SLOW-GROWING COUNTEREXAMPLES TO THE STRONG EREMENKO CONJECTURE

ANDREW P. BROWN

ABSTRACT. Let $f: \mathbb{C} \to \mathbb{C}$ be a transcendental entire function.

In 1989, Eremenko asked the following question concerning the set I(f) of points that tend to infinity under iteration: can every point of I(f) be joined to ∞ by a curve in I(f)? This is known as the *strong Eremenko conjecture* and was disproved in 2011 by Rottenfußer, Rückert, Rempe and Schleicher.

The function has relatively small infinite order: it can be chosen such that $\log \log |f(z)| = (\log |z|)^{1+o(1)}$ as $f(z) \to \infty$. Moreover, f belongs to the Eremenko-Lyubich class \mathcal{B} .

Rottenfußer et al also show that the strong Eremenko conjecture does hold for any $f \in \mathcal{B}$ of *finite* order. We consider how slow a counterexample $f \in \mathcal{B}$ can grow. Suppose that $\Theta: [t_0, \infty) \to [0, \infty)$ satisfies $\Theta(t) \to 0$ and

$$(\log t)^{-\beta\Theta(\log t)}/\Theta(t) \to 0 \quad \text{as } t \to \infty$$

for some $0 < \beta < 1$, along with a certain regularity assumption. Then there exists a counterexample $f \in \mathcal{B}$ as above such that

$$\log \log |f(z)| = O((\log |z|)^{1+\Theta(\log |z|)})$$
 as $f(z) \to \infty$.

The hypotheses are satisfied, in particular, for $\Theta(t) = 1/(\log \log t)^{\alpha}$, for any $\alpha > 0$.

1. INTRODUCTION

Let $f: \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. The escaping set of f is defined as

$$I(f) := \{ z \in \mathbb{C} \colon f^n(z) \to \infty \text{ as } n \to \infty \},\$$

where f^n denotes the *n*-th iterate of f.

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Already in 1926, Fatou [Fat26, p. 369] noticed the existence of curves contained within the escaping sets of certain transcendental entire functions and asked whether this is true of general transcendental entire functions. In 1989, Eremenko [Ere89] was the first to study the set I(f) in general. In particular, he made Fatou's question more precise, stating "It is plausible that the set I(f) always has the following property: every point $z \in I(f)$ can be joined with ∞ by a curve in I(f)."

This is known as the strong Eremenko conjecture and was answered in the negative by Rottenfußer, Rückert, Rempe and Schleicher [RRRS11, Theorem 1.1]. To discuss this example, let us introduce some terminology. An entire function f belongs to the Eremenko-Lyubich class \mathcal{B} if its set of critical and asymptotic values is bounded (see Section 2). Moreover, f has finite order if

$$\log \log |f(z)| = O(\log |z|)$$
 as $|f(z)| \to \infty$,

and *infinite order* otherwise. By [RRRS11, Theorem 1.2], the strong Eremenko conjecture holds for any $f \in \mathcal{B}$ of finite order.

The function f in [RRRS11, Theorem 1.1] belongs to the class \mathcal{B} , so its order must be infinite; it can be checked that

(1.1)
$$\log \log |f(z)| = O((\log |z|)^M) \quad \text{as } |f(z)| \to \infty$$

for some M > 1 [RRRS11, Proposition 8.1]. We investigate how close a counterexample to the strong Eremenko conjecture may be to having finite order of growth.

In [RRRS11], the authors also discuss, without giving all details of the analysis, certain modifications of the construction. Using these, f can be chosen so that (1.1) holds for every M > 1; in other words,

$$\log \log |f(z)| = O\left((\log |z|)^{1+o(1)}\right) \quad \text{as } |f(z)| \to \infty;$$

see [RRRS11, Proposition 8.3] and [Rem13, Theorem 1.10].

The goal of this paper is to give more precise estimates on the possible growth of such counterexamples. More precisely, suppose that Θ is a positive decreasing function (defined for all sufficiently large positive real numbers) such that $\Theta(t) \to 0$ as $t \to \infty$. When does there exist a counterexample $f \in \mathcal{B}$ to the strong Eremenko conjecture such that

(1.2)
$$\log \log |f(z)| = O\left((\log |z|)^{1+\Theta(\log |z|)}\right) \quad \text{as } |f(z)| \to \infty?$$

As f must have infinite order, we may suppose that

(1.3)
$$\Theta(t) \cdot \log t \to \infty \quad \text{as } t \to \infty.$$

Let us also require the following regularity condition:

(1.4)
$$\Theta(t^2)/\Theta(t) \to 1 \text{ as } t \to \infty.$$

Theorem 1.1. Let Φ be a positive decreasing function of one real variable such that $\Phi(t) \to 0$ as $t \to \infty$, and such that (1.3) and (1.4) hold. Suppose that, additionally, for some $0 < \beta < 1$, the function Θ satisfies

(1.5)
$$(\log t)^{-\beta\Theta(\log t)}/\Theta(t) \to 0 \quad as \ t \to \infty.$$

Then there exists $f \in \mathcal{B}$ satisfying (1.2) such that I(f) contains no curve to infinity.

It is plausible that the condition (1.5) is essentially optimal, in the following sense: If $f \in \mathcal{B}$ and there is $\beta > 0$ such that

(1.6)
$$(\log t)^{-\beta\Theta(\log t)}/\Theta(t) \to \infty \text{ as } t \to \infty,$$

then the strong Eremenko conjecture holds for f.

It is easy to check that, for any $\alpha > 0$, the function

$$\Phi(t) := \frac{1}{(\log \log t)^{\alpha}}$$

satisfies the hypotheses of Theorem 1.1.

Let us remark on the condition 1.4. It is a regularity condition, but also implies that Φ tends to zero more slowly than $1/(\log t)^{\alpha}$ for any $\alpha > 0$. It is easy to check that the latter functions do not satisfy (1.5) (in fact, they satisfy (1.6)). In other words, requiring the regularity condition 1.4 in the presence of (1.5) does not impose additional constraints on how fast Φ may tend to 0.

Notation and basic definitions. In this article the Riemann sphere is denoted by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the right half-plane by $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. We also make use of what is known as the *standard estimate* [Mil06, Cor. A.8]. If $V \subset \mathbb{C}$ is a simply connected domain, then the desnity λ_V of the hyperbolic metric on V satisfies

(1.7)
$$\frac{1}{2\operatorname{dist}(z,\partial V)} \le \lambda_V(z) \le \frac{2}{\operatorname{dist}(z,\partial V)}.$$

Structure of the paper. Section 2 introduces the *Eremenko–Lyubich class* \mathcal{B} functions and the logarithmic coordinates that arise naturally. Loosely speaking, we construct domains (tracts) in the logarithmic coordinate system that contain large 'wiggles', introduced in Section 3. These rectangular tracts, similar to those seen in [BR21], are adapted to include small openings at the initial turning point of each wiggle section so that the growth of the function can be maximised at parts of the tract where it has the least impact on the overall order of growth. We then show how such tracts can lead to counterexample functions in Section 4.

Once we have functions satisfying counterexample conditions, we then proceed in Section 5 to demonstrate how to estimate the growth of the function within the lower parts of the 'wiggle' sections, where the rate of growth is the greatest compared with the real part of the argument.

Given the improved growth estimate, we proceed in Section 6 to construct tracts that will be partially predetermined by a recurrence relation using a growth function ϕ between the real part of the end point of one wiggling section and the placement of the next wiggle. This is to aid us in estimating the order of growth of the function by capitalising on the mapping behaviour [RRRS11, Theorem 6.3 and Theorem 8.1]. Once this is determined the work required is to then ensure that the conditions for counterexamples are still met, which reduces to a 'shooting' problem of showing that the small openings included can be selected in a manner which still meets the counterexample conditions of Section 4.

In Section 7, the order of growth of the conformal isomorphism is determined and, with a further assumption on the function ϕ , we show how the tract constructed satisfies counterexample conditions and has the order of growth we desire.

Section 8 is spent on showing, using approximation methods of [Bis15], how the artificially constructed tract and conformal isomorphism in logarithmic coordinates can correspond to a class \mathcal{B} function f that is a counterexample to the strong Eremenko conjecture and has the order of growth desired.

2. EREMENKO–LYUBICH CLASS \mathcal{B} FUNCTIONS

Given a transcendental entire function $f: \mathbb{C} \to \mathbb{C}$ we recall that the set of *critical* values of f is $CV(f) := \{f(z) \in \mathbb{C} : f'(z) = 0\}.$

We say that $a \in \mathbb{C}$ is an *asymptotic value* of f if there exists a curve $\gamma \colon [0,\infty) \to \mathbb{C}$ with $\lim_{t\to\infty} |\gamma(t)| = \infty$ such that $a = \lim_{t\to\infty} f(\gamma(t))$ and we write the following $AV(f) \coloneqq \{a \in \mathbb{C} \colon a \text{ is an asymptotic value of } f\}$.

With these two sets, we define the set of (finite) singular values of f to be $S(f) := \overline{CV(f) \cup AV(f)}$.

We define the *Eremenko-Lyubich class* \mathcal{B} to be the class of functions f where S(f) is bounded as a subset of \mathbb{C} .

Given $f \in \mathcal{B}$ and suppose $D \subset \mathbb{C}$ is a bounded Jordan domain that contains $S(f) \cup \{0, f(0)\}$. Set $W := \mathbb{C} \setminus \overline{D}$ and $\mathcal{V} := f^{-1}(\mathbb{C} \setminus \overline{D})$. The connected components V of \mathcal{V} are called *tracts* of f. It can be shown that these tracts are simply connected and that $f: V \to W$ is a universal covering.

By ensuring $\{0, f(0)\} \subset D$, we know that $0 \notin V$ for all $V \in \mathcal{V}$. This means that we can make a logarithmic change of coordinates.

We now let $\mathcal{U} := \exp^{-1}(\mathcal{V})$ and $H := \exp^{-1}(W)$. We know that there exists an analytic function $F: \mathcal{U} \to H$ such that $\exp(F(z)) = f(\exp(z))$. We call F a *logarithmic transform* of f. The components $U \in \mathcal{U}$ are referred to as the *tracts* of F.

From the construction, we can see that the following conditions hold:

- (a) H is a $2\pi i$ -periodic Jordan domain that contains a right half-plane;
- (b) Every component U of \mathcal{U} is an unbounded Jordan domain with real parts bounded below, but unbounded from above;
- (c) The components of \mathcal{U} have disjoint closures and accumulate only at infinity; that is, if $(z_n)_{n=0}^{\infty} \subset \mathcal{U}$ is a sequence of points all belonging to different tracts, then $z_n \to \infty$ '
- (d) For every component U of $\mathcal{U}, F: U \to H$ is a conformal isomorphism. In particular, F extends continuously to the closure $\overline{\mathcal{U}}$ of \mathcal{U} in \mathbb{C} ;
- (e) For every component $U \in \mathcal{U}$, $\exp|_U$ is injective;
- (f) \mathcal{U} is invariant under translation by $2\pi i$.

