[^1]Let $X, Y$, and $Z$ be normed vector spaces. Recall from [5, 2] that we say that a bilinear operator $T: X \times Y \rightarrow Z$ is a compact bilinear operator if the image $T\left(B_{1}^{X} \times B_{1}^{Y}\right)$ is precompact in $Z$. Several equivalent characterizations of compactness for a bilinear operator $T: X \times Y \rightarrow Z$ are stated in [2, Proposition 1]. In this section, we explore further properties of compact bilinear operators by comparing them with the corresponding properties of compact linear operators. Before proceeding further, let us set some notations. We use $\langle\cdot, \cdot\rangle$ to denote the usual dual pairing; the spaces to which the duality pairing applies will be clear from the context. We define the two transposes of $T$ as $T^{* 1}: Z^{*} \times Y \rightarrow X^{*}$ and $T^{* 2}: X \times Z^{*} \rightarrow Y^{*}$ via

$$
\begin{equation*}
\left\langle T(x, y), z^{*}\right\rangle=\left\langle T^{* 1}\left(z^{*}, y\right), x\right\rangle=\left\langle T^{* 2}\left(x, z^{*}\right), y\right\rangle \tag{2.1}
\end{equation*}
$$

for all $x \in X, y \in Y$ and $z^{*} \in Z^{*}$. Given a bilinear operator $T: X \times Y \rightarrow Z$, we define its section operators $T_{x}: Y \rightarrow Z$ for fixed $x \in X$ and $T_{y}: X \rightarrow Z$ for fixed $y \in Y$ by setting

$$
\begin{equation*}
T_{x}(y)=T(x, y), \quad y \in Y \quad \text { and } \quad T_{y}(x)=T(x, y), \quad x \in X \tag{2.2}
\end{equation*}
$$

Note that bilinearity of $T$ is equivalent to linearity of both $T_{x}$ and $T_{y}$ for any $x \in X$ and $y \in Y$. We say that a bilinear operator $T: X \times Y \rightarrow Z$ is

- separately continuous if $T_{x}$ and $T_{y}$ are continuous linear operators for any $x \in X$ and $y \in Y$,
- separately compact if $T_{x}$ and $T_{y}$ are compact linear operators for any $x \in X$ and $y \in Y$.
If $X$ or $Y$ is Banach, then joint continuity of $T$ is equivalent to separate continuity of $T$; 24 , Theorem 2.17]. The completeness of one of the spaces in the domain of $T$ is crucial for this equivalence. However, the notion of separate compactness is strictly weaker than the notion of (joint) compactness and it turns out that the assumption of completeness of the spaces $X$ and $Y$ is of no importance. In [2, Example 4], an example of a separately compact bilinear operator which is not even continuous (and hence not compact) is provided, where the spaces are not complete in the relevant topologies. In Example 3 below, we present a separately compact bilinear operator which is continuous but not compact, where all the spaces involved are Banach.

We first recall the following characterizations for compact linear operators.
Lemma 2.1. Let $X$ and $Y$ be Banach spaces and $T: X \rightarrow Y$ be a continuous linear operator.
(i) If $T$ is compact, then $T$ maps weakly convergent sequences to strongly convergent sequences. Moreover, by assuming in addition that $X$ is reflexive, if $T$ maps weakly convergent sequences to strongly convergent sequences, then $T$ is compact.
(ii) The operator $T$ is compact if and only if its transpose $T^{*}$ is compact.

As for the first claim in Part (i), see [23, Theorem VI.11].2] The second claim in Part (i) follows from [23, Definition on p.199], saying that $T$ is compact if and only if for any bounded sequence $\left\{x_{n}\right\}_{n \in \mathbb{N}} \subset X$, the sequence $\left\{T\left(x_{n}\right)\right\}_{n \in \mathbb{N}}$ has a convergent subsequence in $Y$, and that a bounded sequence $\left\{x_{n}\right\}_{n \in \mathbb{N}} \subset X$ has a weakly convergent subsequence under the extra assumption that $X$ is reflexive. As for Part (ii), see [23, Theorem VI. 12 (c)].

[^2]By drawing an analogy to the linear case above, we investigate the following questions.
Question 2.2. Let $X, Y$, and $Z$ be Banach spaces and $T: X \times Y \rightarrow Z$ be a continuous bilinear operator. Does any of the following statements hold true in the bilinear setting?
(i) If $T$ is compact, then for every sequence $\left\{\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}} \subset X \times Y$ with $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ weakly convergent in $X$ and $\left\{y_{n}\right\}_{n \in \mathbb{N}}$ weakly convergent in $Y$, the sequence $\left\{T\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ is strongly convergent in $Z$. By assuming in addition that $X$ and $Y$ are reflexive, if for every sequence $\left\{\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}} \subset X \times Y$ with $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ weakly convergent in $X$ and $\left\{y_{n}\right\}_{n \in \mathbb{N}}$ weakly convergent in $Y$, the sequence $\left\{T\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ is strongly convergent in $Z$, then $T$ is compact.
(ii) The operator $T$ is compact if and only if $T^{* 1}$ is compact if and only if $T^{* 2}$ is compact.

As we see below, except for the second statement in Part (i), the answer is negative in general, exhibiting a sharp contrast to the linear case (Lemma 2.1). In the context of bilinear Calderón-Zygmund operators from $L^{p}\left(\mathbb{R}^{d}\right) \times L^{q}\left(\mathbb{R}^{d}\right)$ into $L^{r}\left(\mathbb{R}^{d}\right)$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, however, the answers to Question 2.2(i) and (ii) turn out to be positive (at least in the reflexive case $1<p, q, r<\infty$ ); see Proposition 2.6. While we restrict our attention only to the bilinear case in the following, the discussion (in particular, Propositions 2.3, 2.5, and 2.6) easily extends to the general $m$-linear case.