We denote by \mathcal{B}_{\log} the class of all $F: U \to H$ such that H, \mathcal{U} , and F satisfy properties (a) – (f) regardless of whether they arise from an entire function $f \in \mathcal{B}$ or not. If F is also $2\pi i$ -periodic then we say that F belongs to the class \mathcal{B}_{\log}^p . In this paper, keeping with tradition and to paraphrase, we work in \mathcal{B}_{\log} and harvest in \mathcal{B} with the aid of approximation theory.

From [EL92, Lemma 2.1] we can see that for $F \in \mathcal{B}_{\log}$ there is an $\rho_0 > 0$ such that

$$|F'(z)| \ge 2$$

whenever $\operatorname{Re} F(z) \ge \rho_0$.

For $F \in \mathcal{B}_{\log}$, we say that F is of *disjoint type* if $\overline{\mathcal{U}} \subset H$. If our functions are of disjoint type, iteration is defined for all forward images and we consider the following sets.

Given $F \in \mathcal{B}_{\log}$ define

$$J(F) := \{ z \in \overline{\mathcal{U}} : F^n(z) \text{ is defined and in } \overline{\mathcal{U}} \text{ for all } n \ge 0 \} \text{ and } I(F) := \{ z \in J(F) : \operatorname{Re} F^n(z) \to \infty \text{ as } n \to \infty \}.$$

Note that if $f \in \mathcal{B}$ and F is a logarithmic transform of f then $\exp(J(F)) \subset I(f)$ and if F is of disjoint type then $\exp(J(F)) = J(f)$. It is also known for $f \in \mathcal{B}$ that $J(f) = \overline{I(f)}$ [Ere89, Corollary, p. 344]. Overall, given a disjoint type logarithmic transform F of a class \mathcal{B} function f, if I(F) does not contain a curve to infinity, then I(f) does not contain a curve to infinity also.

3. Tracts

In the construction of our counterexample, we will be considering tracts T that are contained in a half-strip that is 2π in height, that is,

$$T \subset \{z \in \mathbb{C} \colon \operatorname{Re} z > 4, |\operatorname{Im} z| < \pi\}.$$

Definition 3.1. Let \mathcal{T} be the collection of T such that:

- $5 \in T$,
- ∂T is locally connected
- There is only one access to ∞ in T.

Proposition 3.2. For any $T \in \mathcal{T}$, there exists a unique conformal isomorphism $F: T \to \mathbb{H}$ such that F(5) = 5 and F extends continuously to ∞ where

$$\lim_{z \to \infty} F(z) = \infty.$$

Proof. There exists a conformal isomorphism $F: T \to \mathbb{H}$ such that F(5) = 5 which is unique up to postcomposition by a Möbius transformation that fixes 5. By the Carathéodory–Torhorst theorem [Pom92, Theorem 2.1], $F^{-1}: \mathbb{H} \to T$ extends continuously to $\overline{\mathbb{H}} \cup \{\infty\}$. By the unique access to ∞ in the definition of \mathcal{T} , there is precisely one point $\zeta \in \partial \mathbb{H} \cup \{\infty\}$ such that $F^{-1}(\zeta) = \infty$. By postcomposing F with a Möbius transformation we may assume $\zeta = \infty$ which makes F unique. Given that $\overline{\mathbb{H}} \cup \{\infty\}$ is compact, F extends continuously to infinity. \Box

We denote the collection of these conformal isomorphisms by the following:

Definition 3.3. $\mathcal{H} := \{F: T \to \mathbb{H}: T \in \mathcal{T}, F(5) = 5, and F(\infty) = \infty\}.$

From the results in B we can also write the following.

Proposition 3.4. \mathcal{T} is homeomorphic to \mathcal{H} .

Proof. Riemann map from \mathbb{H} to \mathbb{D} and then use appendix results.

Definition 3.5. Given $T \in \mathcal{T}$ with corresponding conformal isomorphism $F \in \mathcal{H}$ and $\rho > 0$ we define $\Gamma_{\rho} := \{z \in T : |F(z)| = \rho\}$. We refer to Γ_{ρ} as a vertical geodesic of T.

We further define the following subclass of \mathcal{H} :

Definition 3.6. For $\nu > 0$,

 $\mathcal{H}_{\nu} := \{ F \in \mathcal{H} : \text{ for all } \rho > 0, \text{ diam}(\Gamma_{\rho}) < \nu \}$

where the diameter is understood to be taken in the Euclidean sense.

Finally define the following subset of T for a given F

Definition 3.7. For $F \in \mathcal{H}$ let $X(F) := \{z \in T : F^n(z) \in T \text{ for all } n \ge 0\}$.

3.1. Wiggles and gates.

Definition 3.8. Let $(r_j)_{j=0}^{\infty}$, $(R_j)_{j=0}^{\infty}$, $(\varepsilon_j)_{j=0}^{\infty}$ and $(\tau_j)_{j=0}^{\infty}$ be sequences of positive real numbers such that, for all $j \ge 0$:

- $r_0 > 6$, $R_j > r_j + 30$, and $r_{j+1} > R_j + 60$.
- $0 < \varepsilon_j \leq 1$,
- $r_j < \tau_j < R_j 1 3\pi$.

Giving: $R_j > 36 + 90j$ and $r_j > 6 + 90j$ We refer to a collection of such information by $\xi_0 := (r_j, R_j, \varepsilon_j, \tau_j)_{j=0}^{\infty}$ and further let Ξ_0 denote the collection of all possible data sets ξ_0 .

We will now construct tracts from data sets.

Definition 3.9. Given $\xi_0 \in \Xi_0$, let

$$L = L^{\xi_0} := \bigcup_{j=0}^{\infty} \Big[\{ r_j + ti \colon t \in [-\pi, \pi/3] \} \cup \{ t + \pi i/3 \colon t \in [r_j, R_j - 1] \} \\ \cup \{ t - \pi i/3 \colon t \in [r_j + 1, R_j] \} \cup \{ R_j + ti \colon t \in [-\pi/3, \pi] \} \\ \cup \{ \tau_j + ti \colon t \in [\pi/3, 2\pi(1 - \varepsilon_j)/3] \} \cup \{ \tau_j + ti \colon t \in [2\pi(1 + \varepsilon_j)/3, \pi] \} \Big].$$

Let $T = T^{\xi_0} := \{ z \in \mathbb{C} : \text{Re } z > 4, |\text{Im } z| < \pi \} \setminus L^{\xi_0}.$



FIGURE 1. A tract with gates.

For a given $\xi_0 \in \Xi_0$, there is a corresponding tract, $T^{\xi_0} \in \mathcal{T}$, and conformal isomorphism, $F^{\xi_0} \in \mathcal{H}$, such that $F^{\xi_0} : T^{\xi_0} \to \mathbb{H}$.

Definition 3.10. Let $\mathcal{T}^{\Xi_0} := \{T^{\xi_0} : \xi_0 \in \Xi_0\}.$

Note that $\mathcal{T}^{\Xi_0} \subset \mathcal{T}$.

Definition 3.11. $\mathcal{H}^{\Xi_0} := \{ F^{\xi_0} : T^{\xi_0} \to \mathbb{H} : T^{\xi_0} \in \mathcal{T}^{\Xi_0} \} \subset \mathcal{H}.$

Proposition 3.12. There exists $\nu_0 > 0$ such that $\mathcal{H}^{\Xi_0} \subset \mathcal{H}_{\nu_0}$.

Proof. This result follows from [BR21, Proposition 8.1].

By invoking the results of Appendix A we can achieve a concrete value of ν_0 in the context of the tracts considered.

Proposition 3.13. $\mathcal{H}^{\Xi_0} \subset \mathcal{H}_{60}$.

Proof. We consider rectangles that have a height:width ratio of 1:4 with the horizontal sides contained in the boundary of T. We know that the vertical geodesic passing through the midpoint must have an endpoint on each of the horizontal sides. The following diagram shows how to contain any possible geodesic by 'bricking up' the section of the tract surrounding any given point.

Throughout the rest of the paper we will refer to ν_0 achieved in Proposition 3.12 rather than using 60 so that it can be seen when we are making use of the fact that our vertical geodesics have bounded diameter.

We will make a simplification to our tracts by fixing the positions of the 'epsilon gates', τ_j , relative to the position of the corresponding R_j .

Definition 3.14. $\Xi := \{\xi_0 \in \Xi_0 : R_j - r_j > 2 + 3\nu_0, \tau_j = R_j - 2 - 2\nu_0 \text{ for all } j \ge 0\} \subset \Xi_0$ and denote the corresponding subclasses of tracts and isomorphisms by \mathcal{T}^{Ξ} and \mathcal{H}^{Ξ} . We refer to the elements of this subclass of data by ξ .

Instead of writing $\xi = (r_j, R_j, \varepsilon_j, R_j - 2 - 2\nu_0)_{j=0}^{\infty}$ we will just write $\xi = (r_j, R_j, \varepsilon_j)_{j=0}^{\infty}$ whenever $\xi \in \Xi$. We will typically suppress the explicit reference to the data set ξ and just write T and F to maintain clarity.

4. Conditions for counterexamples

If we impose a small set of conditions on the spacings between r_j and R_j and geodesics, we are able to produce a set of data that leads to a tract T and conformal isomorphism F where X(F) does not contain a curve to ∞ . In section 8 we will show how this F can be used to determine the existence of an $f \in \mathcal{B}$ that is a counterexample to the strong Eremenko conjecture.

Definition 4.1. Given $T \in \mathcal{T}^{\Xi}$ let us define

$$W_j := \left\{ z \in T \colon r_j \le \operatorname{Re} z \le R_j \text{ and } -\pi \le \operatorname{Im} z \le \frac{\pi}{3} \right\}.$$

This corresponds to the 'bottom two-thirds' of a wiggle.

Let us further define the corresponding subsets

$$W_j^+ := \left\{ z \in W_j : \frac{-\pi}{3} < \operatorname{Im} z < \frac{\pi}{3} \right\},\$$
$$W_j^- := \left\{ z \in W_j : -\pi < \operatorname{Im} z < \frac{-\pi}{3} \right\}$$

Definition 4.2. Suppose, for a set of data $\xi \in \Xi$ with corresponding $T \in \mathcal{T}^{\Xi}$ and $F \in \mathcal{H}^{\Xi}$, that for each $j \geq 0$ there are vertical geodesics C_j and \dot{C}_j and numbers ρ_{j+1} , $\dot{\rho}_{j+1} > 0$ where $C_j := \Gamma_{\rho_{j+1}}$ and $\dot{C}_j := \Gamma_{\dot{\rho}_{j+1}}$ such that the following conditions hold for some constant $\kappa > 0$:

(a) $r_j + 2 \le \rho_j < \rho_j + \kappa < \frac{\dot{\rho}_j}{2} < \dot{\rho}_j < R_j - 2 - 4\nu_0$

(b) C_j and \dot{C}_j have real parts strictly between $R_j - 2 - 3\nu_0$ and $R_j - 2 - \nu_0$

(c) C_j has imaginary parts between $\frac{-\pi}{3}$ and $\frac{\pi}{3}$

(d) \dot{C}_j has imaginary parts between $-\pi$ and $\frac{-\pi}{3}$

We then say that $\xi \in \Xi_{\mathcal{C}}$ and similarly define classes of corresponding tracts, $\mathcal{T}^{\Xi_{\mathcal{C}}}$, and conformal isomorphisms, $\mathcal{H}^{\Xi_{\mathcal{C}}}$.

Theorem 4.3. If $\xi \in \Xi_{\mathcal{C}}$ defines the tract $T \in \mathcal{T}^{\Xi_{\mathcal{C}}}$ and corresponding conformal isomorphism $F \in \mathcal{H}^{\Xi_{\mathcal{C}}}$, X(F) contains no curve to ∞ .

Proof. We proceed by contradiction after proving some initial claims.

Claim 1. If, for some $j \ge 0$, there is a curve $\gamma : [0,1] \to T$ such that $\gamma(0) \in W_j^+$ and $\gamma(1) \in W_j^-$ then there exists a $t^* \in (0,1)$ such that $\gamma(t^*) \in (r_j - \pi i/3, r_j + 1 - \pi i/3)$.

Proof. Assume initially that $\gamma([0,1]) \subset W_j$. The intermediate value theorem implies the existence of a $t^* \in (0,1)$ such that $\operatorname{Im} \gamma(t^*) = -\pi/3$. This must occur on the segment $(r_j - \pi i/3, r_j + 1 - \pi i/3)$ by the construction of T and definition of W_j . In the case where $\gamma([0,1]) \not\subset W_j$ it is possible to find a subcurve that satisfies the initial case by considering the maximal or minimal times when γ re-enters W_j either from above or the right hand side (respectively).

Claim 2. Suppose $w_0 \in X(F) \setminus \{5\}$, let $F(w_0) = w_1$ and let $F(w_j) = w_{j+1}$. There exists $m \in \mathbb{N}$ such that, for all $j \ge 0$, C_{m+j} separates w_j from ∞ and $|w_{j+1}| < \rho_{m+j+1}$.

Proof. We proceed by induction. Given $w_0 \in X(F)$ there exists some $m \ge 0$ such that $|w_0| \le r_m$ and consider the vertical geodesic C_m . Since $F(C_m)$ splits \mathbb{H} into two regions, one bounded and one unbounded, $T \setminus C_m$ is subsequently composed of one bounded region and one unbounded. It can be seen that w_0 is contained in the bounded region and therefore separated from ∞ since it can be connected to 5 by a curve that doesn't intersect C_m .

Now assume that w_j is separated from ∞ by C_{m+j} , that is, part of the bounded segment of $T \setminus C_{m+j}$. This tells us that $F(w_{m+j}) = w_{m+j+1}$ lies in the bounded part of $\mathbb{H} \setminus F(C_{m+j})$, or, $|w_{j+1}| \leq |F(C_{m+j})| = \rho_{m+j+1}$ which proves the claim. \bigtriangleup

Suppose there is a curve $\gamma \subset X(F)$ that tends to ∞ and let $w_0 \in \gamma$. From the claim above there is an $m \geq 0$ such that w_j is separated from infinity by C_{m+j} for all $j \geq 0$. Thus $|w_j| < \rho_{m+j} < \dot{\rho}_{m+j} < \rho_{m+j+1} < \dot{\rho}_{m+j+1}$. This means $F^j(\gamma)$ contains a subcurve connecting $F(C_{m+j})$ and $F(\dot{C}_{m+j})$; which means that $F^{j-1}(\gamma)$ contains a subcurve, $\tilde{\gamma}$ connecting C_{m+j} and \dot{C}_{m+j} . From the first claim there exists a point z_{j-1} in this subcurve that lies on the segment $(r_{j-1} - \pi i/3, r_{j-1} + 1 - \pi i/3)$; and from assumption (a), z_{j-1} is also surrounded by $F(C_{m+j-1})$ and $F(\dot{C}_{m+j-1})$. Let $\tilde{\gamma}$ be this particular subcurve. From an argument analogous to the that in the first claim, $\tilde{\gamma} \cap W_{m+j}^+$ and $\tilde{\gamma} \cap W_{m+j}^-$ both contain further subcurves connecting $F(C_{m+j-1})$ and $F(\dot{C}_{m+j-1})$. By repeating this, we conclude that γ contains 2^j subcurves connecting C_m and \dot{C}_m .

Since the curve can be parameterised, parameterise the subcurve that covers the section of γ between C_m and \dot{C}_m by

$$r\colon [0,1]\to X$$

where r(1) is the final time γ intersects C_m .

Since r is a map from one compact metric space to another it must be uniformly continuous. Here we take the metric on X, $d_X(r(a), r(b))$, be the hyperbolic length of the curve segment connecting r(a) and r(b) in T. As a consequence of uniform continuity, if, for a given $\varepsilon > 0$ and for some x and $y \in [0, 1]$ $d_X(r(x) - r(y)) > \varepsilon$ then there is a

 $\delta > 0$ such that $|x - y| > \delta$. From our assumptions we know that for all $u \in C_m$ and $v \in \dot{C}_m$, the length of any path joining the two must have a Euclidean length at least 2 since $1 < R_j - 2 - 4\nu_0 - (r_j + 1)$ and by invoking the standard estimate (1.7) we can deduce for any $u \in C_m$, and $v \in \dot{C}_m$ that $d_X(u, v) \ge 3/\pi > 1/2$. For any single subarc connectintg C_m and \dot{C}_m , parameterised by r on the subinterval $[a_k, b_k]$, the distance between the endpoints $d_X(r(a_k), r(b_k)) > 1/2$ which means $b_k - a_k > \delta_0$ for some $\delta_0 > 0$. We can write the following:

$$\sum_{k=1}^{2^{j}} (b_k - a_k) > 2^{j} \delta_0.$$

Since this holds for an arbitrary j, this diverges to infinity which contradicts

$$\bigcup_{k=1}^{2^j} [a_k, b_k] \subseteq [0, 1].$$

This proves the theorem.



FIGURE 2. The mapping behaviour of counterexample tracts

We spend the following sections showing how we can in fact produce a set of data that gives us an $F \in \mathcal{H}^{\Xi_c}$ which also satisfies certain growth conditions.

5. Growth of functions in \mathcal{H}^{Ξ}

The following result is analogous to the growth estimate in [BR21] so the proof proceeds in a similar manner.

Theorem 5.1. There exists a C > 1 such that, for any $\xi \in \Xi$ with corresponding tract T and conformal isomorphism $F: T \to \mathbb{H}$, the following inequality holds for all $j \ge 0$ and $z \in W_j$:

$$\frac{1}{C}\left(R_j + \sum_{k=0}^j \log\left(\frac{1}{\varepsilon_k}\right)\right) \le \log|F(z)| \le C\left(R_j + \sum_{k=0}^j \log\left(\frac{1}{\varepsilon_k}\right)\right).$$

Proof. Consider the arc that connects 5 to ∞ within T defined in the following way:

$$\alpha := [5, r_0 - \frac{1}{2}] \cup \bigcup_{k \ge 0} \left(r_k - \frac{1}{2}, i[0, 2\pi/3] \right) \cup \left([r_k - \frac{1}{2}, R_k - \frac{1}{2}] + 2\pi i/3 \right)$$
$$\cup \left(R_k - \frac{1}{2} + i[0, 2\pi/3] \right) \cup [r_k + \frac{1}{2}, R_k - \frac{1}{2}] \cup \left(r_k + \frac{1}{2} + i[-2\pi/3, 0] \right)$$
$$\cup \left([r_k + \frac{1}{2}, R_k + \frac{1}{2}] - 2\pi i/3 \right) \cup \left(R_k + \frac{1}{2} + i[-2\pi/3, 0] \right) \cup [R_k + \frac{1}{2}, r_{k+1} - \frac{1}{2}]$$

We will split α into a part α^0 that consists of the pieces that pass through the gates, and which must hence (if ε_j is small) pass close to the boundary of T, and a complementary part α^1 , which stays away from ∂T by a definite amount. More precisely, write $\alpha = \alpha^0 \cup \alpha^1$ as follows:

$$\alpha^{0} := \bigcup_{k \ge 0} \{ z \in \alpha \colon \tau_{k} - 1 \le \operatorname{Re} z \le \tau_{k} + 1, \operatorname{Im} z = 2\pi/3 \}$$
$$\alpha^{1} := \alpha \setminus \alpha^{0}$$

For $z \in \alpha$, let us denote the part of α that connects 5 to z by α_z . Similarly write $\alpha_z = \alpha_z^0 \cup \alpha_z^1$ where $\alpha_z^0 = \alpha_z \cap \alpha^0$ and $\alpha_z^1 = \alpha_z \cap \alpha^1$. Now let $z \in \alpha \cap W_j$ for some $j \ge 1$. We will estimate the hyperbolic length of α_z in T, $\ell_T(\alpha_z)$, by estimating the hyperbolic lengths of α^0 and α^1 separately.

Since α_z^1 stays away from the boundary, by (1.7) its hyperbolic length is comparable to its Euclidean length, $\ell_E(\alpha_z^1)$, which in turn is comparable to R_j by the definition of α . More precisely:

Claim 1. $\frac{1}{4\pi}R_j \leq \ell_T(\alpha_z^1) \leq 16R_j$. Proof. By definition of α ,

$$\ell_E(\alpha_z^1) \le R_j - 5 + \frac{8\pi}{3}j + 2\pi + 2\sum_{k=0}^j (R_k - r_k - 3)$$
$$\le R_j - 5 + 9(j+1) + 2\sum_{k=0}^j (R_k - r_k)$$
$$\le R_j - 5 + 9(j+1) + 2R_j \le \frac{31R_j}{10} + \frac{2}{5} \le 4R_j$$

Similarly,

$$\ell_E(\alpha_z^1) \ge R_j - \frac{1}{2} - 5 + \frac{8\pi}{3}j + \pi + 2\sum_{k=0}^{j-1} (R_k - r_k - 3) \ge R_j - 5.$$

Overall, since $R_j > 36$, we conclude that

$$\frac{R_j}{2} \le R_j - 5 \le \ell_E(\alpha_z^1) \le 4R_j.$$

Recall that the distance to the boundary of any point $\zeta \in \alpha_z^1$ is at least 1/2 and at most π . So by the standard estimate, we have $1/2\pi \leq \lambda_T(\zeta) \leq 4$. The claim follows. \triangle