The next proposition provides an answer to Question 2.2 (i).
Proposition 2.3. Let $X$ and $Y$ be Banach spaces, $Z$ be a normed vector space, and $T$ : $X \times Y \rightarrow Z$ be a continuous bilinear operator.
(i) In addition, assume that $X$ and $Y$ are reflexive. If for every sequence $\left\{\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}} \subset$ $X \times Y$ with $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ weakly convergent in $X$ and $\left\{y_{n}\right\}_{n \in \mathbb{N}}$ weakly convergent in $Y$, the sequence $\left\{T\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ is strongly convergent in $Z$, then $T$ is compact.
(ii) The converse of Part (i) is false.
(iii) If $T$ is compact, then for every sequence $\left\{\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}} \subset X \times Y$ with $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ weakly convergent in $X$ and $\left\{y_{n}\right\}_{n \in \mathbb{N}}$ weakly convergent in $Y$, the sequence $\left\{T\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ has a strongly convergent subsequence in $Z$.

In Section 3, we will use Proposition 2.3(i) in proving compactness of a bilinear paraproduct; see Proposition 3.1.

Proof. (i) Let $\left\{\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ be a bounded sequence in $X \times Y$. Our goal is to construct a subsequence whose image under $T$ is convergent in $Z$. Since $X$ is a reflexive Banach space and $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ is bounded in $X$, it follows from the Banach-Alaoglu and Eberlein-Šmuljan theorems that there exists a subsequence $\left\{x_{n_{j}}\right\}_{j \in \mathbb{N}}$ that is weakly convergent in $X$. By the reflexivity of $Y$ and the boundedness of $\left\{y_{n_{j}}\right\}_{j \in \mathbb{N}}$, we can extract a further subsequence $\left\{y_{n_{j_{k}}}\right\}_{k \in \mathbb{N}}$ that is weakly convergent in $Y$. Then, by the hypothesis, the sequence $\left\{T\left(x_{n_{j_{k}}}, y_{n_{j_{k}}}\right)\right\}_{k \in \mathbb{N}}$ is strongly convergent in $Z$. Hence, from [2, Proposition 1 (c7)], we conclude that $T$ is compact.
(ii) See Examples 1 and 2 below.
(iii) Fix a sequence $\left\{\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}} \subset X \times Y$ such that $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ is weakly convergent in $X$ and $\left\{y_{n}\right\}_{n \in \mathbb{N}}$ is weakly convergent in $Y$. Then, $\left\{\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ is bounded in $X \times Y$. Hence, it follows from the compactness of $T$ and [2, Proposition 1 (c7)] that there exists a subsequence $\left\{T\left(x_{n_{j}}, y_{n_{j}}\right)\right\}_{j \in \mathbb{N}}$ converging strongly in $Z$.

Remark 2.4. In view of the bilinearity of $T$, in Proposition 2.3 (i), it is enough to verify that for all sequences for which at least one of $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ or $\left\{y_{n}\right\}_{n \in \mathbb{N}}$ converges weakly to 0 , $\left\{T\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ converges strongly to 0 in $Z$, to imply that $T$ is compact. Note that it is not sufficient to assume that both $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ and $\left\{y_{n}\right\}_{n \in \mathbb{N}}$ converge weakly to 0 (and showing that $\left\{T\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ converges strongly to 0 in $\left.Z\right)$.

Part (i) of the next proposition provides a negative answer to Question 2.2 (ii), showing that, regarding compactness. $3^{3}$ the bilinear case is quite different from the linear case (Lemma 2.1(ii)).

Proposition 2.5. (i) There exist Banach spaces $X, Y$, and $Z$ and a compact bilinear operator $T: X \times Y \rightarrow Z$ such that neither $T^{* 1}$ nor $T^{* 2}$ is compact.
(ii) Let $X, Y$, and $Z$ be Banach spaces. A bilinear operator $T: X \times Y \rightarrow Z$ is separately compact if and only if $\left(T^{* 1}\right)_{y}$ and $\left(T^{* 2}\right)_{x}$ are compact for any $(x, y) \in X \times Y$. Here, $\left(T^{* 1}\right)_{y}$ and $\left(T^{* 2}\right)_{x}$ are the section operators (of the transposes) defined in 2.2).

Proof. (i) See Examples 1 and 2 below.
(ii) Suppose that $\left(T^{* 1}\right)_{y}$ and $\left(T^{* 2}\right)_{x}$ are compact for any $(x, y) \in X \times Y$. From (2.1) and (2.2), we have

$$
\left\langle x,\left(T^{* 1}\right)_{y}\left(z^{*}\right)\right\rangle=\left\langle x, T^{* 1}\left(z^{*}, y\right)\right\rangle=\left\langle T(x, y), z^{*}\right\rangle=\left\langle T_{y}(x), z^{*}\right\rangle=\left\langle x,\left(T_{y}\right)^{*}\left(z^{*}\right)\right\rangle
$$

for any $x \in X, y \in Y$, and $z^{*} \in Z^{*}$. Hence, together with a similar computation for $\left(T^{* 2}\right)_{x}$, we have

$$
\begin{equation*}
\left(T^{* 1}\right)_{y}=\left(T_{y}\right)^{*}, \quad y \in Y \quad \text { and } \quad\left(T^{* 2}\right)_{x}=\left(T_{x}\right)^{*}, \quad x \in X \tag{2.3}
\end{equation*}
$$

Then, it follows from Lemma 2.1(ii) with the compactness of $\left(T^{* 1}\right)_{y}$ and $\left(T^{* 2}\right)_{x}$ that $T_{y}$ and $T_{x}$ are compact for any $(x, y) \in X \times Y$, which implies separate compactness of $T$ by definition.

Conversely, if $T$ is separately compact, then $T_{y}$ and $T_{x}$ are compact for any $(x, y) \in X \times Y$. Hence, from Lemma 2.1(ii) with 2.3), we conclude that $\left(T^{* 1}\right)_{y}$ and $\left(T^{* 2}\right)_{x}$ are compact for any $(x, y) \in X \times Y$.

We point out that if $X$ is finite-dimensional, then $T$ being compact implies $T^{* 1}$ is compact. In this case, $X^{*}$ is also finite-dimensional and thus is reflexive. By noting that given a sequence $\left\{\left(z_{n}^{*}, y_{n}\right)\right\}_{n \in \mathbb{N}} \subset B_{1}^{Z^{*}} \times B_{1}^{Y},\left\{T^{*}\left(z_{n}^{*}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ is bounded in $X^{*}$ and hence we can extract a convergent subsequence, which implies compactness of $T^{* 1}$. Similarly, if $Y$ is finitedimensional, then $T$ being compact implies $T^{* 2}$ is compact. As we see in Example 1 , however, finite dimensionality of the target space $Z$ does not yield compactness of $T^{* 1}$ or $T^{* 2}$.

We now present two examples, providing proofs of Proposition 2.3 (ii) and Proposition 2.5 (i).
Example 1. Let $X=Y=L^{2}(\mathbb{T})$ and $Z=\mathbb{C}$. Define a bilinear operator $T: X \times Y \rightarrow Z$ with $\mathbb{T}=\mathbb{R} / \mathbb{Z}$ by setting

$$
T\left(e_{n}, e_{m}\right)=\left\{\begin{array}{ll}
1, & \text { if } n+m=0, \\
0, & \text { otherwise },
\end{array} \quad n, m \in \mathbb{Z}\right.
$$

[^3]and extending the definition bilinearly, where $e_{n}(t)=e^{2 \pi i n t}, t \in \mathbb{T}$. Namely, we have
$$
T(x, y)=\int_{\mathbb{T}} x(t) y(t) d t
$$

Then, by Cauchy-Schwarz's inequality and noting $T\left(e_{n}, e_{-n}\right)=1, n \in \mathbb{Z}$, we have $\|T\|=1$, namely $T$ is bounded. Moreover, $T$ is compact since $T\left(B_{1}^{X} \times B_{1}^{Y}\right)=B_{1}^{Z}$ is compact in $Z=\mathbb{C}$.

We first present a proof of Proposition 2.3(ii). Define a sequence $\left\{\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}} \subset X \times Y=$ $L^{2}(\mathbb{T}) \times L^{2}(\mathbb{T})$ by setting $x_{n}=e_{n}$ and $y_{n}=e_{-n+p(n)}$, where $p(n)$ denotes the "parity" of $n$ given by

$$
p(n)= \begin{cases}1, & \text { if } n \text { is odd }  \tag{2.4}\\ 0, & \text { if } n \text { is even }\end{cases}
$$

By the Riemann-Lebesgue lemma, we see that both $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ and $\left\{y_{n}\right\}_{n \in \mathbb{N}}$ converge weakly to 0 as $n \rightarrow \infty$. On the other hand, we have

$$
T\left(x_{n}, y_{n}\right)= \begin{cases}0, & \text { if } n \text { is odd } \\ 1, & \text { if } n \text { is even }\end{cases}
$$

which shows that $\left\{T\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ is not convergent. This proves Proposition 2.3(ii).
Next, we present a proof of Proposition $2.5(\mathrm{i})$. We only show that $T^{* 1}$ is not compact since non-compactness of $T^{* 2}$ follows from a similar argument. It follows from [2, Proposition 1 (c7)] that if $T^{* 1}$ were compact, then given any bounded sequence $\left\{\left(z_{n}^{*}, y_{n}\right)\right\}_{n \in \mathbb{N}} \subset Z^{*} \times Y$, there would exist a subsequence $\left\{T^{* 1}\left(z_{n_{j}}^{*}, y_{n_{j}}\right)\right\}_{j \in \mathbb{N}}$ that is strongly convergent in $X^{*}$. We will show that this property fails.

Define a bounded sequence $\left\{\left(z_{n}^{*}, y_{n}\right)\right\}_{n \in \mathbb{N}} \subset B_{1}^{Z^{*}} \times B_{1}^{Y}$ by setting $z_{n}^{*}=1$ and $y_{n}=e_{n}$, $n \in \mathbb{N}$. Pick an arbitrary subsequence $\left\{\left(z_{n_{j}}^{*}, y_{n_{j}}\right)\right\}_{j \in \mathbb{N}}$. Then, by the definition of a dual norm and (2.1), we have

$$
\begin{align*}
& \left\|T^{* 1}\left(z_{n_{j}}^{*}, y_{n_{j}}\right)-T^{* 1}\left(z_{n_{k}}^{*}, y_{n_{k}}\right)\right\|_{X^{*}} \\
& \quad=\sup _{x \in B_{1}^{X}}\left|\left\langle T^{* 1}\left(1, y_{n_{j}}\right), x\right\rangle-\left\langle T^{* 1}\left(1, y_{n_{k}}\right), x\right\rangle\right|  \tag{2.5}\\
& \quad=\sup _{x \in B_{1}^{X}}\left|\left\langle T\left(x, y_{n_{j}}\right), 1\right\rangle-\left\langle T\left(x, y_{n_{k}}\right), 1\right\rangle\right| \\
& \quad \geq 1
\end{align*}
$$

for any $j>k \geq 1$, where the last step follows from choosing $x=e_{-n_{j}}$. This shows that the subsequence $\left\{T^{* 1}\left(z_{n_{j}}^{*}, y_{n_{j}}\right)\right\}_{j \in \mathbb{N}}$ is not convergent in $X^{*}$. Since the choice of the subsequence was arbitrary, we conclude that there exists no subsequence of $\left\{T^{* 1}\left(z_{n}^{*}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ that is strongly convergent in $X^{*}$ and therefore, $T^{* 1}$ is not compact. This proves Proposition 2.5(i).