Now we turn to estimating the hyperbolic length of α_z^0 , which is the section of α_z that passes through the epsilon gates. Let $\hat{\alpha}_k$ be the piece of α_z^0 that belongs to the k-th wiggle; i.e., $\hat{\alpha}_k = [\tau_k - 1, \tau_k + 1] + 2\pi i/3$. If we define $\mu_k(t) = \max(|t - \tau_k|, \pi \varepsilon_k/3)$, then for $z \in \hat{\alpha}_k$,

(5.1)
$$\mu_k(\operatorname{Re} z) = \max(|\operatorname{Re} z - \tau_k|, \pi \varepsilon_k/3) \le \operatorname{dist}(z, \partial T)$$
$$\le |\operatorname{Re} z - \tau_k| + \pi \varepsilon_k/3 \le 2\mu_k(\operatorname{Re} z).$$

We have

$$\int_{\hat{\alpha}_k} \frac{|dz|}{\mu_k(z)} = -2 \int_1^{\pi\varepsilon_k/3} \frac{dt}{t} + \int_{-\pi\varepsilon_k/3}^{\pi\varepsilon_k/3} \frac{3dt}{\pi\varepsilon_k} = 2\log\left(\frac{3}{\pi\varepsilon_k}\right) + 2 \le 2\log\left(\frac{1}{\varepsilon_k}\right) + 2.$$

By the standard estimate (1.7) and by (5.1), $1/(4\mu_k(\operatorname{Re} z)) \leq \lambda_T(z) \leq 2/\mu_k(\operatorname{Re} z)$ for $z \in \alpha_k^0$. Hence

$$\frac{1}{2}\log\left(\frac{1}{\varepsilon_k}\right) \le \ell_T(\hat{\alpha}_k) \le 4(\log\left(\frac{1}{\varepsilon_k}\right) + 1).$$

We can then write

$$\frac{1}{2}\sum_{k=0}^{j}\log\left(\frac{1}{\varepsilon_{k}}\right) \leq \ell_{T}(\alpha_{z}^{0}) \leq 4\left(j+1+\sum_{k=0}^{j}\log\left(\frac{1}{\varepsilon_{k}}\right)\right)$$
$$\frac{1}{2}\sum_{k=0}^{j}\log\left(\frac{1}{\varepsilon_{k}}\right) \leq \ell_{T}(\alpha_{z}^{0}) \leq 4\left(\frac{R_{j}}{25}+\sum_{k=0}^{j}\log\left(\frac{1}{\varepsilon_{k}}\right)\right).$$

Overall we find

$$\frac{1}{4\pi}R_j + \frac{1}{2}\sum_{k=0}^j \log\left(\frac{1}{\varepsilon_k}\right) \le \ell_T(\alpha_z) \le 16R_j + 4\left(\frac{R_j}{25} + \sum_{k=0}^j \log\left(\frac{1}{\varepsilon_k}\right)\right).$$

For a general $z \in W_j$, let us consider the vertical geodesic $\Gamma_{|F(z)|}$ and let $\tilde{z} := F^{-1}(|F(z)|)$. The Euclidean length of any path of any curve connecting 5 to z must be at least $R_j - 6$. Using this and considering the minimal contribution from passing through the gates, as seen in the discussion above, we can say

$$\operatorname{dist}_{T}(5,\tilde{z}) \geq \frac{1}{4\pi}(R_{j} - 6 - (j+1)) + \frac{1}{2}\sum_{k=0}^{j}\log\left(\frac{1}{\varepsilon_{k}}\right) \geq \frac{1}{25}\left(R_{j} + \sum_{k=0}^{j}\log\left(\frac{1}{\varepsilon_{k}}\right)\right).$$

We need to provide an upper-bound on $\operatorname{dist}_T(5, \tilde{z})$. From A.1 we know that a vertical geodesic is contained in the rectangular region $\{z \in T : R_j + 1/2 \leq \operatorname{Re} z \leq R_j + 1/2 + 8\pi\}$ that passes through the midpoint $w := R_j + 1/2 + 4\pi \in \alpha$ and also separates z and \tilde{z} from ∞ . We estimate the hyperbolic length of $\alpha_w \setminus \{z \in T : \operatorname{Re} z \leq R_j\}$.

dist_T(
$$R_j - \pi i/3, w$$
) $\leq 4\left(\frac{1}{2} + \frac{2\pi}{3} + 4\pi\right) < 65.$

Thus

$$\operatorname{dist}_{T}(5,\tilde{z}) \leq \operatorname{dist}_{T}(5,z) \leq \operatorname{dist}_{T}(5,w) \leq 16R_{j} + 4\left(\frac{R_{j}}{25} + \sum_{k=0}^{j} \log\left(\frac{1}{\varepsilon_{k}}\right)\right) + 65$$

Since F is a conformal isomorphism standard results in hyperbolic geometry give us the following

$$\operatorname{dist}_T(5,\Gamma_{|F(z)|}) = \operatorname{dist}_T(5,\tilde{z}) = \operatorname{dist}_{\mathbb{H}}(5,|F(z)|) = \log|F(z)| - \log 5$$

We can then say

$$\frac{1}{25} \left(R_j + \sum_{k=0}^j \log\left(\frac{1}{\varepsilon_k}\right) \right) + \log 5 \le \log|F(z)| \le 25 \left(R_j + \sum_{k=0}^j \log\left(\frac{1}{\varepsilon_k}\right) \right) + \log 5 + 65,$$
$$\frac{1}{30} \left(R_j + \sum_{j=0}^j \log\left(\frac{1}{\varepsilon_k}\right) \right) \le \log|F(z)| \le 30 \left(R_j + \sum_{k=0}^j \log\left(\frac{1}{\varepsilon_k}\right) \right).$$
Which follows for suitably large enough choice of r_0 .

Which follows for suitably large enough choice of r_0 .

We will refer to the constant C rather than take any particular value (e.g. 30 in the final steps of the proof) so that it is clear when this growth estimate is being used.

Corollary 5.2. There exists a C > 1 such that for any $z \in U_{j+1} := \{\zeta \in T : R_j < \zeta \in T : R_j < \zeta \in T \}$ $\operatorname{Re} \zeta < R_{j+1} \setminus W_{j+1}$ the following inequality holds:

$$\frac{1}{C} \left(\operatorname{Re} z + \sum_{k=0}^{j+1} \log\left(\frac{1}{\varepsilon_k}\right) \right) \le \log|F(z)| \le C \left(\operatorname{Re} z + \sum_{k=0}^{j+1} \log\left(\frac{1}{\varepsilon_k}\right) \right).$$

Proof. The proof follows analogously to Theorem 5.1.

6. Gate Selection: A Shooting Problem

Taking inspiration from the previous section, in order to find a function with a desired order of growth, we will impose a recurrence relation between r_i and R_i with a target order of growth for the corresponding conformal isomorphism F. With some further restrictions on the growth function and results regarding the conformal isomorphism involved, using the growth estimate in Theorem 5.1, we will be able to derive the existence of suitable data $\xi \in \Xi_{\mathcal{C}}$ such that $\log \operatorname{Re} F(z) = O\left((\operatorname{Re} z)^{1+o(1)}\right)$.

6.1. Introducing our growth function. Let $\Phi: [t_0, \infty) \to [0, \infty)$ (where $t_0 > 0$ depends on the Φ chosen but is always finite) be a strictly decreasing continuous function such that

(6.1)
$$\lim_{t \to \infty} \Phi(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} \Phi(t) \cdot \log t = \infty$$

and set

(6.2)
$$\phi(t) := t^{1+\Phi(t)}.$$

We can deduce that

- There is an $1 < A < \infty$ such that $\phi(t) \leq t^A$ for all $t \geq t_0$,
- $\lim_{t\to\infty} \frac{\phi(t)}{t} = \infty.$

The requirement for $\lim_{t\to\infty} \Phi(t) \cdot \log t = \infty$ is so that the order of growth of the resulting F remains infinite as class \mathcal{B} functions of finite order satisfy the strong Eremenko conjecture, see [RRRS11, Theorem 1.2].

Lemma 6.1. For every $\alpha > 0$ and M > 1 there exists a $t^* > t_0$ such that, for all $t > t^*$, $\phi(t + \alpha) \leq M\phi(t)$.

Proof. Let

$$t^* > \frac{\alpha}{M^{1/(1+\Phi(t_0))} - 1}$$

Which implies

$$M^{\frac{1}{1+\Phi(t_0)}} > 1 + \frac{\alpha}{t^*}.$$

If we suppose $t > t^*$ we can write the following:

$$(1 + \Phi(t + \alpha))\log(t + \alpha) - (1 + \Phi(t))\log t < (1 + \Phi(t_0))\log(1 + \frac{\alpha}{t}) < \log M.$$

Which gives the answer after rearranging.

Choosing r_i and R_i .

Given ϕ in the form of , let $r_0 > \max(6, \exp(\phi(t_0)))$. We know that $\phi(r_0) > r_0 > \log(r_0 + 3) > \phi(t_0)$. We deduce the existence of a $w_0 \in (t_0, r_0)$ such that $\phi(w_0) = \log(r_0 + 3)$. With this, let $R_0 := (r_0 + 3) \cdot \exp(9w_0)$.

Given a ϕ in the form of (6.2), r_0 and R_0 in the manner just described, we will be fixing $(r_j)_{i=0}^{\infty}$ and $(R_j)_{i=0}^{\infty}$ via the following relations for all $j \ge 0$:

(6.3)
$$r_{j+1} := \exp(\phi(R_j)) - 3 \text{ and } \log R_{j+1} := \phi(R_j) + 9R_j.$$

It can be found from a short calculation that this implies the following relation:

(6.4)
$$R_j = (r_j + 3) \cdot \exp(9\phi^{-1}(\log(r_j + 3)))$$

Which has the equivalent form:

$$\log R_j = \left(1 + \frac{9\phi^{-1}(\log(r_j + 3))}{\log(r_j + 3)}\right)\log(r_j + 3).$$

Note: r_0 remains to be chosen. This will be apparent in the following results.

The goal is to now show that, with this ϕ and pair of sequences, $(r_j)_{j=0}^{\infty}$ and $(R_j)_{j=0}^{\infty}$, there also exists a corresponding sequence $(\varepsilon_j)_{j=0}^{\infty}$ such that there is a collection of data $\xi \in \Xi_{\mathcal{C}}$ where the related conformal isomorphism, F, X(F) contains no curve to ∞ .

Standing Assumption: Throughout the rest of the paper, we assume all occurences of Φ , ϕ , R_j and r_j will be of the forms given in (6.1), (6.2) and (6.3) above. Let us also assume $\rho_0 > \max(6, t_0, \exp(\phi(t_0)))$. In recognition of this assumption we will now denote elements of Ξ by $\xi = (\Phi, r_0, (\varepsilon_j)_{i=0}^{\infty})$.