We provide another example, where $Z$ is now infinite-dimensional.
Example 2. Let $X=Y=L^{4}(\mathbb{T})$ and $Z=L^{2}(\mathbb{T})$. Given $s>0$, define a bilinear operator $T: X \times Y \rightarrow Z$ by setting

$$
T(x, y)(t)=\left\langle\partial_{t}\right\rangle^{-s}(x y)(t) .
$$

Here, $\left\langle\partial_{t}\right\rangle^{-s}=\left(1-\partial_{t}^{2}\right)^{-\frac{s}{2}}$ denotes the Bessel potential of order $s>0$ defined by

$$
\left\langle\partial_{t}\right\rangle^{-s} f=\sum_{n \in \mathbb{Z}} \frac{1}{\left(1+4 \pi^{2} n^{2}\right)^{\frac{s}{2}}} \widehat{f}(n) e_{n}
$$

where $e_{n}(t)=e^{2 \pi i n t}$ as above and $\widehat{f}(n)$ denotes the Fourier coefficient of $f$. Then, by Cauchy-Schwarz's inequality, we see that $T(x, y) \in H^{s}(\mathbb{T})$ for any $x \in X$ and $y \in Y$. Here, $H^{s}(\mathbb{T})$ denotes the standard $L^{2}$-based Sobolev space. By the Rellich lemma, the embedding $H^{s}(\mathbb{T}) \hookrightarrow L^{2}(\mathbb{T})$ is compact and hence $T$ is compact.

We first present a proof of Proposition 2.3 (ii). Let $\left\{\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}} \subset X \times Y=L^{4}(\mathbb{T}) \times L^{4}(\mathbb{T})$ by setting $x_{n}=e_{n}$ and $y_{n}=e_{-n+p(n)}$, where $p(n)$ is as in (2.4). Then, we have

$$
T\left(x_{n}, y_{n}\right)= \begin{cases}\frac{1}{\left(1+4 \pi^{2}\right)^{\frac{s}{2}}} e_{1}, & \text { if } n \text { is odd } \\ 1, & \text { if } n \text { is even }\end{cases}
$$

Namely, $\left\{T\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ is not convergent, giving another example for Proposition 2.3(ii).
Next, we present a proof of Proposition 2.5(i). Choose a bounded sequence $\left\{\left(z_{n}^{*}, y_{n}\right)\right\}_{n \in \mathbb{N}} \subset$ $B_{1}^{Z^{*}} \times B_{1}^{Y}$ by setting $z_{n}^{*}=1$ and $y_{n}=e_{n}, n \in \mathbb{N}$. Then, the computation in (2.5) holds by choosing $x=e_{-n_{j}}$, (where the duality pairing is re-interpreted accordingly), which shows that $T^{* 1}$ is not compact. A similar argument shows that $T^{* 2}$ is not compact either.

The next example provides a continuous bilinear operator that is separately compact but is not jointly compact, even in the Hilbert space setting
Example 3. Let $X=Y=Z=\ell^{2}(\mathbb{N})$. Given $n \in \mathbb{N}$, let $\delta^{n}$ be the $n$th basis element in $\ell^{2}(\mathbb{N})$ whose only non-zero entry appears in the $n$th place and is given by 1 . Define a bilinear operator $T: X \times Y \rightarrow Z$ by setting

$$
T(x, y)=\sum_{n=1}^{\infty} x_{n} y_{n} \delta^{n}=\left(x_{1} y_{1}, x_{2} y_{2}, \ldots\right)
$$

for $x=\left\{x_{n}\right\}_{n \in \mathbb{N}}$ and $y=\left\{y_{n}\right\}_{n \in \mathbb{N}}$. By Hölder's inequality and the embedding $\ell^{2}(\mathbb{N}) \subset \ell^{\infty}(\mathbb{N})$, we have

$$
\|T(x, y)\|_{\ell^{2}} \leq\|x\|_{\ell^{\infty}}\|y\|_{\ell^{2}} \leq\|x\|_{\ell^{2}}\|y\|_{\ell^{2}} .
$$

Moreover, we have $T\left(\delta^{n}, \delta^{n}\right)=1, n \in \mathbb{N}$, and thus $T$ is bounded with $\|T\|=1$.
We first show that $T$ is separately compact. Given $N \in \mathbb{N}$, define the projection $\mathbf{P}_{N}$ by setting $\mathbf{P}_{N} x=\sum_{n=1}^{N} x_{n} \delta^{n}$. Then, it follows from the dominated convergence theorem that

$$
\begin{align*}
\left\|T(x, y)-\mathbf{P}_{N} T(x, y)\right\|_{\ell^{2}} & =\left\|\sum_{n=N+1}^{\infty} x_{n} y_{n} \delta^{n}\right\|_{\ell^{2}} \leq\|x\|_{\ell^{2}}\left(\sum_{n=N+1}^{\infty}\left|y_{n}\right|^{2}\right)^{\frac{1}{2}}  \tag{2.6}\\
& \longrightarrow 0,
\end{align*}
$$

as $N \rightarrow \infty$, uniformly in $x \in B_{1}^{\ell^{2}}$. Hence, from (2.2) and (2.6), we see that $T_{y}$ is the limit (in the operator norm topology) of finite rank operators $\left(\mathbf{P}_{N} T\right)_{y}$ for each $y \in Y$, which implies that $T_{y}$ is compact for any $y \in Y$. By symmetry, we deduce that $T_{x}$ is also compact for any $x \in X$. This shows that $T$ is separately compact.

Next, we show that $T$ is not compact. Noting that $T\left(\delta^{n}, \delta^{n}\right)=\delta^{n}, n \in \mathbb{N}$, and that $\left\|\delta^{n}-\delta^{m}\right\|_{\ell^{2}}=\sqrt{2}$ for any $n \neq m$, we see that the sequence $\left\{\left(\delta^{n}, \delta^{n}\right)\right\}_{n \in \mathbb{N}}$ is bounded in
$X \times Y=\ell^{2}(\mathbb{N}) \times \ell^{2}(\mathbb{N})$ but that $\left\{T\left(\delta^{n}, \delta^{n}\right)\right\}_{n \in \mathbb{N}}$ does not have any convergent subsequence in $Z=\ell^{2}(\mathbb{N})$. In view of [2, Proposition $\left.1(\mathrm{c} 7)\right]$, this shows non-compactness of $T$.