The main purpose of this section is to prove that, under these assumptions and for a suitable choice of r_0 , we can construct a set of data ξ with corresponding tract $T \in \mathcal{T}^{\Xi_c}$ and conformal isomorphism $F \in \mathcal{H}^{\Xi_c}$. We need to show that as long as r_0 is chosen to be large enough, we can deduce the existence of a suitable set of parameters $(\varepsilon_j)_{j=0}^{\infty}$, which we do by first showing that for a certain range of values ε_j takes, whenever $z \in W_j$, |F(z)| either 'overshoots' or 'undershoots' its intended target of $\exp(\phi(R_j))$ according to this range of possible ε_j values. We make use of the fact that our vertical geodesics have bounded diameter to ensure we can deduce the existence of some $\zeta_j \in W_j$ such that $F(\zeta_j) = \exp(\phi(R_j))$. We then invoke a corollary of the Poincaré–Miranda Theorem which can be described as multi-dimensional version of the Intermediate Value Theorem

to show that we can achieve this for all target values $(\exp(\phi(R_j)))_{j=1}^{\infty}$ simultaneously. Once this is achieved it is simply a matter of verifying that we satisfy the conditions laid out in Definition 4.2.

Theorem 6.2. Given a function ϕ satisfying our standing assumption, there exists a $\rho_0 > 0$ such that if $r_0 > \rho_0$ then there exists a sequence $(\varepsilon_j)_{j=0}^{\infty}$ such that, for resultant set of data $\xi = (\Phi, r_0, (\varepsilon_j)_{j=0}^{\infty}) \in \Xi$, the corresponding conformal isomorphism F satisfies $\operatorname{Re} F^{-1}(\exp \phi(R_j)) = R_j - 1 - 3\nu_0 \text{ and } \frac{-\pi}{3} < \operatorname{Im} F^{-1}(\exp \phi(R_j)) < \frac{\pi}{3} \text{ for all } j \ge 0.$ Moreover ξ is an element of $\Xi_{\mathcal{C}}$.

Before proving this we will first prove some smaller results and define the range of values that ε_i can take for each $j \ge 0$.

Define the following:

$$a_j := \frac{1}{\exp(2C\phi(R_j))}$$
 and $b_j := \frac{1}{\exp(\phi(R_j)/2C))}$

where C is the constant from Theorem 5.1.

Lemma 6.3. If $\varepsilon_j \in [a_j, b_j]$ for all $j \ge 0$ then, for any M > 0, there is some $\rho_0 > 0$ such that if $r_0 > \rho_0$ then

$$\sum_{k=0}^{j-1} M\phi(R_k) < R_j \quad for \ all \quad j \ge 0.$$

Proof. From the deductions following (6.2) we know that for any A > 1 there is some $\rho_1 > 0$ such that $t < \phi(t) < t^A$ for all $t > \rho_1$.

From the way we defined $(R_j)_{j=0}^{\infty}$ in (6.3) we know that if $R_0 > \rho_1$ then $R_k < \phi(R_k) < \phi(R_k)$ $\log R_j$ for all $0 \le k \le j-1$. Therefore there is some $\rho_2 > 0$ such that; if $R_j > \rho_2$ then $\log R_j > \frac{M}{90}R_j > jM$. There is also a $\rho_3 > 0$ such that if $R_j > \rho_3$ then $R_j > (\log R_j)^{A+1}$. So

$$\sum_{k=0}^{j-1} M\phi(R_k) < \sum_{k=0}^{j-1} MR_k^A < jM(\log(R_j))^A < (\log R_j)^{A+1} < R_j.$$

If we set $\rho_4 = \max\{\rho_1, \rho_2, \rho_3\}$ then the previous inequalities hold whenever $R_0 > \rho_4$. The result follows.

Proposition 6.4. There exists $\rho_0 > 0$ such that if $r_0 > \rho_0$ and if, for $j \ge 0$,

- $\varepsilon_j = a_j$ then $|F(z)| > R_{j+1}$ for $z \in W_j$, $\varepsilon_j = b_j$ then $|F(z)| < r_{j+1}$ for $z \in W_j$.

Proof. First let $\varepsilon_i = a_i$. This means that for $z \in W_j$, according to our growth estimate in Theorem 5.1,

$$\log|F(z)| \ge \frac{1}{C} \left(R_j + \sum_{k=0}^j \log\left(\frac{1}{\varepsilon_k}\right) \right) = \frac{1}{C} \left(R_j + \sum_{k=0}^{j-1} \log\left(\frac{1}{\varepsilon_k}\right) + 2C\phi(R_j) \right)$$
$$\ge 2\phi(R_j) > \phi(R_j) + 9R_j = R_{j+1}.$$

If r_0 is taken large enough.

Now let $\varepsilon_j = b_j$, remembering that $1/\varepsilon_k \leq 1/a_k$,

$$\log|F(z)| \le C\left(R_j + \sum_{k=0}^j \log\left(\frac{1}{\varepsilon_k}\right)\right) = C\left(R_j + \sum_{k=0}^{j-1} \log\left(\frac{1}{\varepsilon_k}\right) + \frac{\phi(R_j)}{2C}\right)$$
$$\le CR_j + \frac{\phi(R_j)}{2} + \sum_{k=0}^{j-1} 2C^2\phi(R_k) \le (C+1)R_j + \frac{\phi(R_j)}{2}$$
$$< \frac{2}{3}\phi(R_j) < \phi(R_j) < \log r_{j+1} < r_{j+1}.$$

Which follows once again for suitably large enough r_0 to satisfy both Theorem 5.1 and Lemma 6.3.

6.2. Tracts with fixed wiggles. The previous result means that we can study a collection of tracts where the positions of the 'wiggles' remain fixed, that is, after fixing Φ , we also fix an initial value of r_0 which then determines the subsequent values of r_j and R_j by the standing assumption. The fundamental mapping properties that we are interested in now only depend on the choice of $(\varepsilon_j)_{j=0}^{\infty}$ and if we ensure $\varepsilon_j \in [a_j, b_j]$ for all $j \geq 0$ then we can reduce our problem to an intermediate value problem in countably many variables. We will write $\Xi(\Phi, r_0) \subset \Xi$ to denote the sets of data $\xi = (\Phi, r_0, (\varepsilon_j)_{j=0}^{\infty})$ defined by a fixed choice of Φ and r_0 under the standing assumption. That is, the data sets only differ in the choice of $(\varepsilon_j)_{j=0}^{\infty}$. The corresponding classes of tracts and conformal isomorphisms will naturally be denoted by $\mathcal{T}_{\Xi(\Phi, r_0)}$ and $\mathcal{H}_{\Xi(\Phi, r_0)}$ respectively.

6.3. Defining signed distance. For a tract $T \in \mathcal{T}_{\Xi}$ we define the following region for each $j \geq 0$

$$\mathcal{Y}_j := \{ z \in T : R_j - 1 - 4\nu_0 \le \operatorname{Re} z \le R_j - 1 - 2\nu_0 \text{ and } -\pi/3 < \operatorname{Im} z < \pi/3 \}.$$

Observe that, for each $j, T \setminus \mathcal{Y}_j$ comprises a bounded component, which we denote by \mathcal{X}_j , and an unbounded component denoted by \mathcal{Z}_j .

We now define a notion of 'signed distance' on T. For $z \in T$ and $j \ge 0$ let

$$\delta(z,j) := \begin{cases} -1 & \text{if } z \in \mathcal{X}_j, \\ \frac{R_j - 1 - 3\nu_0 - \operatorname{Re} z}{\nu_0} & \text{if } z \in \mathcal{Y}_j, \\ 1 & \text{if } z \in \mathcal{Z}_j. \end{cases}$$

Note that for all $\xi \in \Xi(\Phi, r_0)$, these regions in the corresponding T^{ξ} are fixed and independent of the choice of $(\varepsilon_j)_{j=0}^{\infty}$.

We further define $\delta_j(\xi) := \delta(F^{-\xi}(\exp(\phi(R_j))), j)$ where $F^{-\xi}$ is understood to be the inverse of F^{ξ} .

Proposition 6.5. If a sequence of tracts $T_n \in T_{\Xi}$ converges to T with respect to 5 in the sense of Carathéodory kernel convergence then the corresponding sequence of conformal isomorphisms $F_n^{-1} \colon \mathbb{H} \to T_n$ converge locally uniformly to $F^{-1} \colon \mathbb{H} \to T$.

Proof. This is given by [BR21, Proposition 9.2].

The following result is analogous to [Rem13, Lemma 7.3 & Theorem 7.4].

Lemma 6.6. Suppose we are given a Φ satisfying the standing assumption; that $\Delta \subset \mathbb{N}$ and that we are given $(\tilde{\varepsilon}_j)_{j\in\mathbb{N}\setminus\Delta}$ where $\tilde{\varepsilon}_j \in [a_j, b_j]$ for all $j \geq 0$. Then there exists $\rho_0 > 0$ such that if $r_0 > \rho_0$ there is a sequence of $(\varepsilon_j)_{j\in\mathbb{N}}$ such that, for the resulting set of data $\xi = (\Phi, r_0, (\varepsilon_j)_{j=0}^{\infty}) \in \Xi$,

•
$$\varepsilon_i = \tilde{\varepsilon}_i$$
 for $j \notin \Delta$ and

• $\delta_j(\xi) = 0$ for all $j \in \Delta$.

Proof.

Claim. δ_i depends continuously on ξ .

Proof. We can write

$$\varepsilon_j(t) := \frac{1}{2}(a_j(1-t) + b_j(1+t)),$$

which gives a bijection between [-1, 1] and $[a_j, b_j]$. This shows, for $\xi \in \Xi(\Phi, r_0)$, that we have a natural bijection between $[-1, 1]^{\mathbb{N}}$ and $\mathcal{T}_{\Xi(\Phi, r_0)}$ and subsequently one with $\mathcal{H}_{\Xi(\Phi, r_0)}$. Continuity follows if we take the product topology, Carathéodory topology and topology of locally uniform convergence on the respective spaces.

Assume that ρ_0 is taken to be large enough to satisfy Proposition 6.4 and that $r_0 > \rho_0$ throughout. We will make use of the notation and results in Appendix C.

We first prove the case for finite Δ . If $\Delta = \{k_0\}$ then this becomes an application of the intermediate value theorem. Since $\delta_{k_0}(a_{k_0}) = -1$ and $\delta_{k_0}(b_{k_0}) = 1$ there exists some $t_{k_0} \in [-1, 1]$ such that for the corresponding $\varepsilon_{k_0} \in [a_{k_0}, b_{k_0}], \delta_{k_0}(\varepsilon_{k_0}) = 0$.