By working on the Fourier side, the argument above shows that for $X=Y=Z=L^{2}(\mathbb{T})$, the operator $S$ defined by

$$
S(x, y)(t)=x * y(t)=\int_{\mathbb{T}} x(t-s) y(s) d s
$$

is continuous and separately compact but is not (jointly) compact.
We conclude this section by discussing the case of bilinear Calderón-Zygmund operators. In the reflexive case $1<p, q, r<\infty$, the following proposition (together with Proposition 2.3 (i)) provides positive answers to Question 2.2 (i) and (ii).
Proposition 2.6. Let $T: \mathcal{S}\left(\mathbb{R}^{d}\right) \times \mathcal{S}\left(\mathbb{R}^{d}\right) \rightarrow \mathcal{S}^{\prime}\left(\mathbb{R}^{d}\right)$ be a bilinear singular integral operator with a standard Calderón-Zygmund kernel. Then, the following statements hold for any $1<p, q, r<\infty$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$.
(i) The operator $T: L^{p}\left(\mathbb{R}^{d}\right) \times L^{q}\left(\mathbb{R}^{d}\right) \rightarrow L^{r}\left(\mathbb{R}^{d}\right)$ is compact if and only if $T^{* 1}: L^{r^{\prime}}\left(\mathbb{R}^{d}\right) \times$ $L^{q}\left(\mathbb{R}^{d}\right) \rightarrow L^{p^{\prime}}\left(\mathbb{R}^{d}\right)$ is compact if and only if $T^{* 2}: L^{p}\left(\mathbb{R}^{d}\right) \times L^{r^{\prime}}\left(\mathbb{R}^{d}\right) \rightarrow L^{q^{\prime}}\left(\mathbb{R}^{d}\right)$.
(ii) Suppose that $T$ is compact from $L^{p}\left(\mathbb{R}^{d}\right) \times L^{q}\left(\mathbb{R}^{d}\right)$ to $L^{r}\left(\mathbb{R}^{d}\right)$. Then, for every sequence $\left\{\left(f_{n}, g_{n}\right)\right\}_{n \in \mathbb{N}} \subset L^{p}\left(\mathbb{R}^{d}\right) \times L^{q}\left(\mathbb{R}^{d}\right)$ with $\left\{f_{n}\right\}_{n \in \mathbb{N}}$ weakly convergent in $L^{p}\left(\mathbb{R}^{d}\right)$ and $\left\{g_{n}\right\}_{n \in \mathbb{N}}$ weakly convergent in $L^{q}\left(\mathbb{R}^{d}\right)$, the sequence $\left\{T\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ is strongly convergent in $L^{r}\left(\mathbb{R}^{d}\right)$.

Proof. (i) We only prove that compactness of $T$ implies compactness of $T^{* 1}$ and $T^{* 2}$. We first note that the hypotheses on the kernels and the weak compactness property in Theorem A are symmetric for $T, T^{* 1}$, and $T^{* 2}$. Moreover, by noting that $4^{4}\left(T^{* 1}\right)^{* 1}=T$ and $\left(T^{* 1}\right)^{* 2}=\left(T_{\text {flip }}\right)^{* 1}$, where $T_{\text {flip }}(f, g)=T(g, f)$, and that if $T$ is compact from $L^{p}\left(\mathbb{R}^{d}\right) \times L^{q}\left(\mathbb{R}^{d}\right)$ to $L^{r}\left(\mathbb{R}^{d}\right)$, then $T_{\text {flip }}$ is compact from $L^{q}\left(\mathbb{R}^{d}\right) \times L^{p}\left(\mathbb{R}^{d}\right)$ to $L^{r}\left(\mathbb{R}^{d}\right)$, it follows from Theorem A that $T^{* 1}(1,1)$, $\left(T^{* 1}\right)^{* 1}(1,1)=T(1,1)$, and $\left(T^{* 1}\right)^{* 2}(1,1)=\left(T_{\text {flip }}\right)^{* 1}(1,1)$ are all in $C M O$. Hence, by applying Theorem A in the reversed direction, we conclude that $T^{* 1}$ is compact from $L^{r^{\prime}}\left(\mathbb{R}^{d}\right) \times L^{q}\left(\mathbb{R}^{d}\right)$ into $L^{p^{\prime}}\left(\mathbb{R}^{d}\right)$. A similar argument shows that $T^{* 2}$ is compact from $L^{p}\left(\mathbb{R}^{d}\right) \times L^{r^{\prime}}\left(\mathbb{R}^{d}\right)$ into $L^{q^{\prime}}\left(\mathbb{R}^{d}\right)$.
(ii) Fix a sequence $\left\{\left(f_{n}, g_{n}\right)\right\}_{n \in \mathbb{N}} \subset L^{p}\left(\mathbb{R}^{d}\right) \times L^{q}\left(\mathbb{R}^{d}\right)$ such that $\left\{f_{n}\right\}_{n \in \mathbb{N}}$ converges weakly to some $f$ in $L^{p}\left(\mathbb{R}^{d}\right)$ and $\left\{g_{n}\right\}_{n \in \mathbb{N}}$ converges weakly to some $g$ in $L^{q}\left(\mathbb{R}^{d}\right)$. Our goal is to show that the sequence $\left\{T\left(x_{n}, y_{n}\right)\right\}_{n \in \mathbb{N}}$ is strongly convergent in $L^{r}\left(\mathbb{R}^{d}\right)$.

In view of the bilinearity of $T$, we have

$$
\begin{equation*}
T\left(f_{n}, g_{n}\right)-T(f, g)=T\left(f, g_{n}-g\right)+T\left(f_{n}-f, g\right)+T\left(f_{n}-f, g_{n}-g\right) \tag{2.7}
\end{equation*}
$$

Since $T$ is separately compact, the first two terms on the right-hand side of 2.7 converge to 0 in $L^{r}\left(\mathbb{R}^{d}\right)$ as $n \rightarrow \infty$. Therefore, it suffices to prove that if $f_{n}$ converges weakly to 0 in $L^{p}\left(\mathbb{R}^{d}\right)$ and $g_{n}$ converges weakly to 0 in $L^{q}\left(\mathbb{R}^{d}\right)$, then $T\left(f_{n}, g_{n}\right)$ converges to 0 in $L^{r}\left(\mathbb{R}^{d}\right)$.

Fix a subsequence $\left\{T\left(f_{n_{j}}, g_{n_{j}}\right)\right\}_{j \in \mathbb{N}}$. We show that it has a further subsequence that converges to 0 in $L^{r}\left(\mathbb{R}^{d}\right)$. For simplicity of notations, set $X=L^{p}\left(\mathbb{R}^{d}\right), Y=L^{q}\left(\mathbb{R}^{d}\right)$, and $Z=L^{r}\left(\mathbb{R}^{d}\right)$. Without loss of generality, we assume that $f_{n} \in B_{1}^{X}$ and $g_{n} \in B_{1}^{Y}$ for any

[^4]$n \in \mathbb{N}$. Note that the closed unit ball $B_{1}^{Z^{*}}$ is equicontinuous as a collection of continuous linear functionals on $Z$. Indeed, for any $h, h^{\prime} \in Z$ and $h^{*} \in Z^{*}$ with $\left\|h^{*}\right\|_{Z^{*}} \leq 1$, we have
$$
\left|\left\langle h^{*}, h\right\rangle-\left\langle h^{*}, h^{\prime}\right\rangle\right| \leq\left\|h^{*}\right\|_{Z^{*}}\left\|h-h^{\prime}\right\|_{Z} \leq\left\|h-h^{\prime}\right\|_{Z}
$$

Hence, the restriction of $B_{1}^{Z^{*}}$ to a compact set $E:=\overline{T\left(B_{1}^{X} \times B_{1}^{Y}\right)}$, denoted by $\left.\left(B_{1}^{Z^{*}}\right)\right|_{E}$, is a pointwise-bounded, equicontinuous collection of functions on a compact set $E$. By the ArzelàAscoli theorem, we obtain that $\left.\left(B_{1}^{Z^{*}}\right)\right|_{E}$ is a precompact subset of the space of continuous linear functionals on $E$.

Fix small $\varepsilon>0$. Then, for each $j \in \mathbb{N}$, there exists $\left.h_{j} \in\left(B_{1}^{Z^{*}}\right)\right|_{E}$ such that

$$
\begin{align*}
\left\|T\left(f_{n_{j}}, g_{n_{j}}\right)\right\|_{L^{r}} & =\sup _{h \in B_{1}^{Z^{*}}}\left|\left\langle T\left(f_{n_{j}}, g_{n_{j}}\right), h\right\rangle\right| \leq\left|\left\langle T\left(f_{n_{j}}, g_{n_{j}}\right), h_{j}\right\rangle\right|+\varepsilon  \tag{2.8}\\
& \leq\left\|T^{* 1}\left(h_{j}, g_{n_{j}}\right)\right\|_{X^{*}}+\varepsilon .
\end{align*}
$$

By the precompactness of $\left.\left(B_{1}^{Z^{*}}\right)\right|_{E}$, we can extract a subsequence $\left.\left\{h_{j_{k}}\right\}_{k \in \mathbb{N}} \subset\left(B_{1}^{Z^{*}}\right)\right|_{E}$ converging to $h_{\infty}$. Thus, there exists $N_{1}=N_{1}(\varepsilon) \in \mathbb{N}$ such that

$$
\begin{equation*}
\left\|T^{* 1}\left(h_{j_{k}}, g_{n_{j_{k}}}\right)-T^{* 1}\left(h_{\infty}, g_{n_{j_{k}}}\right)\right\|_{X^{*}} \lesssim\left\|T^{* 1}\right\|\left\|h_{j_{k}}-h_{\infty}\right\|_{Z^{*}}<\varepsilon \tag{2.9}
\end{equation*}
$$

for any $k \geq N_{1}$. Lastly, from the compactness of $T$ and Part (i) of this proposition, we see that $T^{* 1}$ is compact and thus is separately compact. Since $\left\{g_{n_{j_{k}}}\right\}_{k \in \mathbb{N}}$ converges weakly to 0 in $Y$, it follows from Lemma 2.1 (i) that $T^{* 1}\left(h_{\infty}, g_{n_{j_{k}}}\right)$ converges strongly to 0 in $X^{*}$ as $k \rightarrow \infty$. In particular, there exists $N_{2}=N_{2}(\varepsilon) \in \mathbb{N}$ such that

$$
\begin{equation*}
\left\|T^{* 1}\left(h_{\infty}, g_{n_{j_{k}}}\right)\right\|_{X^{*}}<\varepsilon \tag{2.10}
\end{equation*}
$$

for any $k \geq N_{2}$.
Therefore, putting (2.8), (2.9), and (2.10) together, we conclude that

$$
\left\|T\left(f_{n_{j_{k}}}, g_{n_{j_{k}}}\right)\right\|_{L^{r}}<3 \varepsilon
$$

for any $k \geq \max \left(N_{1}, N_{2}\right)$. Since the choice of $\varepsilon$ was arbitrary, we then conclude that the subsubsequence $\left\{T\left(f_{n_{j_{k}}}, g_{n_{j_{k}}}\right)\right\}_{k \in \mathbb{N}}$ converges strongly to 0 in $L^{r}\left(\mathbb{R}^{d}\right)$. This shows that the original sequence $T\left(f_{n}, g_{n}\right)$ converges to 0 in $L^{r}\left(\mathbb{R}^{d}\right)$ as $n \rightarrow \infty$.

## 3. Bilinear paraproducts with $C M O$ symbols

We first recall the definition of $B M O\left(\mathbb{R}^{d}\right)$, the space of functions of bounded mean oscillation. Given a locally integrable function $f$ on $\mathbb{R}^{d}$, its $B M O$-seminorm is given by

$$
\|f\|_{B M O}=\sup _{Q} \frac{1}{|Q|} \int_{Q}\left|f(x)-f_{Q}\right| d x
$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^{d}$ and $f_{Q}$ stands for the mean of $f$ over $Q$, namely

$$
f_{Q}=\frac{1}{|Q|} \int_{Q} f(x) d x
$$

We say that $f$ is of bounded mean oscillation if $\|f\|_{B M O}<\infty$, and denote

$$
B M O\left(\mathbb{R}^{d}\right)=\left\{f \in L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{d}\right):\|f\|_{B M O}<\infty\right\}
$$

As usual, we view this space as a space of equivalent classes of functions modulo additive constants. The closure of $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ in the $B M O$ topology is called the space of functions of
continuous mean oscillation, and it is denoted by $\operatorname{CMO}\left(\mathbb{R}^{d}\right)$. In the following, we suppress the underlying space $\mathbb{R}^{d}$ from our notation.