 $t_{k_0} \in [-1, 1]$ such that for the corresponding $\varepsilon_{k_0} \in [a_{k_0}, b_{k_0}], \ \delta_{k_0}(\varepsilon_{k_0}) = 0.$ Now consider $\Delta = \{k_0, k_1, \dots, k_{N-1}\}$ such that $|\Delta| = N > 1$. Let $p_j(t) := \delta_j(\varepsilon_j(t)), t_{\Delta} := (t_{k_0}, t_{k_1}, \dots, t_{k_{N-1}}) \in [-1, 1]^N$ and define $p_{\Delta} \colon \Lambda^N \to \Lambda^N$ to be

$$p_{\Delta}(t_{\Delta}) := (p_{k_0}(t_{k_0}), p_{k_1}(t_{k_1}), \dots, p_{k_{N-1}}(t_{k_{N-1}})).$$

For a given $t_{\Delta} \in \Lambda^N$, it can be seen by Proposition 6.4 that $p_{\Delta}(\Lambda_{k_i}^+) \subset \Lambda_{k_i}^+$ and $p_{\Delta}(\Lambda_{k_i}^-) \subset \Lambda_{k_i}^-$ for all $k_i \in \Delta$. By Corollary C.3, p_{Δ} is then surjective on Λ^N . Therefore there exists a $t_{\Delta}^* \in \Lambda^N$ such that $p_{\Delta}(t_{\Delta}^*) = 0$, which proves the result in the case of finite Δ .

If $\Delta = \{k_0, k_1, \ldots\}$ is infinite then we take an increasing sequence of finite subsets $\Delta_n := \{k_0, \ldots, k_{n-1}\}$ that exhausts Δ . At each step we find a

$$t_{\Delta_n}^* = (t_{\Delta_n,k_0}^*, t_{\Delta_n,k_1}^*, \dots, t_{\Delta_n,n-1}^*) \in \Lambda^n$$

as per the previous step which can be used to define

$$\mathfrak{t}_{n} := (t^{*}_{\Delta_{n}}, 0, 0, \ldots) = (t^{*}_{\Delta_{n}, k_{0}}, t^{*}_{\Delta_{n}, k_{1}}, \ldots, t^{*}_{\Delta_{n}, n-1}, 0, 0, \ldots) \in \Lambda^{\mathbb{N}}.$$

 $[-1,1]^{\mathbb{N}}$ is sequentially compact with the product topology so there exists a convergent subsequence with the limit $\mathfrak{t} = (t^*_{\Delta,k_0}, t^*_{\Delta,k_1}, \ldots) \in \Lambda^{\mathbb{N}}$ such that $\delta_{k_i}(t^*_{\Delta,k_i}) = 0$ for all $k_i \in \Delta$, thus proving the infinite case.

Proof of Theorem 6.2.

Let $\varepsilon_k = \frac{a_k + b_k}{2}$ for all $k \ge 0$, let $\Delta = \mathbb{N}$ and apply Lemma 6.6.

Lemma 6.7. The data set ξ in Theorem 6.2 belongs to $\Xi_{\mathcal{C}}$.

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Proof. This requires us to show that the conditions of Definition 4.2 are satisfied. First let us declare $\zeta_j := F^{-1}(\exp(\phi(R_j)))$ and by the previous lemma, we know that $\operatorname{Re} \zeta_j = R_j - 1 - 3\nu_0$. In accordance with the notation of Definition 4.2, $\dot{\zeta}_j := R_j - 1 - 3\nu_0 - 2\pi i/3$. Let $\rho_{j+1} := |F(\zeta_j)|, \ \dot{\rho}_{j+1} := |F(\dot{\zeta}_j)|, \ C_j := \Gamma_{\rho_{j+1}}$ and $\dot{C}_j := \Gamma_{\dot{\rho}_{j+1}}$.

Claim. $r_j + 2 \le \rho_j < \rho_j + \kappa < \frac{\dot{\rho}}{2} < \dot{\rho}_j < R_j - 2 - 4\nu_0.$

Proof. The first inequality holds by (6.3) and the second holds immediately for any choice of $\kappa > 0$. For this next inequality, we make use of the fact that

$$\log \dot{\rho}_{j+1} - \log \rho_{j+1} = \log |F(\dot{\zeta}_j)| - \log |F(\zeta_j)| = \operatorname{dist}_T(C_j, \dot{C}_j).$$

The geodesic that connects C_j and \dot{C}_j remains in W_j and so the distance of any point from ∂T is at most $\pi/3$ and the Euclidean length of such a curve is greater than $2(R_j - r_j - 4\nu_0 - 2)$. Using the standard estimate we find

$$\log \dot{\rho}_{j+1} - \log \rho_{j+1} \ge \frac{3}{\pi} (R_j - r_j - 4\nu_0 - 2) > \log 3$$

for suitably large enough r_0 . The third and fourth inequalities follow. Considering the distance between C_j and \dot{C}_j once more, we provide an upper bound in the following way. Consider the path α from the proof of Theorem 5.1 again, specifically the segment travelling between C_j and \dot{C}_j . The distance from the boundary is at least 1/2 and the Euclidean length can be bounded above by $2(R_j - r_j) + 2\pi/3$. Making use of the standard estimate once more we can write

$$\log \dot{\rho}_{j+1} - \log \rho_{j+1} < 8(R_j - r_j) + \frac{8\pi}{3} < 8R_j$$

and

$$\log R_{j+1} - \log \dot{\rho}_{j+1} = \log R_{j+1} - \log \rho_{j+1} - (\log \dot{\rho}_{j+1} - \log \rho_{j+1})$$
$$= 9R_j - (\log \dot{\rho}_{j+1} - \log \rho_{j+1}) > R_j.$$

Both of which follow immediately since $r_0 \geq 6$. This proves the final inquality. \triangle

The remaining conditions are immediately satisfied by the results of Theorem 6.2 and the application of Appendix A.1. $\hfill \Box$

We denote the subclass of $\Xi_{\mathcal{C}}$ satisfying Theorem 6.2 by $\Xi_{\mathcal{D}}$ and similarly define the classes of corresponding tracts and conformal isomorphisms by $\mathcal{H}^{\Xi_{\mathcal{D}}}$ and $\mathcal{T}^{\Xi_{\mathcal{D}}}$ respectively.

7. Achieving desired orders of growth

We continue to assume the properties of ϕ and Φ as per 6.2 but now make the following additional assumption.

(7.1)
$$\lim_{t \to \infty} \frac{\Phi(t^2)}{\Phi(t)} = 1$$

An immediate corollary to this follows

Corollary 7.1. For any $\alpha > 1$ and for any M > 1 there exists some $t^* > t_0$ such that for all $t > t^*$, $\Phi(t/M) \le \alpha \Phi(t)$.

Proposition 7.2. Given ϕ and any M > 1 there exists w_0 such that for all $w > w_0$, where $\phi(t) = w$, then

$$w^{1-M\Phi(w)} \le t \le w^{1-\frac{\Phi(w)}{M}}.$$

Proof. Note that

$$t = w^{\frac{1}{1 + \Phi(t)}} = w^{1 - \frac{\Phi(t)}{1 + \Phi(t)}}$$

and we can see that

$$\frac{\Phi(t)}{1+\Phi(t)} \cdot \frac{1}{\Phi(t)} \to 1 \text{ as } t \to \infty.$$

 $\phi(t)$ is strictly increasing so if $\phi(t) = w$ then w > t which means $\Phi(t) > \Phi(w)$. For any given $\varepsilon > 0$ there is some $t_0 > a$ such that for all $t > t_0$, $\phi(t) = t^{1+\Phi(t)} = w < t^{1+\varepsilon}$ which then gives $\Phi(t^{1+\varepsilon}) < \Phi(w)$ for all $t \ge t_0$. Therefore, overall, for any $\varepsilon > 0$ there is some $t_0 > a$ such that for all $t \ge t_0$,

$$\Phi(t^{1+\varepsilon}) \le \Phi(w) \le \Phi(t) \frac{\Phi(t^{1+\varepsilon})}{\Phi(t)} \le \frac{\Phi(w)}{\Phi(t)} \le 1.$$

Therefore

$$\frac{\Phi(w)}{\Phi(t)} \to 1 \text{ as } t \to \infty.$$

Given M > 1 let $1 + \tilde{\delta} < M$. There is some $t_0 > a$ such that for $w > t > t_0$, $1 + \Phi(t) < M$ and

$$M > 1 + \tilde{\delta} \ge \frac{\Phi(t)}{\Phi(w)} \ge \frac{\Phi(t)}{\Phi(w)} \cdot \frac{1}{1 + \Phi(t)} \ge \frac{\Phi(t)}{\Phi(w)} \cdot \frac{1}{M}$$

Therefore, for all $t > t_0$,

$$-M\Phi(w) \le -\frac{\Phi(t)}{1+\Phi(t)} \le -\frac{\Phi(w)}{M}$$

The result follows.

Proposition 7.3. For a given M > 1, there exists $\mu > 1$ such that for all $t > t_0$,

$$\phi(Mt) < \mu\phi(t).$$

Proof. By letting $\mu = M^{1+\Phi(t_0)}$ we can see that

$$\phi(Mt) = M^{1+\Phi(Mt)} t^{1+\Phi(Mt)} \le M^{1+\Phi(t_0)} t^{1+\Phi(Mt)} \le M^{1+\Phi(t_0)} t^{1+\Phi(t)} = \mu \phi(t).$$

Proposition 7.4. For every $\mu > 1$ there exists $t^* > t_0$ such that

$$\phi(t) + 9t \le \mu \phi(t)$$

for all $t > t_0$.

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Proof. Given $\mu > 1$, let $t^* > t_0$ be large enough so that $\log(9/(\mu - 1)) \le \Phi(t) \log t$ for all $t > t^*$. The result follows readily from the definitions and the standing assumption. \Box

Definition 7.5. Given a ϕ as per the standing assumption we define

$$\Psi(t) := \frac{9\phi^{-1}(\log t)}{\log t}$$

and

$$\psi(t) := t^{1+\Psi(t)}.$$

We also define the following for the purposes of approximation, following from Proposition 7.2. Given $0 < \alpha < 1$ let

$$\Psi_{\alpha}(t) := 20(\log t)^{-\alpha\Phi(\log t)}$$

and

$$\psi_{\alpha}(t) := t^{1+\Psi_{\alpha}(t)}$$

First we show the following results.

Lemma 7.6. For any $0 < \alpha < 1$ there exists $r_0 > 6$ such that, for all $j \ge 0$, $R_j \le \psi_{\alpha}(r_j)$.

Proof. We achieve this result after proving some small claims.