Let $\varphi, \psi \in C_{c}^{\infty}$ be radial functions such that $\operatorname{supp}(\varphi) \subset B_{1}, \widehat{\psi}(0)=0$, and

$$
\begin{equation*}
\int_{0}^{\infty}\left|\widehat{\psi}\left(t e_{1}\right)\right|^{2} \frac{d t}{t}=1 \tag{3.1}
\end{equation*}
$$

where $e_{1}=(1,0, \ldots, 0) \in \mathbb{R}^{d}$. For $t \in \mathbb{R}_{+}$, we also define the linear convolution operators $P_{t}$ and $Q_{t}$ by $P_{t} f=\varphi_{t} * f$ and $Q_{t} f=\psi_{t} * f$, where $h_{t}=t^{-d} h\left(t^{-1} \cdot\right)$ for a function $h$ on $\mathbb{R}^{d}$. Then, the Calderón reproducing formula [5] states the following

$$
\begin{equation*}
\int_{0}^{\infty} Q_{t}^{2} f \frac{d t}{t}=f \tag{3.2}
\end{equation*}
$$

in $L^{2}$, where $Q_{t}^{2} f=Q_{t}\left(Q_{t} f\right)=\psi_{t} * \psi_{t} * f$; see also [28].
Given $b \in B M O$, we now define a bilinear paraproduct $\Pi_{b}$ by ${ }^{5}$

$$
\begin{equation*}
\Pi_{b}(f, g)=\int_{0}^{\infty} Q_{t}\left(\left(Q_{t} b\right)\left(P_{t} f\right)\left(P_{t} g\right)\right) \frac{d t}{t} \tag{3.3}
\end{equation*}
$$

We have the following compactness result on the bilinear paraproduct $\Pi_{b}$; see also [16, Proposition 5.2].

Proposition 3.1. Let $1<p, q<\infty$ and $\frac{1}{2}<r<\infty$ be such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. If $b \in C M O$, then $\Pi_{b}$ defined in (3.3) is a compact bilinear Calderón-Zygmund operator from $L^{p} \times L^{q}$ into $L^{r}$, satisfying

$$
\begin{equation*}
\Pi_{b}(1,1)=b \quad \text { and } \quad \Pi_{b}^{* j}(1,1)=0, \quad j=1,2 . \tag{3.4}
\end{equation*}
$$

Proof. Fix $b \in C M O$. Since $b \in B M O$, it follows from [18, Lemma 5.1] that $\Pi_{b}$ is a bilinear Calderón-Zygmund operator, satisfying (3.4), that is bounded from $L^{p} \times L^{q}$ into $L^{r}$ for any $1<p, q<\infty$ and $\frac{1}{2}<r<\infty$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. In the following, we show that under the stronger assumption $b \in C M O$, the bilinear paraproduct $\Pi_{b}$ is indeed a compact bilinear operator from $L^{p} \times L^{q}$ into $L^{r}$.

Fix $1<p, q<\infty$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. We first show that $\Pi_{b}$ is compact from $L^{p} \times L^{q}$ into $L^{2}$. Let $\left\{\left(f_{n}, g_{n}\right)\right\}_{n \in \mathbb{N}} \subset L^{p} \times L^{q}$ such that $f_{n}$ converges weakly in $L^{p}$ and $g_{n}$ converges weakly in $L^{q}$. Moreover, we assume that either $f_{n}$ converges weakly to 0 or $g_{n}$ converges weakly to 0 as $n \rightarrow \infty$. Then, our goal is to show that $\left\|\Pi_{b}\left(f_{n}, g_{n}\right)\right\|_{L^{2}}$ converges to 0 as $n \rightarrow \infty$.

We first note that, since $b \in C M O$, the non-negative measure $\mu$ defined by

$$
\begin{equation*}
d \mu(x, t)=\left|Q_{t} b(x)\right|^{2} d x \frac{d t}{t} \tag{3.5}
\end{equation*}
$$

is a vanishing Carleson measure on $\mathbb{R}^{d+1}=\mathbb{R}^{d} \times \mathbb{R}_{+}$; see [14, Definition 1.3 and Remark 3.2].

[^5]Let $h \in L^{2}$ with $\|h\|_{L^{2}} \leq 1$. By using Hölder's inequality (in $t$ ), the square function estimate $]^{6]}$

$$
\begin{equation*}
\left\|\left(\int_{0}^{\infty}\left|Q_{t} h\right|^{2} \frac{d t}{t}\right)^{\frac{1}{2}}\right\|_{L^{2}} \lesssim\|h\|_{L^{2}} \leq 1, \tag{3.6}
\end{equation*}
$$

and (3.5), we obtain

$$
\begin{align*}
& \left|\left\langle\Pi_{b}\left(f_{n}, g_{n}\right), h\right\rangle\right| \\
& \leq \int_{0}^{\infty} \int_{\mathbb{R}^{d}}\left|\left(Q_{t} b(x) P_{t} f_{n}(x) P_{t} g_{n}(x)\right) Q_{t} h(x)\right| d x \frac{d t}{t} \\
& \leq\left(\int_{0}^{\infty} \int_{\mathbb{R}^{d}}\left|P_{t} f_{n}(x)\right|^{2}\left|P_{t} g_{n}(x)\right|^{2}\left|Q_{t} b(x)\right|^{2} d x \frac{d t}{t}\right)^{\frac{1}{2}}\left\|\left(\int_{0}^{\infty}\left|Q_{t} h\right|^{2} \frac{d t}{t}\right)^{\frac{1}{2}}\right\|_{L^{2}}  \tag{3.7}\\
& \lesssim\left(\int_{0}^{\infty} \int_{\mathbb{R}^{d}}\left|P_{t} f_{n}(x)\right|^{p}\left|Q_{t} b(x)\right|^{2} d x \frac{d t}{t}\right)^{\frac{1}{p}}\left(\int_{0}^{\infty} \int_{\mathbb{R}^{d}}\left|P_{t} g_{n}(x)\right|^{q}\left|Q_{t} b(x)\right|^{2} d x \frac{d t}{t}\right)^{\frac{1}{q}} \\
& =\left\|P_{t} f_{n}(x)\right\|_{L^{p}\left(\mathbb{R}_{+}^{d+1}, d \mu\right)}\left\|P_{t} g_{n}(x)\right\|_{L^{q}\left(\mathbb{R}_{+}^{d+1}, d \mu\right)},
\end{align*}
$$

uniformly in $h \in L^{2}$ with $\|h\|_{L^{2}} \leq 1$. Since $d \mu$ is a vanishing Carleson measure, it follows from [14, Theorem 2.1] that the convolution operator $P_{t}$ is compact from $L^{p}\left(\mathbb{R}^{d}\right)$ to $L^{p}\left(\mathbb{R}_{+}^{d+1} ; d \mu\right)$ for $1<p<\infty$. In view of the weak convergence of $f$ or $g$ to 0 , we then have

$$
\begin{equation*}
\left\|P_{t} f_{n}(x)\right\|_{L^{p}\left(\mathbb{R}_{+}^{d+1}, d \mu\right)} \longrightarrow 0 \quad \text { or } \quad\left\|P_{t} g_{n}(x)\right\|_{L^{q}\left(\mathbb{R}_{+}^{d+1}, d \mu\right)} \longrightarrow 0 \tag{3.8}
\end{equation*}
$$

as $n \rightarrow \infty$. From (3.7) and (3.8), we see that $\Pi_{b}\left(f_{n}, g_{n}\right)$ converges strongly to 0 in $L^{2}$. Hence, from Proposition 2.3 (i) and Remark 2.4 , we conclude that the bilinear paraproduct $\Pi_{b}$ is compact from $L^{p} \times L^{q}$ to $L^{2}$ with $1<p, q<\infty$ satisfying $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$.

Finally, recalling that $\Pi_{b}$ is also bounded from $L^{p} \times L^{q}$ to $L^{r}$ for all $1<p, q<\infty$ and $\frac{1}{2}<r<\infty$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, we conclude from interpolation of bilinear compactness [10, Theorem 5.2] (see also the proof of [10, Theorem 6.1]) that $\Pi_{b}$ is in fact compact from $L^{p} \times L^{q}$ to $L^{r}$ for all $1<p, q<\infty$ and $\frac{1}{2}<r<\infty$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$.
Remark 3.2. In the proof of Proposition 3.1, we needed to assume $p, q<\infty$ in applying [14, Theorem 2.1] on the compactness of $P_{t}$ from $L^{p}\left(\mathbb{R}^{d}\right)$ to $L^{p}\left(\mathbb{R}_{+}^{d+1} ; d \mu\right)$ and [10, Theorem 5.2] on interpolation of bilinear compactness. Compare this with the situation in [16], where the upper endpoint ( $p=\infty$ or $q=\infty$ ) is allowed; see [16, Remark 3.5].

Remark 3.3. In the linear case, the compact $T(1)$ theorem in [22, Theorem 1.1] provides an $L^{2}$-characterization of compact linear Calderón-Zygmund operators. By noting that a Calderón-Zygmund operator is $L^{p}$-bounded for all $1<p<\infty$, we see from Krasnosel'skii's interpolation result [21] that the compact $T(1)$ theorem in [22] is in fact a characterization of $L^{p}$-compactness for all $1<p<\infty$; see also [27, Remark 2.22]. See [9] for a discussion of interpolation results for compact linear operators between more general Banach spaces.
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Árpád Bényi, Department of Mathematics, 516 High St, Western Washington University, Bellingham, WA 98225, USA.

Email address: benyia@wwu.edu
Guopeng Li, School of Mathematics, The University of Edinburgh, and The Maxwell Institute for the Mathematical Sciences, James Clerk Maxwell Building, The King's Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom

Email address: guopeng.li@ed.ac.uk
Tadahiro Oh, School of Mathematics, The University of Edinburgh, and The Maxwell Institute for the Mathematical Sciences, James Clerk Maxwell Building, The King's Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom

Email address: hiro.oh@ed.ac.uk
Rodolfo H. Torres, Department of Mathematics, University of California, Riverside, 200 University Office Building, Riverside, CA 92521, USA

Email address: rodolfo.h.torres@ucr.edu


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[^1]:    ${ }^{1}$ While it may be more appropriate to use the term "extrapolation" as in [6, 19, 20, we follow [9, 10] and use the term "interpolation".

[^2]:    ${ }^{2}$ For this part, we do not need to assume that $X$ and $Y$ are Banach spaces. The result holds for normed vector spaces $X$ and $Y$.

[^3]:    ${ }^{3}$ Compare this with continuity; $T$ is continuous if and only if $T^{* 1}$ is continuous if and only if $T^{* 2}$ is continuous.

[^4]:    ${ }^{4}$ Here, we used the reflexivity of $L^{p}\left(\mathbb{R}^{d}\right), 1<p<\infty$.

[^5]:    ${ }^{5}$ Hereafter, as it is customary, we avoid a detailed explanation on the sense in which the integrals based on Calderón's formula converge to the represented objects. The interested reader can consult [28, 18] for precise explanations and [3] for further references.

[^6]:    ${ }^{6}$ In fact, in the current $L^{2}$ setting, by using (3.2), the first inequality in (3.6) is indeed an equality. One may also prove this fact via Plancherel's identity and the normalizing condition (3.1) with the radiality of $\psi$; see [25, p. 27]. For the general $L^{p}$ setting, $1<p<\infty$, see [25, Subsection I.8.3].