Claim. For any $0 < \alpha < 1$ there exists $r_0 > 6$ such that for all $j \ge 0$

$$\phi^{-1}(\log(r_j+3)) \le 2(\log(r_j))^{1-\alpha\Phi(\log(r_j))}$$

Proof. Given α , take any β such that $\alpha < \beta < 1$. Then, by our assumption there exists some $r_0 > 6$ such that for all $j \ge 0$,

$$\alpha < \frac{\Phi(\log(r_j+3))}{\Phi(\log(r_j))}\beta, \quad \text{equivalently}, \quad 1 - \beta\Phi(\log(r_j+3)) < 1 - \alpha\Phi(\log(r_j)).$$

We also know from Proposition 7.2 that for a large enough r_0 , the following holds for all $j \ge 0$.

$$\phi^{-1}(\log(r_j+3)) \le (\log(r_j+3))^{1-\beta\Phi(\log(r_j+3))}$$

The following deduction can be made:

$$\phi^{-1}(\log(r_j+3)) \le (\log(r_j+3))^{1-\beta\Phi(\log(r_j+3))} \le 2(\log(r_j))^{1-\beta\Phi(\log(r_j+3))} \le 2(\log(r_j))^{1-\alpha\Phi(\log(r_j))}$$

Claim. There is some $r_0 > 6$ such that for all $j \ge 0$

$$\log R_j \le \left(1 + \frac{10\phi^{-1}(\log(r_j + 3))}{\log(r_j)}\right)\log(r_j).$$

Proof. We know that there exists some $r_0 > 6$ such that

$$\left(\frac{1}{\phi^{-1}(\log(r_j+3))} + \frac{9}{\log(r_j)}\right)\log 2 \le 1$$

which implies

$$\left(1 + \frac{9\phi^{-1}(\log(r_j+3))}{\log(r_j)}\right)\log(2r_j) \le \left(1 + \frac{10\phi^{-1}(\log(r_j+3))}{\log(r_j)}\right)\log(r_j).$$

The result follows after comparing the left-hand side of the final inequality with the logarithmic formulation of (6.4). \triangle

Putting these results together we find:

$$\log R_{j} = \left(1 + \frac{9\phi^{-1}(\log(r_{j}+3))}{\log(r_{j}+3)}\right)\log(r_{j}+3) \le \left(1 + \frac{9\phi^{-1}(\log(r_{j}+3))}{\log(r_{j})}\right)\log(2r_{j})$$
$$\le \left(1 + \frac{10\phi^{-1}(\log(r_{j}+3))}{\log(r_{j})}\right)\log(r_{j}) \le \left(1 + 20\left(\log(r_{j})\right)^{-\alpha\Phi(\log(r_{j}))}\right)\log(r_{j})$$
$$= \log\left(\psi_{\alpha}(r_{j})\right).$$

Proposition 7.7. If

$$\frac{\Psi(t)}{\Phi(t)} \to 0 \quad as \quad t \to \infty$$

then for any $\gamma > 1$ there is some $r_0 > 6$ such that for all $j \ge 0$, $\log R_{j+1} = O\left((r_j)^{1+\gamma \Phi(r_j)}\right)$. Proof. We prove an initial claim first.

Claim. For any $0 < \alpha < 1$ there exists an $r_0 > 6$ such that for all $j \ge 0$,

$$\phi(r_j)^{\Psi_\alpha(r_j)} \le 2\phi(r_j).$$

Proof. Given that $\Psi_{\alpha} \to 0$ as $t \to \infty$, we can declare the existence of an $r_0 > 6$ such that

$$(\Psi_{\alpha}(r_j) - 1) \log \phi(r_j) \le \log 2$$

The result follows after rearranging.

Now suppose we are given a $\gamma > 1$. We know that for any $0 < \alpha < 1$ there is an $r_0 > 6$ such that the following holds

$$\log \log R_{j+1} \leq \log \mu \phi(R_j) \leq \log \mu \phi(\psi_\alpha(r_j))$$

$$\leq \log \mu + (1 + \Psi_\alpha(r_j))(1 + \Phi(r_j)) \log r_j$$

$$\leq \log \mu + (1 + \Psi_\alpha(r_j))(1 + \Phi(r_j)) \log r_j$$

$$\leq \log \mu + (1 + \gamma \Phi(r_j)) \log r_j \leq \log \mu(r_j)^{1 + \gamma \Phi(r_j)}.$$

Where the μ comes from Proposition 7.4. The result follows directly.

Theorem 7.8. For every Φ satisfying our standing assumption that also satisfies $\lim_{t\to\infty} \frac{\Psi(t)}{\Phi(t)} = 0$; there then exists a $\rho_0 > 0$ such that if $r_0 > \rho_0$ then there is a set of data $\xi \in \Xi_D$ such that the corresponding conformal isomorphism F satisfies

$$\log \operatorname{Re} F(z) = O(\phi(\operatorname{Re} z)).$$

Proof. If Φ satisfies the standing assumption then so does $\Phi/2$. With this we can find a $\rho_0 > 0$ such that if $r_0 > \rho_0$ there also exists a set of data $\xi = (\Phi/2, r_0, (\varepsilon_j)_{j=0}^{\infty})$ that satisfies both the conditions of Theorem 6.2 and Proposition 7.7, which we will apply taking $\gamma = 2$. Taking $z \in W_j$ we see

$$\log \operatorname{Re} F(z) \le \log |F(z)| \le \log R_{j+1} \le \mu \theta_2(r_j) = \mu \phi(r_j).$$

 \triangle

 \square

We conclude by showing that this order of growth is not exceeded in any other part of the tract derived from $\hat{\phi}$. Recalling Corollary 5.2 we can see for $z \in U_{j+1}$ where $\operatorname{Re} z < R_{j+1} - 3 - 2\nu_0$, that

$$\log|F(z)| \le C\left(\operatorname{Re} z + \sum_{k=0}^{j} \log\left(\frac{1}{\varepsilon_k}\right)\right) \le C\left(\operatorname{Re} z + \hat{\phi}(R_j)\right) \le C'\hat{\phi}(\operatorname{Re} z) \le C'\phi(\operatorname{Re} z).$$

Where C' > 1 is chosen suitably. For $z \in U_{j+1}$ where $R_{j+1} - 3 - 2\nu_0 < \text{Re} \, z < R_{j+1}$ we note

$$\log|F(z)| \le C\left(R_{j+1} + \sum_{k=0}^{j+1}\log\left(\frac{1}{\varepsilon_k}\right)\right) \le C\left(R_{j+1} + \sum_{k=0}^{j}2C\hat{\phi}(R_k) + 2C\hat{\phi}(R_{j+1})\right) \le C\left(R_{j+1} + (2C+1)\hat{\phi}(R_{j+1})\right) \le C'\hat{\phi}(R_j - 3 - 2\nu_0) \le C'\phi(R_j - 3 - 2\nu_0).$$

Once again for a suitable choice of C', noting the use of Lemma 6.1 in the penultimate inequality. The result follows since $\log \operatorname{Re} F(z)$ is defined for all $z \in T$ and never exceeds $\log |F(z)|$.

To provide a concrete example we can let

$$\Phi(t) = \frac{1}{(\log \log t)^{\alpha}}$$

for any $\alpha \geq 1$.

Definition 7.9. Denote the subclass of $\Xi_{\mathcal{D}}$ satisfying Theorem 7.8 by $\Xi_{\mathcal{E}}$.

8. Realising our models with Class $\mathcal B$ functions

In this section we will use methods of [Bis15], using a similar approach to [BR21, Proposition 11.1], to show that our tracts in the previous section correspond to a transcendental entire function g in class \mathcal{B} that is a counterexample to the strong Eremenko Conjecture and has the same order of growth as our constructed F.

8.1. **Bishop Models.** In order to use the results of [Bis15], we will introduce the relevant terminology first.

Suppose $\Omega = \bigcup_{j=0}^{\infty} \Omega_j \subset \mathbb{C}$ is a disjoint union of unbounded simply connected domains satisfying the following conditions.

- (1) Sequences of components of Ω accumulate only at infinity.
- (2) The set $\partial \Omega_j$ is connected for each j (as a subset of \mathbb{C}).

Such an Ω is called a *model domain*. If $\overline{\Omega} \cap \overline{\mathbb{D}} = \emptyset$ then we say that the model domain is of *disjoint type*.

Given a model domain, suppose that $\sigma \colon \Omega \to \mathbb{H}$ is holomorphic and that the following conditions hold.

- (1) The restriction of σ to each Ω_j is a conformal map $\sigma_j \colon \Omega_j \to \mathbb{H}$.
- (2) If $(z_n)_{n=0}^{\infty}$ is a sequence in Ω and $\sigma(z_n) \to \infty$, then $z_n \to \infty$.

Given such a $\sigma \colon \Omega \to \mathbb{H}$ we call $G(z) := \exp(\sigma(z))$ a model function.

A choice of both a model domain Ω and a model function G on Ω will be called a *model*. If Ω is of disjoint type then we call the overall model (Ω, G) a disjoint-type model with disjoint-type function G.

Given a model (Ω, G) and $\rho > 0$, we let

$$\Omega(\rho) := \{ z \in \Omega \colon |G(z)| > e^{\rho} \} = \sigma^{-1}(\{ x + iy \colon x > \rho \})$$

and

$$\Omega(\delta, \rho) := \{ z \in \Omega \colon e^{\delta} < |G(z)| < e^{\rho} \} = \sigma^{-1}(\{ x + iy \colon \delta < x < \rho \})$$

We also let $X(G) := \{z \in \Omega : G^n(z) \in \Omega \text{ for all } n \ge 0\}$ and $I(G) := \{z \in X(G) : \operatorname{Re} G^n(z) \to \infty \text{ as } n \to \infty\}$. The following result follows from [Bis15, Theorem 1.1] and [LV73, II §4.2].

Theorem 8.1. Suppose that (Ω, G) is a model. Then there is an $f \in \mathcal{B}$ and a homeomorphism $q: \mathbb{C} \to \mathbb{C}$ so that $G = f \circ q$ on $\Omega(2)$. In addition, the following conditions hold.

- (1) $S(f) \subset D(0, e)$.
- (2) We have $|f \circ q| \leq e^2$ off $\Omega(2)$ and $|f \circ q| \leq e$ off $\Omega(1)$. Thus the components of $\{z: |f(z)| > e\}$ are in a 1-to-1 correspondence with the components of Ω via q.
- (3) q is Hölder continuous with exponent 1/K, for some K > 1 independent of G and Ω , in every compact subset of Ω .
- (4) The map q^{-1} is conformal except on the set $\Omega(1/2, 2)$.

This implies the following result, as pointed out by Rempe [Bis15, Page 205]. Presented in this form by use of [Ere89, Corollary to Theorem 4].

Theorem 8.2. If G is any disjoint-type model, then there is a disjoint-type function $g \in \mathcal{B}$ and a homeomorphism $p: \mathbb{C} \to \mathbb{C}$ so that

 $g \circ p = p \circ G,$

on an open set that contains both $\overline{I(G)}$ and $\overline{I(g)}$.

8.2. Models from tracts.

Proof of Theorem 1.1. Suppose we are given ϕ satisfying our standing assumptions. The corresponding θ_4 also satisfies the standing assumption and so we can let $\xi \in \Xi_{\mathcal{E}}$ be the data derived from θ_4 according to Theorem 7.8; where the corresponding conformal isomorphism F satisfies

$$\log \operatorname{Re} F(z) = O((\operatorname{Re} z)^{1 + \Phi(\operatorname{Re} z)/2}.$$

$$\Omega = \exp(T) \quad \text{and} \quad G \colon w \mapsto \exp(F(\log(w))).$$

We let (Ω, G) be our disjoint-type model and by Theorem (8.1) we deduce the existence of corresponding homeomorphism q and $f \in \mathcal{B}$ where the Hölder exponent of q is supported on $\Omega(1, 2)$ and f is bounded on $\mathbb{C} \setminus \Omega(2)$.

Claim. I(f) contains no curve to ∞ .

Proof. Take the p and g from Theorem 8.2. We can say that $I(G) = \exp(I(F)) \subset \exp(X(F))$ which we know does not contain any curve to ∞ , implying that I(g) also does not contain any curve to ∞ . We also know that $p^{-1}g(u) = f(q(p^{-1}(u)))$ and from this we can deduce the claim. \bigtriangleup

Claim. f satisfies the desired growth condition.

Proof.

$$\begin{split} \log \log |f(\zeta)| &= \log \log |(\exp(F(z)))| \\ &\leq \log \operatorname{Re} F(z) \\ &= O((\operatorname{Re} z)^{1+\Phi(\operatorname{Re} z)/2}) \\ &= O((\log |q^{-1}(\zeta)|)^{1+\Phi(\log |q^{-1}(\zeta)|)/2}) \\ &\leq O((K \log |\zeta|)^{1+\Phi(\log |\zeta|)/2}) \\ &\leq O((K \log |\zeta|)^{1+\Phi(\log |\zeta|)/2}) \\ &\leq O(K^{1+\Phi(t_0)}(\log |\zeta|)^{1+\Phi(\log |\zeta|)}) \\ &= O((\log |\zeta|)^{1+\Phi(\log |\zeta|)}). \end{split}$$

Thereby proving the theorem.

APPENDIX A. GEOMETRY OF GEODESICS

This section reproduces results of Appendix A in [RRRS11].

Lemma A.1. (Geometry of geodesics) Consider the rectangle

 $Q = \{ z \in \mathbb{C} \colon |\operatorname{Re} z| < 4, |\operatorname{Im} z| < 1 \}$

and let $Y \subset \widehat{\mathbb{C}}$ be a simply connected Jordan domain with $Q \subsetneq Y$ such that $\partial Q \cap \partial Y$ consists exactly of the two horizontal boundary sides of Q. Let $P, R, P', R' \in \partial Y$ be four distinct boundary points in this cyclic order, subject to the condition that P and P'are in the boundary of different components of $Y \setminus Q$ and so that the quadrilateral Ywith the marked points P, R, P', R' has modulus 1.

Let γ be the hyperbolic geodesic in Y connecting R with R'. If $0 \in \gamma$, then the two endpoints of γ are on the horizontal boundaries of Q, one endpoint each on the upper and lower boundary.

Corollary A.2. For a marked quadrilateral $Y \subset \widehat{\mathbb{C}}$ of modulus 1 and a rectangle $Q \subsetneq Y$ where the vertical sides and the horizontal sides are in a ratio of 1: 4, suppose $\partial Q \cap \partial Y$ consists exactly of the two horizontal boundary sides of Q. The hyperbolic geodesic γ that passes through the midpoint of Q has endpoints on the horizontal sides of Q and is completely contained within Q.

Appendix B. Topology of Locally Uniform Convergence and Carathéodory Kernel Convergence

Here we recall some topological notions that are used throughout the paper. Much of the exposition follows that of [RL20], [McM94, 5.1] and [Car52, 119-123].

 \triangle

Definition B.1. Let Ω be a simpy-connected domain in $\widehat{\mathbb{C}}$. We say that a sequence of functions $f_n: \Omega \to \mathbb{C}$ converges locally uniformly to $f: \Omega \to \mathbb{C}$ on Ω if, for each $z_0 \in \Omega$ there exists a neighbourhood of $z_0 \ \Omega_0 \subset \Omega$ such that $f_n \to f$ uniformly on Ω_0 .

An easy exercise is the following:

Theorem B.2. $f_n \to f$ locally uniformly on Ω if and only if $f_n \to f$ uniformly on every compact subset of Ω .

The following definition is adapted from [Mil06, §3].

Definition B.3. Given a domain $\Omega \subset \mathbb{C}$ let $Map(\Omega, \mathbb{C})$ denote the set of continuous maps $f: \Omega \to \mathbb{C}$. Given a compact $K \subset \Omega$ and $\varepsilon > 0$ we define $N_{K,\varepsilon}(f) := \{g \in Map(\Omega, \mathbb{C}): |f(z) - g(z)| < \varepsilon$ for all $z \in K\}$. We say that $U \subset Map(\Omega, \mathbb{C})$ is open if, and only if, for every $f \in U$ there exists compact $K \subset \Omega$ and $\varepsilon > 0$ such that $N_{K,\varepsilon}(f) \subset U$. This characterisation of open sets is how we define the topology of locally uniform convergence on $Map(\Omega, \mathbb{C})$.

Definition B.4. A disk is any simpy-connected region in \mathbb{C} , possibly \mathbb{C} itself. \mathcal{D} is the set of pointed disks, (U,u), where U is a disk and $u \in U$. Let \mathcal{E} be the subspace of pointed disks (U,u) where $U \neq \mathbb{C}$.

Definition B.5. The Carathéodory topology on \mathcal{D} is defined in the following way. The sequence of pointed disks $(U_n, u_n) \to (U, u)$ if, and only if,

- (1) $u_n \to u$ in the usual sense.
- (2) For any compact $K \subset U$, there exists some $m \ge 0$ such that $K \subset U_n$ for all $n \ge m$.
- (3) For any open connected N containing u, if $N \subset U_n$ for infinitely many n then $N \subset U$.

Equivalently, convergence means $u_n \to u$ and for any subsequence such that $\mathbb{C} \setminus U_n \to K$ in the Haudorff topology on compact subsets of the sphere, U is equal to the component of $\widehat{\mathbb{C}} \setminus K$ that contains u.

The version of this taken from [Pom92, 1.4] is

Definition B.6. Let $u \in \mathbb{C}$ be given and let U_n be domains with $u \in U_n \subset \mathbb{C}$. We say that $U_n \to U$ as $n \to \infty$ with respect to u in the sense of kernel convergence if

- (1) either $U = \{u\}$, or U is a domain not equal to \mathbb{C} with $u \in U$ such that some neighbourhood of every $u \in U$ lies in U_n for large n;
- (2) for $v \in \partial U$ there exist $v_n \in \partial U_n$ such that $v_n \to v$ as $n \to \infty$.

We recall the following, taken from [Fal14, II.9]

Definition B.7. If $\mathcal{C}(\widehat{\mathbb{C}}) := \{V \subset \widehat{\mathbb{C}} : V \text{ is compact}\}$ and if $V_{\delta} := \{z \in \widehat{\mathbb{C}} : |z - v| \leq \delta \text{ for some } v \in V\}$ then $d_H(A, B) := \inf\{\delta \geq 0 : A \subset B_{\delta} \text{ and } B \subset A_{\delta}\}$ is a metric on $\mathcal{C}(\widehat{\mathbb{C}})$.

With this we can define the following

Definition B.8. Let (U, u) and $(V, v) \in \mathcal{E}$. Define $D((U, u), (V, v)) := \max\{d_H(\widehat{\mathbb{C}} \setminus U, \widehat{\mathbb{C}} \setminus V), |u - v|\}$. This defines a metric on \mathcal{E} with which we retrieve the Carathéodory topology.

If we let \mathcal{L} be the space of conformal maps $f: \mathbb{D} \to \mathbb{C}$ such that f'(0) > 0 and equip it with the topology of locally uniform convergence. There is a natural bijection $\pi: \mathcal{E} \to \mathcal{L}$ whereby each (U, u) is associated to the unique Riemann map $f: (\mathbb{D}, 0) \to (U, u)$ such that f(0) = u and f'(0) > 0, the existence of which comes from [Ahl78, Theorem 6.1].

Theorem B.9. $\pi: \mathcal{E} \to \mathcal{L}$ is a homeomorphism.

Proof. See [RL20] and [Car52].

The following result is quoted from [Pom92, Theorem 1.8] and [Sch93, 2.11], which themselves follow [Car52].

Theorem B.10. Given the pointed disk (U, u) and a sequence of pointed disks (U_n, u) , let f_n map \mathbb{D} conformally onto U_n with $f_n(0) = u$ and $f'_n(0) > 0$. If $U = \{u\}$ then let $f(z) \equiv u$. Otherwise, let f map \mathbb{D} conformally onto U with f(0) = u and f'(0) > 0. Then, as $n \to \infty$

 $f_n \to f$ locally uniformally in \mathbb{D} if, and only if $U_n \to U$ with respect to u.

Appendix C. The Poincaré–Miranda Theorem

This appendix draws heavily upon [Maw19] and [Kul97] where interesting historical context and background to the theorem are provided. The Poincaré–Miranda theorem, conjectured by Poincaré and then shown to be equivalent to the Brouwer Fixed Point Theorem by Miranda. The proof of Poincaré–Miranda theorem is reproduced from [Maw19] and the corollary, which is crucial in the proof of ?? comes from [Kul97].

The theorem itself can be thought of as a generalisation of the intermediate value theorem and we introduce the following notation.

We will be studying the *n*-dimensional cube $\Lambda^n := [-1, 1]^n$ and will be referring to its 'faces' so let

$$\Lambda_k^+ := \{ x \in \Lambda^n \colon x_k = 1 \}$$
$$\Lambda_k^- := \{ x \in \Lambda^n \colon x_k = -1 \}$$

denote the k-th pair of opposite faces.

Theorem C.1. The Poincaré–Miranda Theorem Let $p: \Lambda^n \to \mathbb{R}^n$, $p = (p_1, p_2, ..., p_n)$ be a continuous map such that for each $j \leq n$, $p_k(\Lambda_k^-) \subset (-\infty, 0]$ and $p_k(\Lambda_k^+) \subset [0, \infty)$. Then there exists a point $c \in \Lambda^n$ such that p(c) = 0.

Theorem C.2. Coincidence Theorem If maps $p, q: \Lambda^n \to \Lambda^n$ are continuous and if $p(\Lambda_k^-) \subset \Lambda_k^-$ and $p(\Lambda_k^+) \subset \Lambda_k^+$ for each k = 1, ..., n then there exists a point $c \in \Lambda^n$ such that q(c) = p(c).

Proof. Let r(x) := p(x) - q(x), this map satisfies the conditions of the Poincaré–Miranda theorem so there exists a point $c \in \Lambda^n$ such that r(c) = 0, that is, p(c) = q(c).

Corollary C.3. A continuous map $p: \Lambda^n \to \Lambda^n$ is surjective if $p(\Lambda_k^-) \subset \Lambda_k^-$ and $p(\Lambda_k^+) \subset \Lambda_k^+$ for each k = 1, ..., n.

Proof. Given $a \in \Lambda^n$, let $q_a(x) := a$. We can apply the Coincidence Theorem to deduce the existence of $c_a \in \Lambda^n$ such that $p(c_a) = q_a(c_a) = a$. This can be achieved for all $a \in \Lambda^n$ thus giving surjectivity.

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